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EXPONENTIAL DECAY OF SOLUTIONS TO A FOURTH-ORDER VISCOELASTIC EVOLUTION EQUATION IN $\mathbb{R}^n$

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ABSTRACT. In this article, we consider a Cauchy problem for a viscoelastic wave equation of fourth order. Under suitable conditions on the initial data and the relaxation function, we show that the rate of decay is exponential.

1. INTRODUCTION

In this work concerns the Cauchy problem

$$\begin{gathered} u_{tt} - \Delta u + u + \int_0^t g(t-s)(\Delta u(s) - u(s))ds - \Delta u_{tt} = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where u_0, u_1 are initial data and g is the relaxation function subjected to some conditions to be specified later. This type of evolution equations of fourth order arises in the study of strain solitary waves [7, 26] and in the theory of viscoelasticity when the material density depends on u_t, see [11, 23].

Hrusa and Nohel [14] studied the one-dimensional nonlinear viscoelastic equation

$$u_{tt} = (\phi(u_x(x,t)))_x - \int_0^t a'(t-s)(\psi(u_x(x,s)))_x ds = 0 \tag{1.2}$$

in $\mathbb{R}^n$. They proved, under reasonable conditions on ϕ, ψ and smallness condition on the initial data, the existence of a unique global classical solution. They also established an asymptotic result but no rate of decay was given. Dassios and Zafiropoulus [9] showed that for the same kernel the decay is of order $t^{-3/2}$, if the material is occupying the whole space $\mathbb{R}^3$. Muñoz [18] extended the result of Dassios and Zafiropoulus to $\mathbb{R}^n$. Precisely, he showed that if the kernel is decaying exponentially then the solution decays exponentially for material occupying bounded domains whereas the decay is of the order $t^{-n/2}$ for material occupying the whole n-dimensional space.

For nonexistence and formation of singularities, we mention the work by Dafermos [8] in 1985. Recently, Kafini and Messaoudi [15] considered the Cauchy problem

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$$\begin{gathered} u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)ds+u_t=|u|^{p-1}u, \quad x\in\mathbb{R}^n,\ t>0 \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\mathbb{R}^n\,. \end{gathered} \tag{1.3}$$

They showed that if the initial energy is negative and

$$\int_0^\infty g(s)ds<\frac{2p-2}{2p-1}, \quad \int_{\mathbb{R}^n}u_0u_1dx\ge 0,$$

then the solution blows up in finite time. Also, in [16], the same authors showed their blow-up result for the coupled system

$$\begin{gathered} u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)ds=f_1(u,v), \quad \text{in } \mathbb{R}^n\times(0,\infty) \\ v_{tt}-\Delta v+\int_0^t h(t-s)\Delta v(x,s)ds=f_2(u,v), \quad \text{in } \mathbb{R}^n\times(0,\infty) \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\mathbb{R}^n \\ v(x,0)=v_0(x), \quad v_t(x,0)=v_1(x), \quad x\in\mathbb{R}^n. \end{gathered} \tag{1.4}$$

For more results related to stability and asymptotic behavior of viscoelastic equations, we refer the reader to the books by Renardy et al. [24], Muñoz and Oquendo [19], Fabrizio and Morro [12], and Baretto et al. [1].

Most of the works [2, 3, 4, 5, 6] concerning the linear case of viscoelastic wave equations use assumptions of the form

$$1-\int_0^\infty g(s)ds=l>0, \tag{1.5}$$

and, for $a>0$,

$$g'(t)\le -ag^p(t), \quad 1\le p<3/2, \quad t\ge 0. \tag{1.6}$$

Lately, a few papers [13, 20, 21, 25] appeared with alternative conditions. For instance, Furati and Tatar [13] proved that for sufficiently small g and g' can give also an exponential decay. Namely, they assumed $g(t)e^{\alpha t}$ and $g'(t)e^{\alpha t}$ have small L^1-norms. Conditions like (1.5) or (1.6) are not imposed. In particular, g is not necessarily always negative. Recently, Messaoudi and Tatar [22] improved some earlier results concerning the exponential decay. They showed that the weak dissipation induced by the convolution term is sufficient to drive the system to rest with an exponential rate. Precisely, they established their result under the conditions

$$g'(t)\le 0 \quad \text{and} \quad \int_0^\infty g(t)e^{\alpha t}dt<+\infty \tag{1.7}$$

for some large positive constant α.

Our aim, in this paper, is to establish a rate of exponential decay for the energy of solutions to (1.1), under the same conditions on g and g' as in [22] but in $\mathbb{R}^n$. Unlike in the bounded domain case, Poincaré's inequality and some embedding inequalities are no longer valid. To overcome this difficulty, more recently, Kafini and Messaoudi [17], exploited the nature of the wave propagation. In our problem, we want to achieve our goal without using such property. We will define functionals with special type that are equivalent to the energy functional. We remark that our proof is also valid for bounded domains ($\Omega\subset\mathbb{R}^n$). Only it is needed to add the condition $u=0$ on $\partial\Omega\times(0,\infty)$ to the original system.

This paper is organized as follows. In section 2, we state the conditions needed on g, and present, without proof, a global existence result. Section 3 starts with five technical lemmas before the statement and proof of the main result.

2. Preliminaries

In this section we present some material needed for the proof of our result. For this goal, we use the assumptions:

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$1 - \int_0^\infty g(s)ds = l > 0, \quad t \geq 0.$$

(G2) $g'(t) \leq 0$ and $\int_0^\infty g(t)e^{\alpha t}dt < +\infty$, for some large positive α.

Proposition 2.1. *Assume that* (G1), (G2) *hold,* $u_0 \in H^1(\mathbb{R}^n)$*, and* $u_1 \in L^2(\mathbb{R}^n)$*, with compact support. Then* (1.1) *has a unique local solution*

$$u \in C([0,\infty); H^1(\mathbb{R}^n)), \quad u_t \in C([0,\infty); L^2(\mathbb{R}^n)) \cap L^2([0,\infty) \times \mathbb{R}^n).$$

Now, we introduce the "modified" energy functional

$$\begin{aligned} E(t) = \frac{1}{2}\Big[&\int_{\mathbb{R}^n} (|u_t|^2 + |\nabla u_t|^2)dx + \Big(1 - \int_0^t g(s)ds\Big) \int_{\mathbb{R}^n} |\nabla u|^2 dx \\ &+ \Big(1 - \int_0^t g(s)ds\Big) \int_{\mathbb{R}^n} |u|^2 dx + (g \boxdot u)(t)\Big] \end{aligned}$$

where

$$(g \boxdot u)(t) = \int_0^t g(t-s) \int_{\mathbb{R}^n} [|\nabla u(t) - \nabla u(s)|^2 + |u(t) - u(s)|^2]\,dx\,ds.$$

Lemma 2.2. *If u is a solution of* (1.1)*, then the "modified" energy satisfies*

$$E'(t) = \frac{1}{2}(g' \boxdot u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \leq \frac{1}{2}(g' \boxdot u) \leq 0. \tag{2.1}$$

Proof. By multiplying the equation in (1.1) by u_t and integrating over $\mathbb{R}^n$, using integration by parts and repeating the same computations as in [22], we obtain the result. □

In this paper, we use the notation

$$\overline{f} = \int_0^\infty |f(s)|ds.$$

3. Decay of solutions

In this section, we establish four lemmas, and then we state and prove our main result. to this end, we introduce the following functionals:

$$\Phi_1(t) := \int_{\mathbb{R}^n} \int_0^t G(t-s) \left[|\nabla u(t) - \nabla u(s)|^2 + |u(t) - u(s)|^2\right]\, ds\,dx \tag{3.1}$$

with $G(t) := e^{-\alpha t} \int_t^\infty e^{\alpha s} g(s)ds$,

$$\Phi_2(t) := \Big(\int_{\mathbb{R}^n} u u_t dx + \int_{\mathbb{R}^n} \nabla u . \nabla u_t dx\Big), \tag{3.2}$$

$$\begin{aligned}\Phi_3(t) := -\Big[\int_{\mathbb{R}^n} \nabla u_t . \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\ + \int_{\mathbb{R}^n} u_t \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\Big]\end{aligned} \tag{3.3}$$

$$F(t) := E(t) + \sum_{i=1}^{3} \gamma_i \Phi_i(t), \quad t \geq 0. \tag{3.4}$$

Lemma 3.1. *Assume* (G1), (G2) *hold. Then, for small enough* γ_2 *and* γ_3*, there exist two positive constants* ξ_1, ξ_2 *such that*

$$\xi_1 E(t) \leq F(t) \leq \xi_2 [E(t) + \Phi_1(t)]. \tag{3.5}$$

Proof. We estimate the terms in the above functionals using Young's inequality as follows

$$\int_{\mathbb{R}^n} u u_t dx \leq \|u\|_2^2 + \frac{1}{4}\|u_t\|_2^2, \tag{3.6}$$

$$\int_{\mathbb{R}^n} \nabla u . \nabla u_t dx \leq \|\nabla u\|_2^2 + \frac{1}{4}\|\nabla u_t\|_2^2, \tag{3.7}$$

$$\begin{aligned}&\int_{\mathbb{R}^n} \nabla u_t . \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\ &\leq \|\nabla u_t\|_2^2 + \frac{\overline{g}}{4}\int_0^t g(t-s)\int_{\mathbb{R}^n} |\nabla u(t)-\nabla u(s)|^2\,dx\,ds,\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\int_{\mathbb{R}^n} u_t \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\ &\leq \|u_t\|_2^2 + \frac{\overline{g}}{4}\int_0^t g(t-s)\int_{\mathbb{R}^n} |u(t)-u(s)|^2\,dx\,ds.\end{aligned} \tag{3.9}$$

By inserting (3.6)-(3.9) in (3.4), we obtain for some positive constant ξ_2,

$$\begin{aligned}F(t) &\leq \gamma_1\Phi_1(t) + (\frac{1}{2} + \frac{\gamma_2}{4} + \gamma_3)\int_{\mathbb{R}^n} |u_t|^2 dx \\ &\quad + (\frac{l}{2} + \gamma_2)\int_{\mathbb{R}^n} |\nabla u|^2 dx + (\frac{l}{2} + \gamma_2)\int_{\mathbb{R}^n} |u|^2 dx \\ &\quad + [\frac{1}{2} + \frac{\gamma_2}{4} + \gamma_3]\int_{\mathbb{R}^n} |\nabla u_t|^2 dx + [\frac{1}{2} + \frac{\overline{g}\gamma_3}{4}](g \boxdot u) \\ &\leq \xi_2[E(t) + \Phi_1(t)].\end{aligned} \tag{3.10}$$

Moreover, the same estimates give

$$\begin{aligned}F(t) &\geq (\frac{1}{2} - \frac{\gamma_2}{4} - \gamma_3)\int_{\mathbb{R}^n} |u_t|^2 dx \\ &\quad + (\frac{l}{2} - \gamma_2)\int_{\mathbb{R}^n} |\nabla u|^2 dx + (\frac{l}{2} - \gamma_2)\int_{\mathbb{R}^n} |u|^2 dx \\ &\quad + [\frac{1}{2} - \frac{\gamma_2}{4} - \gamma_3]\int_{\mathbb{R}^n} |\nabla u_t|^2 dx + [\frac{1}{2} - \frac{\overline{g}\gamma_3}{4}](g \boxdot u).\end{aligned}$$

By taking γ_2 and γ_3 small enough, we arrive, for some positive constant ξ_1,

$$F(t) \geq \xi_1 E(t). \tag{3.11}$$

Combining of (3.10) and (3.11), the result follows. ∎

Lemma 3.2. *If* (G1), (G2) *hold. Then* $\Phi_1(t)$ satisfies*, for any* $\delta_1, \delta_2 > 0$,

$$\Phi_1'(t) \le -\big(\alpha - \frac{2\overline{G}}{\delta_1} - \frac{2\overline{G}}{\delta_2}\big)\Phi_1(t) - (g \boxdot u) + \delta_1\|\nabla u_t\|_2^2 + \delta_2\|u_t\|_2^2. \tag{3.12}$$

Proof. We obtain the result by differentiating (3.1) and using Young's inequality as follows

$$\begin{aligned}\Phi_1'(t) &= -\alpha\Phi_1(t) - (g \boxdot u) + 2\int_{\mathbb{R}^n} \nabla u_t. \int_0^t G(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &\quad + 2\int_{\mathbb{R}^n} u_t \int_0^t G(t-s)(u(t)-u(s))\,ds\,dx \\ &\le -\alpha\Phi_1(t) - (g \boxdot u) + \delta_1\|\nabla u_t\|_2^2 + \frac{2}{\delta_1}\overline{G}\Phi_1(t) + \delta_2\|u_t\|_2^2 + \frac{2}{\delta_2}\overline{G}\Phi_1(t),\end{aligned}$$

where

$$\overline{G} = \int_0^\infty G(s)ds = \int_0^\infty \Big(e^{-\alpha t}\int_t^\infty e^{\alpha s}g(s)ds\Big)dt \le \frac{1}{\alpha}\int_0^\infty e^{\alpha s}g(s)ds < \infty.$$

□

Lemma 3.3. *Assume* (G1), (G2) *hold. Then along the solution of* (1.1)*, for any* $\delta_3, \delta_4 > 0$*, the function* $\Phi_2(t)$ *satisfies*

$$\Phi_2'(t) \le \|u_t\|_2^2 + \|\nabla u_t\|_2^2 - (l-\delta_3)\|\nabla u\|_2^2 - (l-\delta_4)\|u\|_2^2 + \frac{\overline{g}}{4\delta_5}(g \boxdot u). \tag{3.13}$$

Proof. By differentiating (3.2), we have

$$\Phi_2'(t) = \int_{\mathbb{R}^n} |u_t|^2dx + \int_{\mathbb{R}^n} uu_{tt}dx + \int_{\mathbb{R}^n} |\nabla u_t|^2dx + \int_{\mathbb{R}^n} \nabla u.\nabla u_{tt}dx. \tag{3.14}$$

Along (1.1), we find

$$\begin{aligned}&\int_{\mathbb{R}^n} uu_{tt}dx + \int_{\mathbb{R}^n} \nabla u.\nabla u_{tt}dx \\ &= -\int_{\mathbb{R}^n} |\nabla u|^2dx + \int_{\mathbb{R}^n} \nabla u. \int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ &\quad - \int_{\mathbb{R}^n} |u|^2dx - \int_{\mathbb{R}^n} u\int_0^t g(t-s)u(s)\,ds\,dx;\end{aligned}$$

thus (3.14) becomes

$$\begin{aligned}\Phi_2'(t) &= \int_{\mathbb{R}^n} |u_t|^2dx + \int_{\mathbb{R}^n} |\nabla u_t|^2dx - \int_{\mathbb{R}^n} |\nabla u|^2dx - \int_{\mathbb{R}^n} |u|^2dx \\ &\quad + \int_{\mathbb{R}^n} \nabla u. \int_0^t g(t-s)\nabla u(s)\,ds\,dx - \int_{\mathbb{R}^n} u\int_0^t g(t-s)u(s)\,ds\,dx.\end{aligned} \tag{3.15}$$

Using the estimates

$$\begin{aligned}&\int_{\mathbb{R}^n} \nabla u(t). \int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ &\le \delta_3\|\nabla u\|_2^2 + \frac{\overline{g}}{4\delta_3}\int_{\mathbb{R}^n}\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx + \overline{g}\|\nabla u\|_2^2\end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} u(t)\int_0^t g(t-s)u(s)\,ds\,dx \\
&\leq \delta_4\|u\|_2^2+\frac{\overline{g}}{4\delta_4}\int_{\mathbb{R}^n}\int_0^t g(t-s)|u(t)-u(s)|^2\,ds\,dx+\overline{g}\|u\|_2^2.
\end{aligned}
\tag{3.17}
$$

Adding (3.16) and (3.17), for $\delta_5=\min\{\delta_3,\delta_4\}$, yields

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u(t).\int_0^t g(t-s)\nabla u(s)\,ds\,dx-\int_{\mathbb{R}^n} u(t)\int_0^t g(t-s)u(s)\,ds\,dx \\
&\leq \delta_3\|\nabla u\|_2^2+\overline{g}\|\nabla u\|_2^2+\delta_4\|u\|_2^2+\overline{g}\|u\|_2^2+\frac{\overline{g}}{4\delta_5}(g\boxdot u).
\end{aligned}
\tag{3.18}
$$

Inserting (3.18) in (3.15) gives the desired result (3.13). □

Lemma 3.4. *Suppose* (G1), (G2) *hold. Then along the solution of* (1.1)*, for any* $\delta_6,\delta_7,\delta_9,\delta_{10},\delta_{12},\delta_{13}>0$*, the function* $\Phi_3(t)$ *satisfies*

$$
\begin{aligned}
\Phi_3'(t) \leq &-\Big(\int_0^t g(s)ds-\delta_9\Big)\|\nabla u_t\|_2^2-\Big(\int_0^t g(s)ds-\delta_{10}-\delta_{13}\Big)\|u_t\|_2^2 \\
&+(\delta_6+\delta_{12})\|\nabla u\|_2^2+\delta_7\|u\|_2^2+\Big(\frac{1}{4\delta_8}+\frac{1}{4\delta_{14}}+1\Big)\overline{g}(g\boxdot u) \\
&-\frac{g(0)}{4\delta_{11}}(g'\boxdot u).
\end{aligned}
\tag{3.19}
$$

Proof. Differentiation of (3.3) yields

$$
\begin{aligned}
\Phi_3'(t) = &-\int_{\mathbb{R}^n} \nabla u_{tt}.\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&-\int_{\mathbb{R}^n} \nabla u_t.\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&-\int_{\mathbb{R}^n} u_{tt}\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&-\int_{\mathbb{R}^n} u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \\
&-\Big(\int_0^t g(s)ds\Big)\|u_t\|_2^2-\Big(\int_0^t g(s)ds\Big)\|\nabla u_t\|_2^2.
\end{aligned}
\tag{3.20}
$$

Along (1.1), we find

$$
\begin{aligned}
&\int_{\mathbb{R}^n} u_{tt} \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&\quad + \int_{\mathbb{R}^n} \nabla u_{tt}. \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&= -\int_{\mathbb{R}^n} \nabla u. \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\quad - \int_{\mathbb{R}^n} u \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&\quad + \int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)\nabla u(s)ds. \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)dx \\
&\quad - \int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)u(s)ds. \int_0^t g(t-s)(u(t)-u(s))ds\Big)dx.
\end{aligned} \tag{3.21}
$$

The first two terms in the right side of (3.21) can be estimated as follows

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u. \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\le \delta_6\|\nabla u\|_2^2 + \frac{\overline{g}}{4\delta_6}\int_{\mathbb{R}^n}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx,
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n} u \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&\le \delta_7\|u\|_2^2 + \frac{\overline{g}}{4\delta_7}\int_{\mathbb{R}^n}\int_0^t g(t-s)|u(t)-u(s)|^2\,ds\,dx,
\end{aligned} \tag{3.23}
$$

from these two estimates, for $\delta_8 = \min\{\delta_6, \delta_7\}$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u. \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx + \int_{\mathbb{R}^n} u \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&\le \delta_6\|\nabla u\|_2^2 + \delta_7\|u\|_2^2 + \frac{\overline{g}}{4\delta_8}(g \boxdot u).
\end{aligned} \tag{3.24}
$$

Using

$$
\int_{\mathbb{R}^n} \Big|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big|^2 dx \le \overline{g}\int_{\mathbb{R}^n}\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx,
$$

we estimate the last two terms in (3.21), for $\delta_{14} = \min\{\delta_{12}, \delta_{13}\}$, as follows

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)\nabla u(s)ds. \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)dx \\
&\quad - \int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)u(s)ds. \int_0^t g(t-s)(u(t)-u(s))ds\Big)dx \\
&\le \int_{\mathbb{R}^n} \nabla u(t). \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\
&\quad + \int_{\mathbb{R}^n} u(t) \int_0^t g(t-s)(u(t)-u(s))\,ds\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^n}\Big|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big|^2dx\\
&+\int_{\mathbb{R}^n}\Big|\int_0^t g(t-s)(u(t)-u(s))ds\Big|^2dx\\
&\le \delta_{12}\|\nabla u\|_2^2+\delta_{13}\|u_t\|_2^2+\frac{\overline{g}}{4\delta_{14}}(g\boxdot u)+\overline{g}(g\boxdot u).
\end{aligned}
$$

Similarly, the second and the fourth term of (3.20) can be handled as follows

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\le \delta_9\|\nabla u_t\|_2^2-\frac{g(0)}{4\delta_9}\int_{\mathbb{R}^n}\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^n}u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\\
&\le \delta_{10}\|u_t\|_2^2-\frac{g(0)}{4\delta_{10}}\int_{\mathbb{R}^n}\int_0^t g'(t-s)|u(t)-u(s)|^2\,ds\,dx,
\end{aligned}
\tag{3.26}
$$

from the (3.25) and (3.26), for $\delta_{11}=\min\{\delta_9,\delta_{10}\}$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\nabla u_t.\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx+\int_{\mathbb{R}^n}u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\\
&\le \delta_9\|\nabla u_t\|_2^2+\delta_{10}\|u_t\|_2^2-\frac{g(0)}{4\delta_{11}}(g'\boxdot u).
\end{aligned}
\tag{3.27}
$$

Combining (3.20)-(3.27), the result follows. □

Theorem 3.5. *Assume* (G1), (G2) *hold for large* α. *Then, for any* $t_0>0$, *there exist two positive constants* K *and* k *such that*

$$
E(t)\le Ke^{-kt}.
$$

Proof. Differentiating (3.4) and using (2.1) yields

$$
F'(t)=E'(t)+\sum_{i=1}^3\gamma_i\Phi_i'(t)\le\frac{1}{2}(g'\boxdot u)+\sum_{i=1}^3\gamma_i\Phi_i'(t).
\tag{3.28}
$$

Since g is continuous and $g(0)>0$ then, for any $t\ge t_0>0$, we have

$$
\int_0^t g(s)ds\ge\int_0^{t_0}g(s)ds=g_0>0.
$$

By inserting (3.12), (3.13) and (3.19) in (3.28), we obtain

$$
\begin{aligned}
F'(t)\le&-\big(\alpha-\frac{2\overline{G}}{\delta_1}-\frac{2\overline{G}}{\delta_2}\big)\gamma_1\Phi_1(t)+[\frac{1}{2}-\frac{\gamma_3 g(0)}{4\delta_{11}}](g'\boxdot u)\\
&-\big[\gamma_1-\overline{g}(\frac{\gamma_2}{4\delta_5}+\gamma_3(\frac{1}{4\delta_8}+\frac{1}{4\delta_{14}}+1))\big](g\boxdot u)\\
&-[\gamma_2(l-\delta_3)-\gamma_3(\delta_6+\delta_{12})]\|\nabla u\|_2^2\\
&-[\gamma_3(g_0-\delta_9)-\gamma_2-\gamma_1\delta_1]\|\nabla u_t\|_2^2\\
&-[\gamma_3(g_0-\delta_{10}-\delta_{13})-\gamma_2-\gamma_1\delta_2]\|u_t\|_2^2-[\gamma_2(l-\delta_4)-\gamma_3\delta_7]\|u\|_2^2.
\end{aligned}
\tag{3.29}
$$

At this point, we fix $\delta_3 = \delta_4 < l$, $\delta_9 = \delta_{10} + \delta_{13} < g_0$. Then any choice of γ_2, γ_3 so that

$$\frac{(\delta_6 + \delta_7 + \delta_{12})}{(l - \delta_3) + (l - \delta_4)}\gamma_3 < \gamma_2 < \gamma_3 \frac{(g_0 - \delta_9) + (g_0 - \delta_{10} - \delta_{13})}{2}, \tag{3.30}$$

for $\delta_6 + \delta_7 + \delta_{12} < \lambda = [(l - \delta_3) + (l - \delta_4)][(g_0 - \delta_9) + (g_0 - \delta_{10} - \delta_{13})]/2$, will make

$$\begin{gathered}
\gamma_2(l - \delta_4) - \gamma_3\delta_7 > 0 \\
\gamma_2(l - \delta_3) - \gamma_3(\delta_6 + \delta_{12}) > 0 \\
\gamma_3(g_0 - \delta_9) - \gamma_2 = k_1 > 0 \\
\gamma_3(g_0 - \delta_{10} - \delta_{13}) - \gamma_2 = k_2 > 0.
\end{gathered}$$

So we choose $\delta_6 + \delta_7 + \delta_{12} < \lambda$ and γ_3 small enough so that (3.5) and (3.30) remain valid and

$$\frac{1}{2} - \gamma_3\Big(\frac{g(0)}{4\delta_{11}}\Big) > 0.$$

Then we pick γ_1 large enough so that

$$\gamma_1 - \overline{g}\Big(\frac{\gamma_2}{4\delta_5} + \gamma_3\big(\frac{1}{4\delta_8} + \frac{1}{4\delta_{14}} + 1\big)\Big) > 0,$$

and δ_1, δ_2 small enough so that

$$k_1 - \gamma_1\delta_1 > 0, \quad k_2 - \gamma_1\delta_2 > 0.$$

Therefore if α is large enough so that $\alpha - \frac{2\overline{G}}{\delta_1} - \frac{2\overline{G}}{\delta_2} > 0$, then ,for all $t \geq t_0$, (3.29) becomes

$$F'(t) \leq -c[E(t) + \Phi_1(t)] \leq \frac{-c}{\xi_2}F(t).$$

Integrating over (t_0, t) yields

$$F(t) \leq F(t_0)e^{ct_0/\xi_2}e^{-ct/\xi_2}.$$

The equivalence in (3.5) completes the proof for $K = \frac{F(t_0)}{\xi_1}e^{ct_0/\xi_2}$ and $k = c/\xi_2$. □

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