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ALMOST PERIODIC SOLUTIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS ON HILBERT SPACES

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ABSTRACT. We find necessary and sufficient conditions for the differential equation

$$u^{(n)}(t) = Au(t) + f(t), \quad t \in \mathbb{R}$$

to have a unique almost periodic solution. Some applications are also given.

1. INTRODUCTION

In this article, we study the almost periodicity of solutions to the differential equation

$$u^{(n)}(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where A is a linear, closed operator on a Hilbert space H and f is a function from $\mathbb{R}$ to H. The asymptotic behavior and, in particular, the almost periodicity of solutions of (1.1) has been a subject of intensive study for recent decades, see e.g. [2, 6, 7, 11, 12, 13, 14] and references therein. A particular condition for almost periodicity is the countability of the spectrum of the solution. In this paper we investigate the almost periodicity of mild solutions of Equation (1.1), when A is a linear, unbounded operator on a Hilbert space H. We use the Hilbert space $AP(\mathbb{R}, H)$ introduced in [5], defined by follows: Let $(,)$ be the inner product of H and let $AP_b(\mathbb{R}, E)$ be the space of all almost periodic functions from $\mathbb{R}$ to H. The completion of $AP_b(\mathbb{R}, E)$ is then a Hilbert space with the inner product defined by:

$$\langle f, g\rangle := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (f(s), g(s))ds.$$

First, we establish the relationship between the Bohr transforms of the almost periodic solutions of (1.1) and those of the inhomogeneity f. We then give a necessary and sufficient condition so that (1.1) admits a unique almost periodic solution for each almost periodic inhomogeneity f. As applications, in Section 4 we show a short proof of the Gearhart's Theorem: If A is generator of a strongly continuous semigroup $T(t)$, then $1 \in \varrho(T(1))$ if and only if $2k\pi i \in \varrho(A)$ and $\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty$.

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2. Hilbert space of almost periodic functions

Let us fix some notation. Define $S(t)f$ as $(S(t)f)(s) = f(s+t)$. Recall that a bounded, uniformly continuous function f from $\mathbb{R}$ to a Banach space H is almost periodic, if the set $\{S(t)f : t \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R}, H)$, the space of bounded uniformly continuous functions with sup norm topology. Let H be now a complex Hilbert space with $(,)$ and $\|\cdot\|$ be the inner product and the norm in H, respectively. Let $AP_b(\mathbb{R}, H)$ be the space of all almost periodic functions from $\mathbb{R}$ to H. In $AP_b(\mathbb{R}, H)$ the following expression

$$\langle f, g\rangle := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (f(s), g(s))ds$$

exists and defines an inner product. Hence, $AP_b(\mathbb{R}, H)$ is a pre-Hilbert space and its completion, denoted by $AP(\mathbb{R}, H)$, is a Hilbert space. The inner product and the norm in $AP(\mathbb{R}, H)$ are denoted by $\langle,\rangle$ and $\|\cdot\|_{AP}$, respectively.

For each function $f \in AP(\mathbb{R}, H)$, the Bohr transform is defined by

$$a(\lambda, f) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s)e^{-i\lambda s}ds.$$

The set

$$\sigma(f) := \{\lambda \in \mathbb{R} : a(\lambda, f) \neq 0\}$$

is called the Bohr spectrum of f. It is well known that $\sigma(f)$ is countable for each function $f \in AP(\mathbb{R}, H)$. The Fourier-Bohr series of f is

$$\sum_{\lambda\in\sigma(f)} a(\lambda, f)e^{i\lambda t}$$

and it converges to f in the norm topology of $AP(\mathbb{R}, H)$. The following Parseval's equality also holds:

$$\|f\|^2_{AP(\mathbb{R},H)} = \sum_{\lambda\in\sigma(f)} \|a(\lambda, f)\|^2.$$

For more information about the almost periodic functions and properties of the Hilbert space $AP(\mathbb{R}, H)$, we refer readers to [5, 6, 12].

Let $W^k(AP)$ be the space consisting of all almost periodic functions f, such that f', f'', ..., $f^{(k)}$ are in $AP(\mathbb{R}, H)$. $W^k(AP)$ is then a Hilbert space with the norm

$$\|f\|^2_{W^k(AP)} := \sum_{i=0}^{k} \|f^{(i)}\|^2_{AP(\mathbb{R},H)}.$$

Note that, for $k \geq 0$, the $W^{k+1}(AP)$-topology is stronger than the sup-norm topology in $C^k_b(\mathbb{R}, H)$, the space of k-times continuously differentiable functions with all derivatives until order k inclusively bounded (see [15]). We will use the following lemma in the sequel. (See also [11, Lemma 2.1]).

Lemma 2.1. *If F is a function in $W^1(AP)$ and $f = F'$, then we have*

$$a(\lambda, f) = \lambda i \cdot a(\lambda, F). \tag{2.1}$$

Proof. If $\lambda \neq 0$, using integration by part we have

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda s} f(s)ds &= \frac{1}{2T}F(t)e^{-i\lambda t}|_{-T}^{T} + \frac{i\lambda}{2T}\int_{-T}^{T} F(s)e^{-i\lambda s}ds \\ &= \frac{F(T)e^{-i\lambda T} - F(-T)e^{i\lambda T}}{2T} + i\lambda\frac{1}{2T}\int_{-T}^{T} F(s)e^{-i\lambda s}ds.\end{aligned}$$

Let $T \to \infty$, and note that $F(t)$ is bounded, we have (2.1). If $\lambda = 0$, then

$$a(0,f) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(s)ds = \lim_{T\to\infty}\frac{F(T)-F(-T)}{2T} = 0,$$

which also satisfies (2.1). □

Finally, for a linear and closed operator A in a Hilbert space H, we denote the domain, the range, the spectrum and the resolvent set of A by $D(A)$, $Range(A)$, $\sigma(A)$ and $\varrho(A)$, respectively.

3. Almost periodic mild solutions of differential equations

We now turn to the differential equation

$$u^{(n)}(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $n \in \mathbb{N}^+$ and A is a linear and closed operator on H. First we define two types of solutions to Equation (3.1). Let $I : C(\mathbb{R}, H) \to C(\mathbb{R}, H)$ be the operator defined by $If(t) := \int_0^t f(s)ds$ and $I^n f := I(I^{n-1}f)$.

Definition 3.1. (a) We say that $u : \mathbb{R} \to H$ is a classical solution of (3.1), if u is n-times continuously differentiable , $u(t) \in D(A)$ and (3.1) is satisfied for all $t \in \mathbb{R}$.

(b) For $f \in C(\mathbb{R}, H)$, a continuous function u is called a mild solution of (3.1), if $I^n u(t) \in D(A)$ and there exist n points v_0, v_1, …, v_{n-1} in H such that

$$u(t) = \sum_{j=0}^{n-1}\frac{t^j}{j!}v_j + AI^n u(t) + I^n f(t) \tag{3.2}$$

for all $t \in \mathbb{R}$.

Remark. Using the standard argument, we can prove the following:

(i) If a mild solution u is m times differentiable, $0 \leq m < n$, then v_i, $(i = 0, 1, \dots, m)$, are the initial values, i.e. $u(0) = v_0$, $u'(0) = v_1$, …, and $u^{(m)}(0) = v_m$.

(ii) If $n = 1$ and A is the generator of a C_0 semigroup $T(t)$, then a continuous function $u : \mathbb{R} \to E$ is a mild solution of (3.1) if and only if it has the form

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r)dr$$

for $t \geq s$.

(iii) If u is a bounded mild solution of (3.1) corresponding to a bounded inhomogeneity f and $\phi \in L^1(\mathbb{R}, E)$ then $u * \phi$ is a mild solution of (3.1) corresponding to $f * \phi$.

The mild solution to (3.1) defined by (3.2) is really an extension of classical solution in the sense that every classical solution is a mild solution and conversely, if a mild solution is n-times continuously differentiable, then it is a classical solution. That statement is actually contained in the following lemma (see also [10]).

Lemma 3.2. *Suppose $0 \le m \le n$ and u is a mild solution of (3.1), which is m-times continuously differentiable. Then for all $t \in \mathbb{R}$ we have $I^{n-m}u(t) \in D(A)$ and*

$$u^{(m)}(t) = \sum_{j=m}^{n-1} \frac{t^{j-m}}{(j-m)!} v_j + AI^{n-m}u(t) + I^{n-m}f(t), \tag{3.3}$$

where $v_m, \dots, v_{n-1}$ are given in Definition 3.1(b).

Proof. If $m = 0$, then (3.3) coincides with (3.2). We prove for $m = 1$: Let $v(t) := AI^n u(t)$. Then, by (3.2), v is continuously differentiable and

$$v'(t) = u'(t) - \sum_{j=1}^{n-1} \frac{t^{j-1}}{(j-1)!} v_j - I^{n-1}f(t).$$

Let $h > 0$ and put

$$v_h := \frac{1}{h} \int_t^{t+h} I^{n-1}u(s)ds.$$

Then $v_h \to (I^{n-1}u)(t)$ for $h \to 0$ and

$$\begin{aligned}
\lim_{h\to 0} Av_h &= \lim_{h\to 0} \frac{1}{h}\Big(A\int_0^{t+h} I^{n-1}u(s)ds - A\int_0^t I^{n-1}u(s)ds\Big) \\
&= \frac{1}{h}(v(t+h) - v(t)) \\
&= v'(t).
\end{aligned}$$

Since A is a closed operator, we obtain that $I^{n-1}u(t) \in D(A)$ and

$$AI^{n-1}u(t) = u'(t) - \sum_{j=1}^{n-1} \frac{t^{j-1}}{(j-1)!} v_j - I^{n-1}f(t),$$

from which (3.3) with $m = 1$ follows. If $m > 1$, we obtain (3.3) by repeating the above process $(m-1)$ times. ☐

In particular, if the mild solution u is n-times continuously differentiable, then (3.3) becomes $u^{(n)}(t) = Au(t) + f(t)$; i.e. u is a classical solution of (3.1).

We now consider the mild solutions of (3.1), which are $(n-1)$ times continuously differentiable. The following proposition describes the connection between the Bohr transforms of such solutions and those of $f(t)$.

Proposition 3.3. *Suppose A is a linear and closed operator on H, $f \in AP(\mathbb{R}, H)$ and u is an almost periodic mild solution of (3.1), which belongs to $C_b^{n-1}(\mathbb{R}, H)$. Then*

$$[(\lambda i)^n - A]a(\lambda, u) = a(\lambda, f) \tag{3.4}$$

for every $\lambda \in \mathbb{R}$.

Proof. Suppose u is an almost periodic mild solution of (3.1), which belongs to $C_b^{n-1}(\mathbb{R}, H)$ and λ is a real number. Using (3.3) with $m = n-1$ we have

$$u^{(n-1)}(t) = u^{(n-1)}(0) + AIu(t) + If(t). \tag{3.5}$$

For $\lambda \neq 0$, multiplying each side of (3.5) with $e^{-i\lambda t}$ and taking definite integral from $-T$ to T on both sides, we have

$$\begin{aligned}\int_{-T}^{T} e^{-i\lambda t} u^{(n-1)}(t)dt = \int_{-T}^{T} e^{-i\lambda t} u^{(n-1)}(0)dt + A\int_{-T}^{T} e^{-i\lambda t}\int_0^t u(s)\,ds\,dt \\ + \int_{-T}^{T} e^{-i\lambda t}\int_0^t f(s)\,ds\,dt.\end{aligned} \tag{3.6}$$

Here we used the fact that $\int_a^b Au(t)dt = A\int_a^b u(t)dt$ for a closed operator A. It is easy to see that

$$\int_{-T}^{T} e^{-i\lambda t} u^{(n-1)}(0)dt = -\frac{e^{-i\lambda T}u^{(n-1)}(0) - e^{i\lambda T}u^{(n-1)}(0)}{i\lambda}$$

and, applying integration by part for any integrable function $g(t)$, we have

$$\begin{aligned}\int_{-T}^{T} e^{-i\lambda t}\int_0^t g(s)\,ds\,dt &= -\frac{1}{i\lambda}e^{-i\lambda t}\int_0^t g(s)ds\Big|_{-T}^{T} + \frac{1}{i\lambda}\int_{-T}^{T} e^{-i\lambda t}g(t)dt \\ &= -\frac{1}{i\lambda}e^{-i\lambda T}\int_0^T g(t)dt + \frac{1}{i\lambda}e^{i\lambda T}\int_0^{-T} g(t)dt \\ &\quad + \frac{1}{i\lambda}\int_{-T}^{T} e^{-i\lambda t}g(t)dt.\end{aligned} \tag{3.7}$$

Using (3.7) for $g(t) = u(t)$ and $g(t) = f(t)$ in (3.6), respectively, we have

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}u^{(n-1)}(t)dt \\ &= -\frac{e^{-i\lambda T}u^{(n-1)}(0) - e^{i\lambda T}u^{(n-1)}(0)}{2i\lambda T} - \frac{e^{-i\lambda T}}{2i\lambda T}\Big(A\int_0^T u(t)dt + \int_0^T f(t)dt\Big) \\ &\quad + \frac{e^{i\lambda T}}{2i\lambda T}\Big(A\int_0^{-T} u(t)dt + \int_0^{-T} f(t)dt\Big) \\ &\quad + \frac{1}{i\lambda 2T}\Big(A\int_{-T}^{T} e^{-i\lambda t}u(t)dt + \int_{-T}^{T} e^{-i\lambda t}f(t)dt\Big) \\ &= I_1 + I_2 + I_3,\end{aligned} \tag{3.8}$$

where

$$I_1 = -\frac{e^{-i\lambda T}u^{(n-1)}(0) - e^{i\lambda T}u^{(n-1)}(0)}{2i\lambda T} \to 0$$

as $T \to \infty$;

$$\begin{aligned}I_2 &= -\frac{e^{-i\lambda T}}{2i\lambda T}\Big(A\int_0^T u(t)dt + \int_0^T f(t)dt\Big) + \frac{e^{i\lambda T i\lambda}}{2i\lambda T}\Big(A\int_0^{-T} u(t)dt + \int_0^{-T} f(t)dt\Big) \\ &= -\frac{e^{-i\lambda T}}{2i\lambda T}\Big(u^{(n-1)}(T) - u^{(n-1)}(0)\Big) + \frac{e^{i\lambda T}}{2i\lambda T}\Big(u^{(n-1)}(-T) - u^{(n-1)}(0)\Big) \\ &\quad \to 0 \quad \text{as } T \to \infty,\end{aligned}$$

and

$$I_3 = \frac{1}{i\lambda}\Big(\frac{1}{2T}A\int_{-T}^{T} e^{-i\lambda t}u(t)dt + \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}f(t)dt\Big). \tag{3.9}$$

Let $u_T := \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}u(t)dt$. It is clear that

$$\lim_{T\to\infty} u_T = a(\lambda, u) \tag{3.10}$$

and from (3.9) we have

$$\begin{aligned} Au_T &= \frac{1}{2T}A\int_{-T}^{T} e^{-i\lambda t}u(t)dt \\ &= i\lambda I_3 - \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}f(t)dt \\ &= i\lambda\Big(\frac{1}{2T}\int_{-T}^{T} u^{(n-1)}(t)dt - I_1 - I_2\Big) - \frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}f(t)dt \\ &\to i\lambda a(\lambda, u^{(n-1)}) - a(\lambda, f) \quad \text{as } T\to\infty. \end{aligned} \tag{3.11}$$

Since A is a closed operator, from (3.10) and (3.11), we obtain $a(\lambda, u) \in D(A)$ and

$$Aa(\lambda, u) = i\lambda a(\lambda, u^{(n-1)}) - a(\lambda, f) = (i\lambda)^n a(\lambda, u) - a(\lambda, f),$$

from which (3.4) follows. Next, if $\lambda = 0$, using Formula (3.5), we have

$$\begin{gathered} u^{(n-1)}(T) = v_{n-1} + A\int_0^T u(t)dt + \int_0^T f(t)dt, \\ u^{(n-1)}(-T) = v_{n-1} + A\int_0^{-T} u(t)dt + \int_0^{-T} f(t)dt. \end{gathered}$$

Hence,

$$\frac{u^{(n-1)}(T) - u^{(n-1)}(-T)}{2T} = A\frac{1}{2T}\int_{-T}^{T} u(t)dt + \frac{1}{2T}\int_{-T}^{T} f(t)dt. \tag{3.12}$$

Let $u_T = \frac{1}{2T}\int_{-T}^{T} u(s)ds$. Then $\lim_{t\to\infty} u_T = a(0, u)$, and by (3.12),

$$\begin{aligned} Au_T &= \frac{1}{2T}A\int_{-T}^{T} u(s)ds \\ &= \frac{u^{(n-1)}(T) - u^{(n-1)}(-T)}{2T} - \frac{1}{2T}\int_{-T}^{T} f(s)ds \to -a(0, f) \quad \text{as } T\to\infty. \end{aligned}$$

Again, since A is a closed operator, it implies $a(0, u) \in D(A)$ and $Aa(0, u) = -a(0, f)$, from which (3.4) follows, and this completes the proof. □

Note that Proposition 3.3 also holds in a Banach space. We are now going to look for conditions that Equation (3.1) has an almost periodic mild solution.

Theorem 3.4. *Suppose A is a linear and closed operator and f is a function in $AP(\mathbb{R}, H)$. Then the following statements are equivalent*

(i) *Equation* (3.1) *has an almost periodic mild solution, which is in $W^n(AP)$;*

(ii) *For every* $\lambda \in \sigma(f)$, $a(\lambda, f) \in \mathrm{Range}((i\lambda)^n - A)$ *and there exists a set* $\{x_\lambda\}_{\lambda\in\sigma(f)}$ *in* H *satisfying* $((i\lambda)^n - A)x_\lambda = a(\lambda, f)$, *for which the following inequalities*

$$\sum_{\lambda\in\sigma(f)} |\lambda|^{2k}\|x_\lambda\|^2 < \infty \tag{3.13}$$

hold for $k = 0, 1, 2, \dots, n$.

Proof. (i) $\Rightarrow$ (ii): Let $u(t)$ be an almost periodic solution to (3.1), which is in $W^n(AP)$. By Proposition 3.3, $((i\lambda)^n - A)a(\lambda, u) = a(\lambda, f)$. Hence $a(\lambda, f) \in Range((i\lambda)^n - A)$ for all $\lambda \in \sigma(f)$. Put now $x_\lambda := a(\lambda, u)$ for $\lambda \in \sigma(f)$. Then it satisfies $((i\lambda)^n - A)x_\lambda = a(\lambda, f)$. Moreover, by Lemma 2.1, $(i\lambda)^k x_\lambda = a(\lambda, u^{(k)})$. Hence, for $0 \le k \le n$ we have

$$\begin{aligned}
\sum_{\lambda\in\sigma(f)} |\lambda|^{2k}\|x_\lambda\|^2 &= \sum_{\lambda\in\sigma(f)} |a(\lambda, u^{(k)})|^2\\
&\le \sum_{\lambda\in\sigma(f)\cup\sigma(u^{(k)})} |a(\lambda, u^{(k)})|^2\\
&= \sum_{\lambda\in\sigma(u^{(k)})} |a(\lambda, u^{(k)})|^2\\
&= \|u^{(k)}\|^2_{AP},
\end{aligned}$$

from which (3.13) follows.

(ii) $\Rightarrow$ (i): Let $\{x_\lambda\}_{\lambda\in\sigma(f)}$ be a set in H satisfying $((i\lambda)^n - A)x_\lambda = a(\lambda, f)$, for which (3.13) holds. Put

$$f_N(t) := \sum_{\lambda\in\sigma(f),|\lambda|<N} e^{i\lambda t}a(\lambda, f), \quad u_N(t) := \sum_{\lambda\in\sigma(f),|\lambda|<N} e^{i\lambda t}x_\lambda.$$

It is then easy to find their norms

$$\|u_N^{(k)}\|^2_{AP} = \sum_{\lambda\in\sigma(f),|\lambda|<N} |\lambda|^{2k}\|x_\lambda\|^2.$$

From (3.13) it implies that $u_N^{(k)} \to U_k$ as $N \to \infty$ for some functions U_k ($k = 0, 1, 2, \dots, n$) in the topology of $AP(\mathbb{R}, H)$. Since the differential operator is closed, we obtain $U_k' = U_{k-1}$ and $\lim_{N\to\infty} u_N = U_0$ in the topology of $W^n(AP)$. It remains to show that U_0 is a mild solution of (3.1). In order to do that, note u_N is a classical, and hence, a mild solution of (3.1) corresponding to f_N; i.e.,

$$u_N(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!}u_N^{(i)}(0) + AI^n u_N(t) + I^n f_N(t). \tag{3.14}$$

For each $t \in \mathbb{R}$ we have

$$\lim_{N\to\infty}\int_0^t f_N(s)ds = \int_0^t f(s)ds, \quad \lim_{N\to\infty}\int_0^t u_N(s)ds = \int_0^t U_0(s)ds.$$

Hence,

$$\lim_{N\to\infty} I^k u_N(t) = I^k U_0(t), \quad \lim_{N\to\infty} I^k f_N(t) = I^k f(t)$$

for $k = 0, 1, 2, \dots, n$. Using Equation (3.14), we have

$$\begin{aligned}\lim_{N\to\infty} A(I^n u_N(t)) &= \lim_{N\to\infty} \Big(u_N(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} u_N^{(i)}(0) - I^n f_N(t) \Big) \\ &= U_0(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} U_0^{(i)}(0) - I^n f(t).\end{aligned}$$

Since A is a closed operator, we obtain $I^n U_0(t) \in D(A)$ and

$$A(I^n U_0(t)) = u(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} U_0^{(i)}(0) - I^n f(t),$$

which shows that U_0 is a mild solution of (3.1) and the proof is complete. □

Note that if condition (ii) in Theorem 3.4 holds, Equation (3.1) may have two or more almost periodic mild solutions. We are going to find conditions such that for each almost periodic function f, Equation (3.1) has a unique almost periodic mild solution. We are now in the position to state the main result.

Theorem 3.5. *Suppose A is a linear and closed operator on a Hilbert space H and M is a closed subset of $\mathbb{R}$. For $0 \le k \le n$, the following statements are equivalent*

- (i) *For each function $f \in W^k(AP)$ with $\sigma(f) \subset M$, Equation (3.1) has a unique almost periodic mild solution u in $W^n(AP)$ with $\sigma(u) \subseteq M$.*
- (ii) *For each $\lambda \in M$, $(i\lambda)^n \in \varrho(A)$ and*

$$\sup_{\lambda\in M} |\lambda|^m \|((i\lambda)^n - A)^{-1}\| < \infty \tag{3.15}$$

for all $m = 0, 1, 2, \dots, n - k$.

Proof. $(i) \Rightarrow (ii)$: Let $W^k(AP)_{|M}$ be the subspace of all functions f in $W^k(AP)$ with $\sigma(f) \subset M$. Then $W^k(AP)_{|M}$ is a Hilbert space by nature. Let x be any vector in H, λ be a number in M and let $f(t) = e^{i\lambda t}x$. Then $f \in W^k(AP)_{|M}$ and hence, Equation (3.1) has a unique almost periodic solution u in $W^n(AP)_{|M}$. By Theorem 3.4, $x = a(\lambda, f) \in \mathrm{Range}((i\lambda)^n - A)$, hence $((i\lambda)^n - A)$ is surjective for all $\lambda \in M$. If $((i\lambda)^n - A)$ were not injective, i.e., there exists a nonzero vector $y \in H$ such that $((i\lambda)^n - A)y = 0$, we show that $u_2(t) = u(t) + e^{i\lambda t}y$, would be an other almost periodic mild solution to (3.1) with $\sigma(u_2) = \sigma(u) \subseteq M$. In deed, since u is $(n-1)$ times differentiable, we can use formula (3.3) to obtain

$$\begin{aligned}u_2^{(n-1)}(t) &= u^{(n-1)}(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds + (i\lambda)^{(n-1)} e^{i\lambda t} y \\ &= u^{(n-1)}(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds + \frac{e^{i\lambda t}}{i\lambda} Ay \\ &= (u^{(n-1)}(0) + \frac{Ay}{i\lambda}) + A\int_0^t (u(s) + e^{i\lambda s}y)ds + \int_0^t f(s)ds \\ &= u_2^{(n-1)}(0) + A\int_0^t u_2(s)ds + \int_0^t f(s)ds,\end{aligned}$$

which means u_2 is another mild solution of (3.1) corresponding to f, contradicting to the uniqueness of the solution. Therefore, $((i\lambda)^n - A)$ is bijective and $(i\lambda)^n \in \varrho(A)$ for all $\lambda \in M$.

We now define the operator $L : W^k(AP)_{|M} \to W^n(AP)_{|M}$ by follows: For each $f \in W^k(AP)_{|M}$, Lf is the unique almost periodic mild solution to (3.1) corresponding to f. By the assumption, L is everywhere defined. We will prove that L is a bounded operator by showing L is closed. Let $f_n \to f$ in $W^k(AP)_{|M}$ and $Lf_n \to u$ in $W^n(AP)_{|M}$, where

$$(Lf_n)^{(n-1)}(t) = (Lf_n)^{(n-1)}(0) + A\int_0^t (Lf_n)(s)ds + \int_0^t f_n(s)ds. \tag{3.16}$$

For each $t \in \mathbb{R}$, we have $\lim_{n\to\infty}(Lf_n)^{(n-1)}(t) = u^{(n-1)}(t)$, $\lim_{N\to\infty}\int_0^t f_n(s)ds = \int_0^t f(s)ds$ and $\lim_{n\to\infty}\int_0^t Lf_n(s)ds = \int_0^t u(s)ds$. Moreover, from (3.16) we have

$$\begin{aligned} A\int_0^t (Lf_n)(s)ds &= (Lf_n)^{(n-1)}(t) - (Lf_n)^{(n-1)}(0) - \int_0^t f_n(s)ds \\ &\to u^{(n-1)}(t) - u^{(n-1)}(0) - \int_0^t f(s)ds, \quad \text{as } n \to \infty, \end{aligned}$$

for each $t \in \mathbb{R}$. Since A is a closed operator, $\int_0^t u(s)ds \in D(A)$ and

$$A\int_0^t u(s)ds = u(t) - u(0) - \int_0^t f(s)ds,$$

which means u is a mild solution to (3.1) corresponding to f. Thus, $f \in D(L)$, $Lf = u$ and hence, L is closed.

Next, for any $x \in H$ and $\lambda \in M$, put $f(t) = e^{i\lambda t}x$, then $u(t) = e^{i\lambda t}((i\lambda)^n - A)^{-1}x$ is the unique almost periodic solution to (3.1), i.e., $u = Lf$. Using the boundedness of operator L, we obtain

$$\sum_{j=0}^{n} |\lambda|^{2j}\|((i\lambda)^n - A)^{-1}x\|^2 = \|u\|^2_{W^n(AP)} \le \|L\|^2\|f\|^2_{W^k(AP)} = \|L\|^2\sum_{j=0}^{k}|\lambda|^{2j}\|x\|^2,$$

which implies

$$\|((i\lambda)^n - A)^{-1}x\|^2 \le \|L\|^2\frac{\sum_{j=0}^k|\lambda|^{2j}}{\sum_{j=0}^n|\lambda|^{2j}}\cdot\|x\|^2. \tag{3.17}$$

for any $x \in H$ and any $\lambda \in M$. For a real number λ and an integer m with $0 \le m \le n-k$ it is easy to show the inequality

$$\frac{\sum_{j=0}^k|\lambda|^{2j}}{\sum_{j=0}^n|\lambda|^{2j}} \le \frac{1}{|\lambda|^{2m}}.$$

Thus, from (3.17) we have

$$\|((i\lambda)^n - A)^{-1}x\| \le \|L\|\frac{1}{|\lambda^m|}\cdot\|x\|,$$

from which (3.15) follows.

$(ii) \Rightarrow (i)$: Suppose f is in $W^k(AP)_{|M}$. Put $x_\lambda := ((i\lambda)^n - A)^{-1}a(\lambda, f)$. For any integer m with $0 \le m \le n$ we can write $m = m_1 + m_2$ with $0 \le m_1 \le n-k$

and $0 \le m_2 \le k$. We have

$$\begin{aligned}\sum_{\lambda\in\sigma(f)} \lambda^{2m}\|x_\lambda\|^2 &\le \sum_{\lambda\in\sigma(f)} \Big(\Big(|\lambda|^{2m_1}\|((i\lambda)^n - A)^{-1}\|^2\Big)\cdot\Big(|\lambda|^{2m_2}\|a(\lambda,f)\|^2\Big)\Big)\\ &\le \Big(\sup_{\lambda\in\sigma(f)} |\lambda|^{2m_1}\|((i\lambda)^n - A)^{-1}\|^2\Big)\sum_{\lambda\in\sigma(f)} |\lambda|^{2m_2}\|a(\lambda,f)\|^2\\ &= \Big(\sup_{\lambda\in\sigma(f)} |\lambda|^{m_1}\|((i\lambda)^n - A)^{-1}\|\Big)^2\|f^{(m_2)}\|_{AP}^2 < \infty.\end{aligned}$$

By Proposition 3.4, Equation (3.1) has an almost periodic mild solution in $W^n(AP)$. That solution is unique and is in $W^n(AP)_{|M}$, since its Bohr transforms are uniquely determined by $a(\lambda,u) = ((i\lambda)^n - A)^{-1}a(\lambda,f)$ for all $\lambda \in M$. ☐

We can apply Theorem 3.5 to some particular sets for M. First, if $M = \mathbb{R}$ we have

Corollary 3.6. *Suppose A is a linear and closed operator on a Hilbert space H. For $0 \le k \le n$, the following statements are equivalent*

(i) *For each function $f \in W^k(AP)$, Equation* (3.1) *has a unique almost periodic mild solution in $W^n(AP)$.*

(ii) *$(i\mathbb{R})^n \subseteq \varrho(A)$ and*

$$\sup_{\lambda\in\mathbb{R}} |\lambda|^{n-k}\|((i\lambda)^n - A)^{-1}\| < \infty.$$

Let $L_2(0,1)$ be the Hilbert space of integrable functions f from $(0,1)$ to H with the norm

$$\|f\|_{L_2(0,1)}^2 = \int_0^1 \|f(t)\|^2 dt < \infty.$$

If $M = \{2p\pi : p \in \mathbb{Z}\}$, then the space $AP(\mathbb{R},H)_{|M}$ becomes $L_2(0,1)$ and $W^k(AP)$ becomes $W^k(1)$, the space of all periodic functions f of period 1 with $f^{(k)} \in L_2(0,1)$. $W^k(1)$ is then a Hilbert space with the norm

$$\|f\|_{W^k(1)}^2 = \sum_{j=0}^k |f^{(k)}\|_{L_2(0,1)}^2.$$

Note that, since $M = \{2p\pi : p \in \mathbb{Z}\}$, Condition (3.15) is satisfied for all m from 0 to $(n-k)$ if and only if it is satisfied for only $m = n-k$. Hence, we obtain the following corollary, which generalizes a result in [10].

Corollary 3.7 ([10, Theorem 2.6])**.** *Suppose A is a linear and closed operator on a Hilbert space H. For $0 \le k \le n$, the following statements are equivalent*

(i) *For each function $f \in W^k(1)$, Equation* (3.1) *has a unique 1-periodic mild solution in $W^n(1)$.*

(ii) *For each $p \in \mathbb{Z}$, $2pi\pi \in \varrho(A)$ and*

$$\sup_{p\in\mathbb{Z}} |2p\pi|^{(n-k)}\|((2pi\pi)^n - A)^{-1}\| < \infty. \tag{3.18}$$

4. Application: A C_0-semigroup case

If A generates a C_0-semigroup $(T(t))_{t\geq 0}$, then mild solutions of the first order differential equation

$$u'(t) = Au(t) + f(t) \qquad t \in \mathbb{R}, \tag{4.1}$$

can be expressed by

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau \tag{4.2}$$

for $t \geq s$ (see [1, Theorem 2.5]). We obtain the following result.

Corollary 4.1. *Let A generate a C_0-semigroup $(T(t))$ on a Hilbert H and M is a closed subset in $\mathbb{R}$. The following statements are equivalent*

(i) *For each function $f \in W^1(AP)_{|M}$, Equation (4.1) has a unique solution in $W^1(AP)_{|M}$.*
(ii) *For each function $f \in W^1(AP)_{|M}$, Equation (4.1) has a unique almost periodic classical solution u with $\sigma(u) \subset M$.*
(iii) *For each $\lambda \in M$, $\lambda i \in \varrho(A)$ and*

$$\sup_{\lambda\in M} \|(i\lambda - A)^{-1}\| < \infty. \tag{4.3}$$

Proof. The equivalence $(i) \Leftrightarrow (iii)$ is shown in Theorem 3.5, $(ii) \Rightarrow (i)$ is obvious. So, it remains to show the implication $(i) \Rightarrow (ii)$.

Let f be any function in $W^1(AP)$ and $u(t)$ be the unique mild solution of (4.1), which is in $W^1(AP)$. We will show u is a classical solution by showing $u(t_0) \in D(A)$ for every point $t_0 \in \mathbb{R}$. Take any point $s_0 \in \mathbb{R}$ with $s_0 < t_0$. Since for each almost everywhere differentiable function f, the function $g(t) := \int_{s_0}^t T(t-s)f(s)ds$ is continuously differentiable and $g(t) \in D(A)$ for all $t \in [s_0, t_0]$ (see [9]). So, from Formula (4.2), it suffices to show $T(t_0 - s_0)u(s_0) \in D(A)$.

By the assumptions, function $g(t) := T(t-s_0)u(s_0) = u(t) - \int_{s_0}^t T(t-s)f(s)ds$ is almost everywhere differentiable on $[s_0, t_0]$. It follows that $g(t) \in D(A)$ for almost t in $[s_0, t_0]$ (since $t \mapsto T(t)x$ is differentiable at right at t_0 if and only if $T(t_0)x \in D(A)$). Taking a point $s_1 \in (s_0, t_0)$ such that $T(s_1 - s_0)u(s_0) \in D(A)$, then $T(t_0 - s_0)u(s_0) = T(t_0 - s_1)T(s_1 - s_0)u(s_0) \in D(A)$. The uniqueness of this classical solution is obvious and the proof is complete. □

If $M = \{2k\pi : k \in \mathbb{Z}\}$ and f is a 1-periodic function, then it is easy to see that solution u is 1-periodic if and only if $u(1) = u(0)$. Hence, to consider 1-periodic solution, it suffices to consider u in $[0, 1]$ and in this interval we have

$$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq 1. \tag{4.4}$$

From Corollary 4.1 we have a direct consequence below, in which we show the Gearhart's Theorem (the equivalence $(iv) \Leftrightarrow (v)$). For the proof of that Corollary, note that the equivalence $(i) \Leftrightarrow (ii)$ can be easily proved by using standard arguments and $(i) \Leftrightarrow (v)$ has been shown in [8].

Corollary 4.2. *Let A generate a C_0-semigroup $(T(t))$ on a Hilbert H, then the following statements are equivalent*

(i) *For each function* $f \in L_2(0,1)$, *Equation* (4.1) *has a unique 1-periodic mild solution.*
(ii) *For each function* $f \in W^1(1)$, *Equation* (4.1) *has a unique 1-periodic classical solution.*
(iii) *For each function* $f \in W^1(1)$, *Equation* (4.1) *has a unique 1-periodic solution contained in* $W^1(1)$.
(iv) *For each* $k \in \mathbb{Z}$, $2k\pi i \in \varrho(A)$ *and*

$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty.$$

(v) $1 \in \varrho(T(1))$.

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