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A HOLMGREN TYPE THEOREM FOR PARTIAL DIFFERENTIAL EQUATIONS WHOSE COEFFICIENTS ARE GEVREY FUNCTIONS

MASAKI KAWAGISHI

Dedicated to Prof. Takesi Yamanaka on his 77th birthday

Abstract. In this article, we consider a uniqueness theorem of Holmgren type for p-th order Kovalevskaja linear partial differential equations whose coefficients are Gevrey functions. We prove that the only C^p-solution to the zero initial-valued problem is the identically zero function. To prove this result we use the uniqueness theorem for higher-order ordinary differential equations in Banach scales.

1. Introduction

In this article we consider the linear partial differential equation

$$\frac{\partial^p v(t,x)}{\partial t^p} = \sum_{\alpha\in\mathbb{Z}_+^n,\, \lambda|\alpha|+j\le p,\, j\le p-1} a_{\alpha,j}(t,x)\frac{\partial^{j+|\alpha|}v(t,x)}{\partial t^j\partial x^\alpha}, \tag{1.1}$$

where p is an integer ≥ 1, λ is a real constant > 1 and $a_{\alpha,j}(t,x), v(t,x)$ are $\mathbb{C}$-valued functions of $(t,x)\in\mathbb{R}\times\mathbb{R}^n$. We denote by $\mathbb{Z}_+^n$ the set of n-dimensional multi-indices. We write $|\alpha| = \alpha_1+\dots+\alpha_n$ for $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{Z}_+^n$. Each $a_{\alpha,j}(t,x)$ is assumed to be continuous in t and Gevrey function of order λ in x.

The purpose of this paper is to prove that the only C^p-solution of (1.1) which satisfies the following condition

$$v(0,x) = \frac{\partial v(0,x)}{\partial t} = \dots = \frac{\partial^{p-1}v(0,x)}{\partial t^{p-1}} = 0 \tag{1.2}$$

is $v(t,x)\equiv 0$. An exact statement of the above result will be given in §4 as Theorem 4.1.

To prove the result mentioned above, we use the result given in [1] on the uniqueness of the solution of a non-linear ODE in a Banach scale.

The outline of the proof of the Holmgren type uniqueness theorem in this paper is as follows: First we construct a Banach scale consisting of the duals of some normed spaces of Gevrey functions, and define the adjoint equation to the equation (1.1) on the above dual Banach scale. Then it is shown, by the uniqueness, that

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the only solution of 0-initial value problem of the adjoint equation in a Banach scale is identically 0. On the other hand the given solution $v(t,x)$ of the problem (1.1)–(1.2) gives rise to a solution $L(t)$ of the 0-initial value problem for the adjoint equation. From these facts we conclude that $v(t,x) \equiv 0$.

As a preparation for performing our plan as mentioned above, we shall prepare in §2 some important properties of Gevrey functions. Especially, a problem of approximation in Gevrey class is important for our purpose. We shall state that problem in Lemma 2.7 and Theorem 2.8, and at the end of §2 we construct a dual Banach scale. §3 is devoted to showing the uniqueness of the solution of initial value problem in a Banach scale. In §4 we shall state and prove the main result in the paper, i.e., a Holmgren type uniqueness theorem for the initial value problem (1.1)–(1.2).

Here we review the definition of Banach scale. Let J be an interval of real numbers. A family $\{E_\sigma\}_{\sigma\in J}$ of Banach spaces E_σ is called a Banach scale, if $\delta < \sigma$ $(\sigma, \delta \in J)$, then $E_\sigma \subset E_\delta$ and $\|u\|_\delta \le \|u\|_\sigma$ $(u \in E_\sigma)$.

Finally we define the notation for partial differential operators. If $f(x)$ is a function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $\partial_i f(x) = \partial f(x)/\partial x_i$ and $\partial^\alpha f(x) = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f(x) = \partial^{|\alpha|} f(x)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$.

2. Preparation: Gevrey functions

Throughout this paper, λ denotes a fixed real constant > 1 and Ω a fixed open set $\subset \mathbb{R}^n$. We write $\alpha! = \alpha_1! \dots \alpha_n!$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$. A C^∞-function $f : \Omega \to \mathbb{C}$ is said to be in the Gevrey class of order λ if there exists a positive constant σ such that

$$\sup_{\alpha\in\mathbb{Z}_+^n,\, x\in\Omega} |\partial^\alpha f(x)| \frac{\sigma^{|\alpha|}}{(\alpha!)^\lambda} < \infty.$$

It is easily seen that a C^∞-function $f : \Omega \to \mathbb{C}$ is in the Gevrey class of order λ if and only if there exists a positive constant ρ such that

$$\sup_{\alpha\in\mathbb{Z}_+^n,\, x\in\Omega} |\partial^\alpha f(x)| \frac{\rho^{|\alpha|}}{(\alpha!)^\lambda} (1 + |\alpha|)^{2n} < \infty. \tag{2.1}$$

We denote by $\mathcal{G}_{\lambda,\rho}(\Omega)$ the set of all C^∞-functions $f : \Omega \to \mathbb{C}$ which satisfy the condition (2.1) and define the norm $|f|_{\lambda,\rho}$ of $f \in \mathcal{G}_{\lambda,\rho}(\Omega)$ by

$$|f|_{\lambda,\rho} = \sup_{\alpha\in\mathbb{Z}_+^n,\, x\in\Omega} |\partial^\alpha f(x)| \frac{\rho^{|\alpha|}}{(\alpha!)^\lambda} (1 + |\alpha|)^{2n}.$$

Then $\mathcal{G}_{\lambda,\rho}(\Omega)$ is a Banach space with the norm $|\cdot|_{\lambda,\rho}$ and the family $\{\mathcal{G}_{\lambda,\rho}(\Omega)\}_{\rho>0}$ forms a Banach scale.

In the following two theorems some important properties of elements of the Gevrey class $\mathcal{G}_{\lambda,\rho}(\Omega)$ are stated. These results were given in [2] and [5]. However, it may not be considered quite suitable for our present situation without some modification. So we shall state and prove these results as Theorem 2.1 and 2.2

Theorem 2.1. *If $f, g \in \mathcal{G}_{\lambda,\rho}(\Omega)$, then the product fg is again in $\mathcal{G}_{\lambda,\rho}(\Omega)$ and*

$$|fg|_{\lambda,\rho} \le 2^{3n} |f|_{\lambda,\rho} |g|_{\lambda,\rho}. \tag{2.2}$$

Proof. For any $\alpha \in \mathbb{Z}_+^n$, we have

$$
\begin{aligned}
|\partial^\alpha(fg)(x)| &\leq \sum_{\beta\in\mathbb{Z}_+^n,\, \beta\leq\alpha} {}_\alpha\mathrm{C}_\beta |\partial^\beta f(x)||\partial^{\alpha-\beta} g(x)| \\
&\leq |f|_{\lambda,\rho}|g|_{\lambda,\rho}\frac{(\alpha!)^\lambda}{\rho^{|\alpha|}}\frac{1}{(2+|\alpha|)^{2n}}\sum_{\beta\leq\alpha}({}_\alpha\mathrm{C}_\beta)^{1-\lambda}\big\{\frac{1}{1+|\beta|}+\frac{1}{1+|\alpha-\beta|}\big\}^{2n} \\
&\leq |f|_{\lambda,\rho}|g|_{\lambda,\rho}\frac{(\alpha!)^\lambda}{\rho^{|\alpha|}}\frac{2^{2n}}{(1+|\alpha|)^{2n}}\sum_{\beta\leq\alpha}\big\{\frac{1}{1+|\beta|}\big\}^{2n} \\
&\leq |f|_{\lambda,\rho}|g|_{\lambda,\rho}\frac{(\alpha!)^\lambda}{\rho^{|\alpha|}}\frac{2^{2n}}{(1+|\alpha|)^{2n}}\Big(\sum_{k=0}^{\infty}\frac{1}{(1+k)^2}\Big)^n \\
&\leq |f|_{\lambda,\rho}|g|_{\lambda,\rho}\frac{(\alpha!)^\lambda}{\rho^{|\alpha|}}\frac{2^{3n}}{(1+|\alpha|)^{2n}},
\end{aligned}
$$

which shows the theorem. □

In what follows we restrict the range of the scale parameter ρ for the sake of simplicity of calculation. We restrict ρ to the range $m^{-1} \leq \rho \leq em^{-1}$, where m is a positive integer.

Theorem 2.2. *If $f \in \mathcal{G}_{\lambda,\rho}(\Omega)$, $m^{-1} \leq \sigma < \rho \leq em^{-1}$, then, for $\alpha \in \mathbb{Z}_+^n$, $\partial^\alpha f \in \mathcal{G}_{\lambda,\sigma}(\Omega)$ and*

$$
|\partial^\alpha f|_{\lambda,\sigma} \leq (\lambda|\alpha|)^{\lambda|\alpha|}\frac{|f|_{\lambda,\rho}}{(\rho-\sigma)^{\lambda|\alpha|}}. \tag{2.3}
$$

Proof. Note first that, if $a, b \in \mathbb{Z}_+$ and $0 < \sigma < \rho \leq em^{-1}$, then

$$
\begin{aligned}
\Big(\frac{\sigma}{\rho}\Big)^{a+b}\Big(\frac{(a+b)!}{b!}\Big)^\lambda &\leq \sup_{s\in\mathbb{R},\, s\geq 0}\Big(\frac{\sigma}{\rho}\Big)^s s^{\lambda a} \\
&\leq \Big(\frac{\lambda\rho a}{e}\Big)^{\lambda a}\frac{1}{(\rho-\sigma)^{\lambda a}} \\
&\leq \frac{1}{m^{\lambda a}}\frac{(\lambda a)^{\lambda a}}{(\rho-\sigma)^{\lambda a}}.
\end{aligned}
\tag{2.4}
$$

If $\beta \in \mathbb{Z}_+^n$, we have, using (2.4) and the condition $m^{-1} \leq \sigma$,

$$
\begin{aligned}
|\partial^\beta(\partial^\alpha f)(x)| &\leq |f|_{\lambda,\rho}\frac{((\alpha+\beta)!)^\lambda}{\rho^{|\alpha+\beta|}}\frac{1}{(1+|\alpha+\beta|)^{2n}} \\
&\leq |f|_{\lambda,\rho}\frac{(\beta!)^\lambda}{\sigma^{|\beta|}}\frac{1}{(1+|\beta|)^{2n}}\frac{1}{\sigma^{|\alpha|}}\Big(\frac{\sigma}{\rho}\Big)^{|\alpha+\beta|}\Big(\frac{(\alpha+\beta)!}{\beta!}\Big)^\lambda \\
&\leq |f|_{\lambda,\rho}\frac{(\beta!)^\lambda}{\sigma^{|\beta|}}\frac{1}{(1+|\beta|)^{2n}}(\lambda|\alpha|)^{\lambda|\alpha|}\frac{1}{(\rho-\sigma)^{\lambda|\alpha|}},
\end{aligned}
$$

which shows the theorem. □

Let us state the following important fact which is known as that the Gevrey space $\mathcal{G}_{\lambda,\rho}(\Omega)$ is sufficiently rich.

Theorem 2.3. *Let $\lambda > 1C\rho$ a positive constant and Ω an open set in $\mathbb{R}^n$. Let a point $x_0 \in \Omega$ and a neighborhood $V \subset \Omega$ of x_0 be given. Then there exists an element $\varphi \in \mathcal{G}_{\lambda,\rho}(\Omega)$ such that*

$$\operatorname{supp}\varphi \subset V, \quad \varphi(x_0) > 0, \quad \varphi(x) \geq 0 \quad (\forall x \in \Omega),$$

where $\operatorname{supp}\varphi$ *means the support of* φ.

Proof. Here we give a sketch of the proof. Let q be an integer ≥ 2. Define a C^∞-function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(t) = \begin{cases} \exp(-1/t^q) & (t > 0), \\ 0 & (t \leq 0). \end{cases}$$

We can show that, if $1 + q^{-1} < \lambda$, then there exists a positive constant c such that the function

$$f_{c,q}(t) = f_q(ct)$$

belongs to $\mathcal{G}_{\lambda,\rho}(\mathbb{R})$.

Write $x_0 = (x_{01}, \dots, x_{0n})$. Take $r > 0$ such that

$$[x_{01} - r, x_{01} + r] \times \dots \times [x_{0n} - r, x_{0n} + r] \subset V.$$

For $x = (x_1, \dots, x_n) \in \Omega$, define

$$\varphi(x) = \prod_{i=1}^{n} f_{c,q}(x_i - x_{0i} + r) f_{c,q}(x_{0i} + r - x_i).$$

Then $\varphi \in \mathcal{G}_{\lambda,\rho}(\Omega)$ and satisfies all requirements in the theorem. □

Lemma 2.4 and Lemma 2.5 stated below will play important roles when we discuss the problem concerning the approximation of functions on Gevrey spaces (Lemma 2.7). However, the proofs of two lemmas can be performed by a standard method similar to the case of C^∞-functions as can be seen in Treves [3]. So we omit the proofs.

Lemma 2.4. *Let $\varphi \in \mathcal{G}_{\lambda,\rho}(\mathbb{R}^n)$ be a function such that*

$$\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n : \|x\| \leq 1\}, \quad \varphi(0) > 0, \quad \varphi(x) \geq 0 \ (\forall x \in \mathbb{R}^n).$$

For $\varepsilon > 0$, define

$$\varphi_\varepsilon(x) = \varepsilon^{-n} a \varphi(\varepsilon^{-1} x) \quad (x \in \mathbb{R}^n),$$

where

$$a = \Big(\int_{\|x\| \leq 1} \varphi(x) dx \Big)^{-1}.$$

Then φ_ε has the following properties:

(i) $\varphi_\varepsilon \in \mathcal{G}_{\lambda,\varepsilon\rho}(\mathbb{R}^n)$, $|\varphi_\varepsilon|_{\lambda,\varepsilon\rho} \leq \varepsilon^{-n} a |\varphi|_{\lambda,\rho}$,

(ii) $\operatorname{supp}\varphi_\varepsilon \subset \{x \in \mathbb{R}^n : \|x\| \leq \varepsilon\}$, $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = 1$.

Lemma 2.5. *Let $\varphi_\varepsilon \in \mathcal{G}_{\lambda,\varepsilon\rho}(\mathbb{R}^n)$ be the function defined in Lemma 2.4. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that* $\operatorname{supp} f$ *is compact. Then the convolution*

$$\varphi_\varepsilon * f(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(y) f(x - y) dy \ \Big(= \int_{\mathbb{R}^n} \varphi_\varepsilon(x - y) f(y) dy \Big)$$

of φ_ε and f satisfies the following properties:

(i) $\varphi_\varepsilon * f \in \mathcal{G}_{\lambda,\varepsilon\rho}(\mathbb{R}^n)$, $|\varphi_\varepsilon * f|_{\lambda,\varepsilon\rho} \leq \big\{ \int_{\mathbb{R}^n} |f(y)| dy \big\} |\varphi_\varepsilon|_{\lambda,\varepsilon\rho}$.

(ii) $\operatorname{supp}\varphi_\varepsilon * f \subset \{x \in \mathbb{R}^n : d(x, \operatorname{supp} f) \leq \varepsilon\}$, *where the letter d denotes distance.*

Lemma 2.6. *Let $f \in \mathcal{G}_{\lambda,\rho}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ be given. Define the function f_y by $f_y(x) = f(x-y)$. If $0 < \sigma < \rho$, then $f_y \in \mathcal{G}_{\lambda,\sigma}(\mathbb{R}^n)$ and the map*

$$\mathbb{R}^n \ni y \mapsto f_y \in \mathcal{G}_{\lambda,\sigma}(\mathbb{R}^n)$$

is continuous.

Proof. Fix $y_0, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}^n_+$ arbitrarily. Define the function $k(\theta)$ by

$$k(\theta) = \partial^\alpha f(x - y_0 + \theta(y_0 - y)) \quad (0 \le \theta \le 1).$$

Write $y = (y_1, \dots, y_n), y_0 = (y_{01}, \dots, y_{0n})$. Then we obtain

$$\begin{aligned}
\partial^\alpha f_y(x) - \partial^\alpha f_{y_0}(x) &= \partial^\alpha f(x-y) - \partial^\alpha f(x-y_0) = k(1) - (0) \\
&= \int_0^1 k'(\theta) d\theta \\
&= \int_0^1 \Big\{ \sum_{j=1}^n \partial_j (\partial^\alpha f)(x - y_0 + \theta(y_0 - y))(y_j - y_{0j}) \Big\} d\theta.
\end{aligned}$$

On the other hand, by Theorem 2.2, we know that $\partial_j f \in \mathcal{G}_{\lambda,\sigma}(\mathbb{R}^n)$. Hence, from the above equalities, it follows that

$$\begin{aligned}
|\partial^\alpha f_y(x) - \partial^\alpha f_{y_0}(x)| &\le \sum_{j=1}^n \int_0^1 |\partial^\alpha (\partial_j f)(x - y_0 + \theta(y_0 - y))| |y_j - y_{0j}| d\theta \\
&\le \sum_{j=1}^n |\partial_j f|_{\lambda,\sigma} |y_j - y_{0j}| \frac{(\alpha!)^\lambda}{\sigma^{|\alpha|}} \frac{1}{(1+|\alpha|)^{2n}} \\
&\le \Big\{ \sum_{j=1}^n |\partial_j f|_{\lambda,\sigma} \Big\} \|y - y_0\| \frac{(\alpha!)^\lambda}{\sigma^{|\alpha|}} \frac{1}{(1+|\alpha|)^{2n}},
\end{aligned}$$

and we obtain the inequality

$$|f_y - f_{y_0}|_{\lambda,\sigma} \le \Big\{ \sum_{j=1}^n |\partial_j f|_{\lambda,\sigma} \Big\} \|y - y_0\|,$$

which shows the continuity of the map $y \mapsto f_y \in \mathcal{G}_{\lambda,\sigma}(\mathbb{R}^n)$. □

Let us define the subset $\mathcal{G}^c_{\lambda,\rho}(\Omega)$ of $\mathcal{G}_{\lambda,\rho}(\Omega)$ by

$$\mathcal{G}^c_{\lambda,\rho}(\Omega) = \{ f \in \mathcal{G}_{\lambda,\rho}(\Omega) : \operatorname{supp} f \text{ is compact} \},$$

and give $\mathcal{G}^c_{\lambda,\rho}(\Omega)$ the norm $|\cdot|_{\lambda,\rho}$ defined on $\mathcal{G}_{\lambda,\rho}(\Omega)$.

Lemma 2.4, Lemma 2.5 and Lemma 2.6 yield the next lemma.

Lemma 2.7. *If $0 < \delta < \sigma < \rho$, then $\mathcal{G}^c_{\lambda,\rho}(\Omega)$ is a dense subspace of $\mathcal{G}^c_{\lambda,\sigma}(\Omega)$ with respect to the norm of $\mathcal{G}^c_{\lambda,\delta}(\Omega)$. In other words, if $f \in \mathcal{G}^c_{\lambda,\sigma}(\Omega)$ and $\varepsilon > 0$, then there exists an element $g \in \mathcal{G}^c_{\lambda,\rho}(\Omega)$ such that*

$$|f - g|_{\lambda,\delta} < \varepsilon.$$

Proof. Let $f \in \mathcal{G}^c_{\lambda,\sigma}(\Omega)$ and $\varepsilon > 0$ be given. If $0 < \delta < \sigma$, then, by Lemma 2.6, there exists $r > 0$ such that

$$\|y\| < r \Rightarrow |f_y - f|_{\lambda,\delta} < \varepsilon.$$

We can take the number r so that $\{x \in \mathbb{R}^n : d(x, \operatorname{supp} f) \le r\} \subset \Omega$. For the numbers r and ρ taken above, we take a third number s such that $\rho < rs$ and fix it. Next, we take the function $\varphi_r \in \mathcal{G}_{\lambda,rs}(\mathbb{R}^n)$ defined in Lemma 2.4. Then, by Lemma 2.5, the relation

$$\operatorname{supp} \varphi_r * f \subset \{x : d(x, \operatorname{supp} f) \le r\} \subset \Omega$$

holds. It follows that

$$\varphi_r * f \in \mathcal{G}^c_{\lambda,rs}(\Omega) \subset \mathcal{G}^c_{\lambda,\rho}(\Omega).$$

Since

$$\int_{\mathbb{R}^n} \varphi_r(y) dy = \int_{\|y\| \le r} \varphi_r(y) dy = 1,$$

we have

$$\varphi_r * f(x) - f(x) = \int_{\|y\| \le r} \varphi_r(y)(f(x-y) - f(x)) dy$$

and, for any $\alpha \in \mathbb{Z}^n_+$,

$$\begin{aligned} |\partial^\alpha(\varphi_r * f)(x) - \partial^\alpha f(x)| &\le \int_{\|y\| \le r} \varphi_r(y) |\partial^\alpha f_y(x) - \partial^\alpha f(x)| dy \\ &\le \int_{\|y\| \le r} \varphi_r(y) \left\{ |f_y - f|_{\lambda,\delta} \frac{(\alpha!)^\lambda}{\delta^{|\alpha|}} \frac{1}{(1+|\alpha|)^{2n}} \right\} dy \\ &\le \frac{(\alpha!)^\lambda}{\delta^{|\alpha|}} \frac{1}{(1+|\alpha|)^{2n}} \varepsilon, \end{aligned}$$

which shows that $|\varphi_r * f - f|_{\lambda,\delta} \le \varepsilon$. This completes the proof. □

Using Lemma 2.7, we can prove the next theorem.

Theorem 2.8. *For $\sigma > 0$, define a subspace $G^c_{\lambda,\sigma}(\Omega)$ of $\mathcal{G}^c_{\lambda,\sigma}(\Omega)$ by*

$$G^c_{\lambda,\sigma}(\Omega) = \bigcup_{\sigma < \rho} \mathcal{G}^c_{\lambda,\rho}(\Omega).$$

In $G^c_{\lambda,\sigma}(\Omega)$, we adopt the same norm $|\cdot|_{\lambda,\sigma}$ as in $\mathcal{G}_{\lambda,\sigma}(\Omega)$. If $\sigma < \delta$, then

$$G^c_{\lambda,\delta}(\Omega) \subset G^c_{\lambda,\sigma}(\Omega)$$

and $G^c_{\lambda,\delta}(\Omega)$ is a dense subspace of $G^c_{\lambda,\sigma}(\Omega)$.

Proof. The relation $G^c_{\lambda,\delta}(\Omega) \subset G^c_{\lambda,\sigma}(\Omega)$ is clear. In order to see that $G^c_{\lambda,\delta}(\Omega)$ is dense in $G^c_{\lambda,\sigma}(\Omega)$, take $f \in G^c_{\lambda,\sigma}(\Omega)$ and $\varepsilon > 0$. Then $f \in \mathcal{G}^c_{\lambda,\sigma_1}(\Omega)$ for some number σ_1 such that $\sigma < \sigma_1 < \delta$. Take two numbers σ_2 and δ_1 such that $\sigma < \sigma_2 < \sigma_1 < \delta < \delta_1$. Then, by Lemma 2.7, there exists $g \in \mathcal{G}^c_{\lambda,\delta_1}(\Omega)$ such that $|f - g|_{\lambda,\sigma_2} < \varepsilon$. On the other hand, g belongs to $G^c_{\lambda,\delta}(\Omega)$ and satisfies the inequality

$$|f - g|_{\lambda,\sigma} \le |f - g|_{\lambda,\sigma_2} < \varepsilon.$$

Hence, this shows that $G^c_{\lambda,\delta}(\Omega)$ is a dense subspace of $G^c_{\lambda,\sigma}(\Omega)$. □

Concerning the product of two functions and differential operators on a function, the family $\{G^c_{\lambda,\sigma}(\Omega)\}_{\sigma>0}$ defined in Theorem 2.8 has the same properties as those in the Gevrey class $\{\mathcal{G}_{\lambda,\sigma}(\Omega)\}_{\sigma>0}$. In other words, the implication

$$f, g \in G^c_{\lambda,\sigma}(\Omega) \Rightarrow fg \in G^c_{\lambda,\sigma}(\Omega)$$

holds and, if $0 < \sigma < \rho, \alpha \in \mathbb{Z}^n_+$, then the implication

$$f \in G^c_{\lambda,\rho}(\Omega) \Rightarrow \partial^\alpha f \in G^c_{\lambda,\sigma}(\Omega)$$

holds.

Now, it is easy, by Theorem 2.8, to construct a Banach scale consisting of dual Banach spaces. If $(X, \|\cdot\|_X)$ is a normed space, then the dual space X^* of X is defined as usual. Let Y be a linear subspace of X and $(Y, \|\cdot\|_Y)$ a normed space such that $\|y\|_X \le \|y\|_Y$ for $y \in Y$. Then the map $i : Y \ni y \mapsto y \in X$ is continuous, and the adjoint operator $i^* : X^* \to Y^*$ of i satisfies the inequality

$$\|i^*(u)\|_Y \le \|u\|_X \ (u \in X^*). \tag{2.5}$$

Here we want to identify X^* with $i^*(X^*)$. This is possible if Y is dense in X, since i^* is injective in that case. Then we have, by (2.5),

$$\|u\|_Y \le \|u\|_X \ (u \in X^*).$$

We have the following theorem.

Theorem 2.9. *Fix a number $\rho > 0$ arbitrarily. For σ such that $0 < \sigma < \rho$, put*

$$\mathcal{D}_{\lambda,\sigma}(\Omega) = \{G^{c}_{\lambda,\rho-\sigma}(\Omega)\}^*,$$

and denote by $\|\cdot\|_{\lambda,\sigma}$ the norm on $\mathcal{D}_{\lambda,\sigma}(\Omega)$. Then the family $\{\mathcal{D}_{\lambda,\sigma}(\Omega)\}_{0<\sigma<\rho}$ forms a Banach scale.

The proof of the above theorem is obvious from the arguments preceding the theorem.

3. Uniqueness of the solution of the initial value problem in a Banach scale

In this section, we shall prove the uniqueness of the solution of the initial value problem in a Banach scale, and we use this result in showing our main theorem of this paper.

Let $\{E_\sigma\}_{\sigma_0<\sigma\le\delta_0}$ be a scale of Banach spaces, where $0 \le \sigma_0 < \delta_0 < \infty$. Let I be an interval which contains 0 as an inner point. Let F be a map of the form

$$F : I \times \bigcup_{\sigma_0<\sigma\le\delta_0} \underbrace{E_\sigma \times \cdots \times E_\sigma}_{p-\text{times}} \to \bigcup_{\sigma_0<\sigma\le\delta_0} E_\sigma$$

such that if $\sigma_0 < \delta < \sigma \le \delta_0$, then $F(I \times E_\sigma \times \cdots \times E_\sigma) \subset E_\delta$ and the map

$$F : I \times E_\sigma \times \cdots \times E_\sigma \to E_\delta$$

is continuous. Further we assume that there exists a positive constant C such that if $\sigma_0 < \delta < \sigma \le \delta_0$, then the inequality

$$\|F(t,u) - F(t,v)\|_\delta \le C \sum_{j=0}^{p-1} \frac{\|u_j - v_j\|_\sigma}{(\sigma-\delta)^{p-j}} \qquad (\forall t \in I) \tag{3.1}$$

holds for $u = (u_0, \dots, u_{p-1}), v = (v_0, \dots, v_{p-1}) \in E_\sigma \times \cdots \times E_\sigma$.

For such a map F we can consider an initial value problem of the form

$$u^{(p)}(t) = F(t, u(t), u'(t), \dots, u^{(p-1)}(t)), \tag{3.2}$$

$$u(0) = b_0, u'(0) = b_1, \dots, u^{(p-1)}(0) = b_{p-1}, \tag{3.3}$$

where

$$b_0, \dots, b_{p-1} \in E_{\delta_0}.$$

Concerning the above mentioned problem, the author proved, in [1], the existence and the uniqueness of the solution. However, in this paper, we only use the uniqueness of the solution. Here we state and prove a uniqueness result which is simpler than the uniqueness part of the result given in [1].

Theorem 3.1. *Let $T > 0$ such that $[-T,T] \subset I$. Let u, v be C^p- maps from the interval $[-T,T]$ to E_{δ_0}. If u and v are the solutions of the problem* (3.2)-(3.3)*, then for $t \in [-T,T]$*

$$u(t) = v(t).$$

Proof. The problem (3.2)-(3.3) is rewritten as a system of first order equations of the form

$$\begin{cases} u_1'(t) &= u_2(t) \\ &\vdots \\ u_{p-1}'(t) &= u_p(t) \\ u_p'(t) &= F(t, u_1(t), \dots, u_p(t)) \end{cases} \tag{3.4}$$

$$u_1(0) = b_0, \dots, u_p(0) = b_{p-1} \tag{3.5}$$

in the unknown functions $u_1(t), \dots, u_p(t)$. Further the problem (3.4)-(3.5) is equivalent to the integral equation of the form

$$\begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix} = \begin{bmatrix} b_0 \\ \vdots \\ b_{p-1} \end{bmatrix} + \int_0^t \begin{bmatrix} u_2(\tau) \\ \vdots \\ u_p(\tau) \\ F(\tau, u_1(\tau), \dots, u_p(\tau)) \end{bmatrix} d\tau. \tag{3.6}$$

From (3.6) it follows that each $u_j(t)$ satisfies

$$\begin{aligned} u_j(t) = b_{j-1} + tb_j + \dots + \frac{t^{p-j}}{(p-j)!} b_{p-1} \\ + \int_0^t \frac{(t-\tau)^{p-j}}{(p-j)!} F(\tau, u_1(\tau), \dots, u_p(\tau)) d\tau. \end{aligned} \tag{3.7}$$

If $u(t), v(t)$ are C^p-solutions of the problem (3.2)-(3.3) , then $(u_1(t), \dots, u_p(t)) = (u(t), u'(t), \dots, u^{(p-1)}(t))$ and $(v_1(t), \dots, v_p(t)) = (v(t), v'(t), \dots, v^{(p-1)}(t))$ are solutions of the integral equation (3.6). $u_j(t), v_j(t)$ are continuous maps from $[-T,T]$ to the Banach space E_{δ_0}.

To prove the theorem, it is sufficient to prove $u_j(t) = v_j(t)$. If $\sigma_0 < \nu < \mu \le \delta_0$, then, by (3.1) and by (3.7), we have, for $t \in [-T,T]$,

$$\begin{aligned} \|u_j(t) - v_j(t)\|_\nu &\le \int_0^{|t|} \frac{(|t|-\tau)^{p-j}}{(p-j)!} \|F(\tau, u_1(\tau). \dots, u_p(\tau)) \\ &\quad - F(\tau, v_1(\tau). \dots, v_p(\tau))\|_\nu d\tau \\ &\le C \sum_{i=1}^p \frac{1}{(\mu-\nu)^{p+1-j}} \int_0^{|t|} \frac{(|t|-\tau)^{p-j}}{(p-j)!} \|u_i(\tau) - v_i(\tau)\|_\mu d\tau. \end{aligned} \tag{3.8}$$

For the moment we assume that $0 \le t \le T$ and put

$$M = \max_{0 \le t \le T} \sum_{j=1}^p \|u_j(t) - v_j(t)\|_{\delta_0}.$$

Let $\mu = \delta_0, \nu = \delta_0 - \xi$ $(0 < \xi < \delta_0 - \sigma_0)$. We have, by (3.8),

$$\begin{aligned} \|u_j(t) - v_j(t)\|_{\delta_o - \xi} &\le C \sum_{i=1}^{p} \frac{1}{\xi^{p+1-i}} \int_0^t \frac{(t-\tau)^{p-j}}{(p-j)!} \|u_i(\tau) - v_i(\tau)\|_{\delta_0} d\tau \\ &\le MC \sum_{l=1}^{p} \frac{1}{\xi^{j-l}} \frac{1}{\xi^{p+1-j}} \frac{t^{p+1-j}}{(p+1-j)!} \\ &\le MC \sum_{l=1}^{p} \frac{1}{\xi^{j-l}} \sum_{k=1}^{p} \frac{1}{\xi^k} \frac{t^k}{k!}. \end{aligned} \tag{3.9}$$

Next let $\mu = \delta_0 - \xi, \nu = \delta_0 - 2\xi$ $(0 < 2\xi < \delta_0 - \sigma_0)$, we have, by (3.8) and (3.9),

$$\begin{aligned} \|u_j(t) - v_j(t)\|_{\delta_0 - 2\xi} &\le MC^2 \sum_{i=1}^{p} \frac{1}{\xi^{p+1-i}} \int_0^t \frac{(t-\tau)^{p-j}}{(p-j)!} \Big\{ \sum_{l=1}^{p} \frac{1}{\xi^{i-l}} \sum_{k=1}^{p} \frac{1}{\xi^k} \frac{\tau^k}{k!} \Big\} d\tau \\ &= MC^2 \sum_{i=1}^{p} \frac{1}{\xi^{p+1-i}} \Big(\sum_{l=1}^{p} \frac{1}{\xi^{i-l}} \Big) \sum_{k=1}^{p} \frac{1}{\xi^k} \int_0^t \frac{(t-\tau)^{p-j}}{(p-j)!} \frac{\tau^k}{k!} d\tau \\ &= MpC^2 \sum_{l=1}^{p} \frac{1}{\xi^{p+1-l}} \sum_{k=1}^{p} \frac{1}{\xi^k} \frac{t^{p+1+k-j}}{(p+1+k-j)!} \\ &= MpC^2 \sum_{l=1}^{p} \frac{1}{\xi^{j-l}} \sum_{k=1}^{p} \frac{1}{\xi^{p+1+k-j}} \frac{t^{p+1+k-j}}{(p+1+k-j)!} \\ &\le MpC^2 \sum_{l=1}^{p} \frac{1}{\xi^{j-l}} \sum_{k=2}^{2p} \frac{1}{\xi^k} \frac{t^k}{k!}. \end{aligned}$$

Repeating this process, we can show that, if n is a natural number and ξ is a positive number such that $n\xi < \delta_0 - \sigma_0$, then the inequality

$$\|u_j(t) - v_j(t)\|_{\delta_0 - n\xi} \le Mp^{n-1}C^n \sum_{l=1}^{p} \frac{1}{\xi^{j-l}} \sum_{k=n}^{np} \frac{1}{\xi^k} \frac{t^k}{k!} \tag{3.10}$$

holds for $0 \le t \le T$. Writing $\mu = \delta_0 - n\xi$, (3.10) is rewritten as

$$\|u_j(t) - v_j(t)\|_{\mu} \le Mp^{n-1}C^n \sum_{l=1}^{p} \frac{n^{j-l}}{(\delta_0 - \mu)^{j-l}} \sum_{k=n}^{np} \frac{n^k}{(\delta_0 - \mu)^k} \frac{t^k}{k!}. \tag{3.11}$$

In (3.11), μ can be any number such that $\sigma_0 < \mu < \delta_0$. Moreover, since $e^n > n^k/k!$, we see that the inequality

$$\|u_j(t) - v_j(t)\|_{\mu} \le \frac{M}{p} \Big(\frac{peCt}{\delta_0 - \mu} \Big)^n \Big(\sum_{l=1}^{p} \frac{n^{j-l}}{(\delta_0 - \mu)^{j-l}} \Big) \sum_{k=0}^{n(p-1)} \big(\frac{t}{\delta_0 - \mu} \big)^k$$

holds. Let $L = \min\{1/peC, 1\}$. For each $t \in [0, T]$ such that $t < L(\delta_0 - \mu)$, we have

$$\|u_j(t) - v_j(t)\|_{\mu} \le \frac{M}{p} \left(\frac{peCt}{\delta_0 - \mu} \right)^n \Big(\sum_{l=1}^{p} \frac{n^{j-l}}{(\delta_0 - \mu)^{j-l}} \Big) \frac{1}{1 - \frac{t}{\delta_0 - \mu}}. \tag{3.12}$$

Letting $n \to \infty$ in (3.12), we know that $\|u_j(t) - v_j(t)\|_{\mu} = 0$ for $t \in [0, T] \cap [0, L(\delta_0 - \mu))$, since

$$\lim_{n \to \infty} \left(\frac{peCt}{\delta_0 - \mu} \right)^n \frac{n^{j-l}}{(\delta_0 - \mu)^{j-l}} = 0.$$

Thus we have shown that $u_j(t) = v_j(t)$ for $t \in [0,T] \cap [0, L(\delta_0 - \mu))$. Further, since μ can be taken as close to σ_0 as desired, we see that $u_j(t) = v_j(t)$ for $t \in [0,T] \cap [0, L(\delta_0 - \sigma_0)]$. If $L(\delta_0 - \sigma_0) < T$, then by a similar argument as above, we conclude that $u_j(t) = v_j(t)$ for $t \in [0,T]$ such that $L(\delta_0 - \sigma_0) \le t \le 2L(\delta_0 - \sigma_0)$, and hence $u_j(t) = v_j(t)$ for $t \in [0,T] \cap [0, 2L(\delta_0 - \sigma_0)]$. Repeating this argument, we conclude that $u_j(t) = v_j(t)$ for $\forall t \in [0,T]$. Finally, it is clear by (3.8) that a similar argument as above is valid in the case for $t \in [-T, 0]$, too. This completes the proof. $\square$

4. Main theorem

The purpose of this paper is to give a proof of the following theorem.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $I \subset \mathbb{R}$ be an interval of the form $I = [-T, T]$, where T is a positive constant. Let λ be a constant > 1. Write $\rho_0 = m^{-1}(1+e)$, where m is a positive integer. In the differential equation* (1.1)*, assume that each coefficient $a_{\alpha,j}(t,x)$ satisfies the condition that the function $x \mapsto a_{\alpha,j}(t,x)$ belongs to $\mathcal{G}_{\lambda,\rho_0}(\Omega)$ for each fixed $t \in I$ and the map $I \ni t \mapsto a_{\alpha,j}(t,\cdot) \in \mathcal{G}_{\lambda,\rho_0}(\Omega)$ is continuous. Then the only C^p-solution $v(t,x)$ in the domain $(t,x) \in I \times \Omega$ of the initial value problem* (1.1)–(1.2) *is $v(t,x) \equiv 0$.*

Proof. We first introduce an initial value problem on the dual Banach scale which is called 'adjoint' to the problem (1.1)-(1.2).

Let $\Psi \subset \mathbb{R}^n$ be an open set such that its closure $\overline{\Psi}$ is compact and is contained in Ω. For δ such that $m^{-1} < \delta \le m^{-1}e$, put $\mathcal{D}_{\lambda,\delta}(\Psi) = \{G^c_{\lambda,\rho_0-\delta}(\Psi)\}^*$. Then, by Theorem 2.9, $\{\mathcal{D}_{\lambda,\delta}(\Psi)\}_{m^{-1}<\delta\le m^{-1}e}$ forms a Banach scale, and we use this scale throughout the rest of this paper.

Using the *dual scale*, we can define the adjoint equation to the problem (1.1). For δ, σ such that $m^{-1} < \delta < \sigma \le m^{-1}e$, we define a map

$$F : I \times \underbrace{\mathcal{D}_{\lambda,\sigma}(\Psi) \times \dots \mathcal{D}_{\lambda,\sigma}(\Psi)}_{p-\text{times}} \to \mathcal{D}_{\lambda,\delta}(\Psi)$$

by

$$F(t, L_0, \dots, L_{p-1}) = \sum_{\alpha\in\mathbb{Z}^n_+,\, \lambda|\alpha|+j\le p,\, j\le p-1} (-1)^{|\alpha|} A^*_{\alpha,j}(t)(L_j), \tag{4.1}$$

where $A^*_{\alpha,j}(t)$ is the adjoint of the linear map $A_{\alpha,j}(t)$ which is defined by

$$A_{\alpha,j}(t) : G^c_{\lambda,\rho_0-\delta}(\Psi) \to G^c_{\lambda,\rho_0-\sigma}(\Psi),\ A_{\alpha,j}(t)(\varphi)(x) = \partial^\alpha(a_{\alpha,j}(t,x)\varphi(x)).$$

We have to verify that F is well defined. If $m^{-1} < \delta < \sigma \le m^{-1}e$ and $\varphi \in G^c_{\lambda,\rho_0-\delta}(\Psi)$, then, by Theorem 2.1, the product $a_{\alpha,j}(t,\cdot)\varphi(\cdot)$ is in $G^c_{\lambda,\rho_0-\delta}(\Psi)$. Then, by Theorem 2.2, $\partial^\alpha(a_{\alpha,j}(t,\cdot)\varphi(\cdot))$ is in $G^c_{\lambda,\rho_0-\sigma}(\Psi)$. Hence the map $A_{\alpha,j}(t)$ is well defined. The continuity of $A_{\alpha,j}(t)$ as a linear map from $G^c_{\lambda,\rho_0-\delta}(\Psi)$ to $G^c_{\lambda,\rho_0-\sigma}(\Psi)$ follows immediately from Theorem 2.1 and Theorem 2.2. This shows that if $\delta < \sigma$, then the adjoint map

$$A^*_{\alpha,j}(t) : \mathcal{D}_{\lambda,\sigma}(\Psi) \to \mathcal{D}_{\lambda,\delta}(\Psi)$$

of $A_{\alpha,j}(t)$ is well defined, and the map F is well defined, too.

Here, by the above arguments, we can define an ODE on the Banach scale $\{\mathcal{D}_{\lambda,\sigma}(\Psi)\}_{m^{-1}<\sigma\le m^{-1}e}$, by

$$L^{(p)}(t) = F(t, L(t), L'(t), \dots, L^{(p-1)}(t)), \tag{4.2}$$

which is the 'adjoint' equation to the given equation (1.1).

Now we show the following fact as the first step of the proof.

Assertion ($\star$): *The only C^p -solution of the equation* (4.2) *such that*

$$L(0) = L'(0) = \cdots = L^{(p-1)}(0) = 0 \tag{4.3}$$

is $L(t) \equiv 0$.

To prove the above result, we use Theorem 3.1. We first show that the map F is continuous. Fix the numbers δ, σ such that $m^{-1} < \delta < \sigma \le m^{-1}e$. We show the continuity of the adjoint operator

$$A^*_{\alpha,j} : I \times \mathcal{D}_{\lambda,\sigma}(\Psi) \to \mathcal{D}_{\lambda,\delta}(\Psi).$$

For $t \in I$ and $L, L_0 \in \mathcal{D}_{\lambda,\sigma}(\Psi)$, the inequality

$$\begin{aligned} &\|A^*_{\alpha,j}(t)(L - L_0)\|_{\lambda,\delta} \\ &\le \|L - L_0\|_{\lambda,\sigma} \sup\{|A_{\alpha,j}(t)\varphi|_{\lambda,\rho_0-\sigma} : \varphi \in G^c_{\lambda,\rho_0-\delta}(\Psi),\ |\varphi|_{\lambda,\rho_0-\delta} \le 1\} \end{aligned} \tag{4.4}$$

holds. If $|\varphi|_{\lambda,\rho_0-\delta} \le 1$, then, by (2.2) and (2.3) in §2, we obtain

$$\begin{aligned} |A_{\alpha,j}(t)\varphi|_{\lambda,\rho_0-\sigma} &= |\partial^\alpha(a_{\alpha,j}(t,\cdot)\varphi(\cdot))|_{\lambda,\rho_0-\sigma} \\ &\le (\lambda|\alpha|)^{\lambda|\alpha|} \frac{|a_{\alpha,j}(t,\cdot)\varphi(\cdot)|_{\lambda,\rho_0-\delta}}{(\rho_0 - \delta - \rho_0 + \sigma)^{\lambda|\alpha|}} \\ &\le p^p \frac{2^{3n}|a_{\alpha,j}(t,\cdot)|_{\lambda,\rho_0-\delta}}{(\sigma - \delta)^{\lambda|\alpha|}}. \end{aligned} \tag{4.5}$$

Put

$$K = \sup\{|a_{\alpha,j}(t,\cdot)|_{\lambda,\rho_0} : \lambda|\alpha| + j \le p, j \le p-1, t \in I\}.$$

Since the map $t \mapsto a_{\alpha,j}(t,\cdot) \in \mathcal{G}_{\lambda,\rho_0}(\Omega)$ is continuous, we know that K is finite. It follows, from (4.5), that

$$|A_{\alpha,j}(t)\varphi|_{\lambda,\rho_0-\sigma} \le p^p 2^{3n} K \frac{1}{(\sigma - \delta)^{\lambda|\alpha|}}$$

holds. By the last inequality and (4.4), we obtain

$$\|A^*_{\alpha,j}(t)(L - L_0)\|_{\lambda,\delta} \le \|L - L_0\|_{\lambda,\sigma} \frac{2^{3n}p^pK}{(\sigma - \delta)^{\lambda|\alpha|}}. \tag{4.6}$$

Next, we look at the inequality

$$\begin{aligned} &\|(A^*_{\alpha,j}(t) - A^*_{\alpha,j}(t_0))L_0\|_{\lambda,\delta} \\ &\le \|L_0\|_{\lambda,\sigma} \sup\{|A_{\alpha,j}(t)\varphi - A_{\alpha,j}(t_0)\varphi|_{\lambda,\rho_0-\sigma} : \varphi \in G^c_{\lambda,\rho_0-\delta}(\Psi), |\varphi|_{\lambda,\rho_0-\delta} \le 1\}, \end{aligned} \tag{4.7}$$

where $t, t_0 \in I$. If we use (2.2), (2.3) and (4.5), then

$$\begin{aligned} |A_{\alpha,j}(t)\varphi - A_{\alpha,j}(t_0)\varphi|_{\lambda,\rho_0-\sigma} &\le (\lambda|\alpha|)^{\lambda|\alpha|} \frac{|(a_{\alpha,j}(t,\cdot) - a_{\alpha,j}(t_0,\cdot))\varphi(\cdot)|_{\lambda,\rho_0-\delta}}{(\rho_0 - \delta - \rho_0 + \sigma)^{\lambda|\alpha|}} \\ &\le p^p \frac{2^{3n}|a_{\alpha,j}(t,\cdot) - a_{\alpha,j}(t_0,\cdot)|_{\lambda,\rho_0}}{(\sigma - \delta)^{\lambda|\alpha|}}. \end{aligned}$$

The last inequality and (4.7) imply

$$\|(A^*_{\alpha,j}(t) - A^*_{\alpha,j}(t_0))L_0\|_{\lambda,\delta} \le \|L_0\|_{\lambda,\sigma} p^p \frac{2^{3n}|a_{\alpha,j}(t,\cdot) - a_{\alpha,j}(t_0,\cdot)|_{\lambda,\rho_0}}{(\sigma - \delta)^{\lambda|\alpha|}}. \tag{4.8}$$

From (4.6) and (4.8) it follows that

$$\begin{aligned} &\|A^*_{\alpha,j}(t)L - A^*_{\alpha,j}(t_0)L_0\|_{\lambda,\delta} \\ &\le \|A^*_{\alpha,j}(t)(L-L_0)\|_{\lambda,\delta} + \|(A^*_{\alpha,j}(t)-A^*_{\alpha,j}(t_0))L_0\|_{\lambda,\delta} \\ &\le p^p\frac{2^{3n}}{(\sigma-\delta)^{\lambda|\alpha|}}\{K\|L-L_0\|_{\lambda,\sigma} + |a_{\alpha,j}(t,\cdot)-a_{\alpha,j}(t_0,\cdot)|_{\lambda,\rho_0}\|L_0\|_{\lambda,\sigma}\}. \end{aligned} \tag{4.9}$$

Since the map $t \mapsto a_{\alpha,j}(t,\cdot) \in \mathcal{G}_{\lambda,\rho_0}(\Omega)$ is continuous, (4.9) implies that the map

$$A^*_{\alpha,j} : I \times \mathcal{D}_{\lambda,\sigma}(\Psi) \to \mathcal{D}_{\lambda,\delta}(\Psi)$$

is continuous. Hence the map F is also continuous.

Let us show that the map F satisfies the condition (3.1) in §3. Put $L_0 = 0$ in (4.9). Then, for $t \in I$ and $L \in \mathcal{D}_{\lambda,\sigma}(\Psi)$, the inequality

$$\|A^*_{\alpha,j}(t)L\|_{\lambda,\delta} \le p^p 2^{3n} K \frac{\|L\|_{\lambda,\sigma}}{(\sigma-\delta)^{\lambda|\alpha|}} \tag{4.10}$$

holds. For

$$\mathcal{L} = (L_0,\ \dots, L_{p-1}), \mathcal{M} = (M_0,\dots,M_{p-1}) \in \underbrace{\mathcal{D}_{\lambda,\sigma}(\Psi)\times\cdots\times\mathcal{D}_{\lambda,\sigma}(\Psi)}_{p-\text{times}}$$

we have

$$\|F(t,\mathcal{L}) - F(t,\mathcal{M})\|_{\lambda,\delta} \le \sum_{\alpha\in\mathbb{Z}^n_+,\,\lambda|\alpha|+j\le p,\,j\le p-1} \|A^*_{\alpha,j}(t)(L_j - M_j)\|_{\lambda,\delta}.$$

Hence, by the last inequality and (4.10), we obtain

$$\|F(t,\mathcal{L}) - F(t,\mathcal{M})\|_{\lambda,\delta} \le p^p 2^{3n} K \sum_{\alpha,j} \frac{\|L_j - M_j\|_{\lambda,\sigma}}{(\sigma-\delta)^{\lambda|\alpha|}}. \tag{4.11}$$

Here we note that the inequality

$$\frac{1}{(\sigma-\delta)^{\lambda|\alpha|}} = \frac{(\sigma-\delta)^{p-j-\lambda|\alpha|}}{(\sigma-\delta)^{p-j}} \le \frac{e^p}{(\sigma-\delta)^{p-j}}$$

holds and we put

$$C = (ep)^p 2^{3n} K \sum_{\alpha\in\mathbb{Z}^n_+,\,\lambda|\alpha|\le p} 1_\alpha,$$

where $1_\alpha = 1$. Then, by (4.11), we obtain

$$\|F(t,\mathcal{L}) - F(t,\mathcal{M})\|_{\lambda,\delta} \le C\sum_{j=0}^{p-1} \frac{\|L_j - M_j\|_{\lambda,\sigma}}{(\sigma-\delta)^{p-j}},$$

which shows that the map F satisfies the condition (3.1) in §3.

Consequently, we can use the Theorem 3.1 to the problem (4.2)-(4.3). As a result, we conclude that the Assertion ($\star$) is true.

Proof of the identity $v(t,x) \equiv 0$: Let us construct a solution of the problem (4.2)-(4.3) by the given solution $v(t,x)$ of the problem (1.1)-(1.2).

We define

$$L_v(t)(\varphi) = \int_\Psi v(t,x)\varphi(x)dx$$

for $\varphi \in G^c_{\lambda,\rho_0-m^{-1}e}(\Psi) = G^c_{\lambda,m^{-1}}(\Psi)$ and $t \in I$. Then we can see that $L_v(t) \in \mathcal{D}_{\lambda,m^{-1}e}(\Psi)$ $(= \{G^c_{\lambda,m^{-1}}(\Psi)\}^*)$ and the map $I \ni t \mapsto L_v(t) \in \mathcal{D}_{\lambda,m^{-1}e}(\Psi)$ is C^p-class. The derivative $L_v^{(j)}(t)$ of $L_v(t)$ has the form

$$L_v^{(j)}(t)(\varphi) = \int_\Psi \partial_t^j v(t,x)\varphi(x)dx \quad (j = 1,\dots,p-1).$$

These facts can be proved without difficulty by the compactness of $\overline{\Psi}$ and by the uniform continuity of $\partial_t^j v(t,x)$ on $I \times \Psi$.

Now let us show that $L_v(t)$ is a solution of the problem (4.2)-(4.3). It is obvious by (1.2) that $L_v(t)$ satisfies the initial condition (4.3). Let σ be a number such that $m^{-1} < \sigma < m^{-1}e$. Take $\varphi \in G^c_{\lambda,\rho_0-\sigma}(\Psi)$ arbitrarily. We have

$$\begin{aligned} L_v^{(p)}(t)(\varphi) &= \int_\Psi \partial_t^p v(t,x)\varphi(x)dx \\ &= \sum_{\alpha\in\mathbb{Z}_+^n,\, \lambda|\alpha|+j\le p,\, j\le p-1} \int_\Psi \partial_x^\alpha(\partial_t^j v(t,x))a_{\alpha,j}(t,x)\varphi(x)dx. \end{aligned}$$

Further we have, by integration by parts in x,

$$\begin{aligned} L_v^{(p)}(t)(\varphi) &= \sum_{\alpha,j}(-1)^{|\alpha|}\int_\Psi \partial_t^j v(t,\ x)\partial_x^\alpha(a_{\alpha,j}(t,x)\varphi(x))dx \\ &= \sum_{\alpha,j}(-1)^{|\alpha|}L_v^{(j)}(t)(A_{\alpha,j}(t)\varphi) \\ &= \sum_{\alpha,j}(-1)^{|\alpha|}A^*_{\alpha,j}(t)(L_v^{(j)}(t))(\varphi) \\ &= F(t, L_v(t), L_v'(t),\dots,L_v^{(p-1)}(t))(\varphi), \end{aligned}$$

which shows that $L_v(t)$ is a solution of the adjoint equation (4.2). It follows from the Assertion $(\star)$ that $L_v(t) \equiv 0$, which means that for each $\varphi \in G^c_{\lambda,m^{-1}}(\Psi)$

$$\int_\Psi v(t,x)\varphi(x)dx = 0 \quad (\forall t \in I).$$

Hence, by Theorem 2.3, $v(t,x) \equiv 0$ on $I \times \Psi$. Further, since we can let Ψ be as close to Ω as desired, we conclude that $v(t,x) \equiv 0$ on $I \times \Omega$. This completes the proof. □

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Masaki Kawagishi
Department of General Education College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan
E-mail address: masaki@suruga.ge.cst.nihon-u.ac.jp