*Electronic Journal of Differential Equations*, Vol. 2010(2010), No. 75, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# REACTION-DIFFUSION SYSTEM OF EQUATIONS IN NON-STATIONARY MEDIUM AND ARBITRARY NON-SMOOTH DOMAINS

SIKIRU ADIGUN SANNI

Abstract. A system of non-linear partial differential equations describing one-step irreversible reaction, reactant to product, in a non-stationary medium and non-smooth domain is considered. After obtaining the necessary a priori estimates, the existence of a unique local strong solution to the system is proved using a fixed point theorem.

## 1. Introduction

We consider the semilinear parabolic system of partial differential equations

$$\nabla.\bar{v} = 0 \quad \text{in } \Omega_T \tag{1.1}$$

$$\frac{\partial \bar{v}}{\partial t} - \nu\Delta\bar{v} = -\nabla.(\bar{v}\otimes\bar{v}) - \frac{1}{\rho}\nabla p \quad \text{in } \Omega_T \tag{1.2}$$

$$\frac{\partial u}{\partial t} - k\Delta u = -\nabla.(\bar{v}u) + Qwf(u) \quad \text{in } \Omega_T \tag{1.3}$$

$$\frac{\partial w}{\partial t} - d\Delta w = -\nabla.(\bar{v}w) - wf(u) \quad \text{in } \Omega_T \tag{1.4}$$

$$\bar{v} = \bar{0},\ u = w = 0 \quad \text{on } \partial\Omega\times[0,T) \tag{1.5}$$

$$\bar{v}(x,0) = \bar{v}_0(x), \quad u(x,0) = u_0(x), \quad w(x,0) = w_0(x) \tag{1.6}$$

where $\bar{0}$ is the zero vector in $\mathbb{R}^3$, $\otimes$ is the matrix multiplication defined by the tensor $\bar{v}\otimes\bar{v} := v_iv_j$ $(i,j = 1,2,3)$ and $\Omega_T := \Omega\times[0,T)$. Notice then that $\nabla.(\bar{v}\otimes\bar{v}) = \frac{\partial}{\partial x_i}(\bar{v}_i\bar{v}_j) = \frac{\partial}{\partial x_j}(\bar{v}_i\bar{v}_j) = v_i\frac{\partial v_j}{\partial x_i} = \bar{v}.\nabla\bar{v}$ (using (1.1)).

In applications, the system models a single-step irreversible reaction, reactant $\to$ product in non-stationary incompressible medium. $\bar{v}(x,t)$ is the velocity of the medium; $\nu$ and $\rho$ are the kinematic viscosity and the density of the medium respectively. $u(x,t)$ is the temperature in the reaction vessel, $w(x,t)$ is the mass fraction of the reactant, $1-w(x,t)$ is the mass fraction of the product, $k$ the positive thermal conductivity and $d$ the reactant diffusivity. $Qwf(u)$ and $-wf(u)$ are the reaction kinetics, determined by a positive, uniformly bounded and differentiable function

2000 *Mathematics Subject Classification.* 35B40, 35K57, 80A25.
*Key words and phrases.* Irreversible reaction; reactant diffusivity; thermal conductivity; a priori estimates; Banach's fixed point theorem.

Submitted November 27, 2009. Published May 21, 2010.

$f(u)$. Furthermore, $f'(u)$ is assumed to be Lipschitz continuous. It is assumed that $\Omega$ is an open and bounded arbitrary non-smooth domain in $\mathbb{R}^3$. Theoretically, the reactant decomposes at a rate which is proportional to $w(x,t)f(u)$, where $f(u)$ is the approximate number of molecules that have sufficient energy for the reaction to begin. In this paper, we shall assume that

$$0 \le f(u) \le B \tag{1.7}$$

$$|f'(u)| \le B', \quad |f'(u) - f'(\tilde{u})| \le L|u - \tilde{u}| \tag{1.8}$$

For further information on chemical kinetics and combustion, the reader is referred to Buckmaster[3], Buckmaster and Ludford [4], and Frank-Kamenetskii [9].

Several combustion models assumed some smoothness on the boundary vis-a-vis stationary media. Authors of these models include Avrin [1, 2], Daddiouaissa [5], De Oliviera et al [6], Fitzgibbon and Martin [8], Henry [10], Konach [11], Sanni [13], Sattinger [14], and some literature cited in them.

In this paper, we establish the existence of a unique local-in-time strong solution to the system (1.1)-(1.6), in arbitrary non-smooth domains. Clearly, the inclusion of the Navier-Stokes equations in the system implies that the medium is non-stationary.

Using Leray projector [15], the problem (1.1)-(1.6) can be reduced to that of finding only $(\bar{v}, u, w)$ by a variational formulation. We are thus motivated to define:

**Definition 1.1.** We call a solution $(\bar{v}, u, w)$ of the system (1.1)-(1.6) a strong solution, provided $(\bar{v}, u, w) \in X^3$, where $X$ is defined by

$$X := L^\infty[0,T; H_0^1(\Omega)] \cap H^1[0,T; H_0^1(\Omega)] \cap W^{1,\infty}[0,T; L^2(\Omega)] \tag{1.9}$$

## 2. A priori estimates

We will need the following Sobolev embedding theorem, stated and proved in [7, pp. 265-266].

**Theorem 2.1.** *Assume that $\Omega \subset \mathbb{R}^n$ is open and bounded. Suppose $U \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$. Then we have the estimate*

$$\|U\|_{L^q(\Omega)} \le C\|\nabla U\|_{L^p(\Omega)} \tag{2.1}$$

*for each $q \in [1, p*]$, the constant $C$ depending only on $p, q, n$ and $\Omega$, where $p* := \frac{np}{n-p}$ is the Sobolev conjugate.*

Notice that the hypothesis of Theorem 2.1 requires no smoothness assumption on the boundary $\Omega$.

We now set out to obtain a priori estimates required to prove the existence of a unique local strong solution to the system (1.1)-(1.6). We first state and prove the following Lemmas.

**Lemma 2.2.** *Let $u \in H^1(\Omega)$ and $v, w, p \in H_0^1(\Omega)$. Then*

$$\int_\Omega uwp\,dx \le \epsilon\|u\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-1}\|w\|^2_{H_0^1(\Omega)}\|p\|^2_{H_0^1(\Omega)} \tag{2.2}$$

$$\int_\Omega uwp\,dx \le \epsilon\Big(\|u\|^2_{L^2(\Omega)} + \|p\|^2_{H_0^1(\Omega)}\Big) + C(\Omega)\epsilon^{-3}\|w\|^4_{H_0^1(\Omega)}\|p\|^2_{L^2(\Omega)} \tag{2.3}$$

$$\int_\Omega uwp\,dx \le \epsilon\|u\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-1}\|w\|^2_{H_0^1(\Omega)}\Big(T^{-1/2}\|p\|^2_{L^2(\Omega)} + T^{1/2}\|p\|^2_{H_0^1(\Omega)}\Big) \tag{2.4}$$

$$
\begin{aligned}
&\int_\Omega u\,(pw-\tilde p\tilde w)\,dx \\
&\le \epsilon\|u\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-1}\Big[\|p\|^2_{H^1_0(\Omega)}\|w-\tilde w\|^2_{H^1_0(\Omega)} + \|p-\tilde p\|^2_{H^1_0(\Omega)}\|\tilde w\|^2_{H^1_0(\Omega)}\Big]
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\int_\Omega u\,(pw-\tilde p\tilde w)\,dx \\
&\le \epsilon\|u\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-1}\Big[\|p\|^2_{H^1_0(\Omega)}\Big(T^{-1/2}\|w-\tilde w\|^2_{L^2(\Omega)} \\
&\quad + T^{1/2}\|w-\tilde w\|^2_{H^1_0(\Omega)}\Big) + \|p-\tilde p\|^2_{H^1_0(\Omega)}\Big(T^{-1/2}\|\tilde w\|^2_{L^2(\Omega)} + T^{1/2}\|\tilde w\|^2_{H^1_0(\Omega)}\Big)\Big]
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\int_\Omega uvwp\,dx &\le \epsilon\|\nabla u\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-1}\|v\|^4_{H^1_0(\Omega)}\|u\|^2_{L^2(\Omega)} \\
&\quad + C(\Omega)\Big(T^{-1/2}\|p\|^2_{L^2(\Omega)} + T^{1/2}\|p\|^2_{H^1_0(\Omega)}\Big)\|w\|^2_{H^1_0(\Omega)}
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&\int_\Omega u(vwp-\tilde{\tilde v}\tilde w\tilde p)dx \\
&\le C(\Omega)\epsilon^{-1}\Big(\|v\|^4_{H^1_0(\Omega)} + \|\tilde w\|^4_{H^1_0(\Omega)}\Big)\|u\|^2_{L^2(\Omega)} \\
&\quad + \epsilon\|\nabla u\|^2_{L^2(\Omega)} + C(\Omega)\Big[\|w\|^2_{H^1_0(\Omega)}\Big(T^{-1/2}\|p-\tilde p\|^2_{L^2(\Omega)} + T^{1/2}\|p-\tilde p\|^2_{H^1_0(\Omega)}\Big) \\
&\quad + \Big(T^{-1/2}\|\tilde p\|^2_{L^2(\Omega)} + T^{1/2}\|\tilde p\|^2_{H^1_0(\Omega)}\Big)\Big(\|w-\tilde w\|^2_{H^1_0(\Omega)} + \|v-\tilde{\tilde v}\|^2_{H^1_0(\Omega)}\Big)\Big]
\end{aligned}
\tag{2.8}
$$

*Proof.* 1. Proof of (2.2). By Hölder's inequality,

$$
\begin{aligned}
\int_\Omega uwp\,dx &\le \|u\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}\|p\|_{L^4(\Omega)} \\
&\le C(\Omega)\|u\|_{L^2(\Omega)}\|w\|_{H^1_0(\Omega)}\|p\|_{H^1_0(\Omega)}
\end{aligned}
\tag{2.9}
$$

by Sobolev embedding theorem. Then (2.2) follows easily from (2.9) by Cauchy's inequality with $\epsilon$.

2. Proof of (2.3) and (2.4).

$$
\begin{aligned}
\int_\Omega uwp\,dx &\le \epsilon\|u\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon}\int_\Omega p^2w^2dx \quad \text{(by Cauchy's inequality with } \epsilon) \\
&\le \epsilon\|u\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon}\|p\|_{L^2(\Omega)}\|p\|_{L^6(\Omega)}\|w\|^2_{L^6(\Omega)} \quad \text{(by Hölder's inequality)} \\
&\le \epsilon\|u\|^2_{L^2(\Omega)} + C(\Omega)(4\epsilon)^{-1}\|p\|_{L^2(\Omega)}\|p\|_{H^1_0(\Omega)}\|w\|^2_{H^1_0(\Omega)},
\end{aligned}
\tag{2.10}
$$

by Sobolev embedding theorem. Then (2.3) and (2.4) follow by applying Cauchy's inequality with $\epsilon^2$ and $T^{1/2}$, respectively, to the appropriate factors of the second term on the right side of (2.10).

3. Proof of (2.5) and (2.6).

$$
\int_\Omega u(pw-\tilde p\tilde w)dx = \int_\Omega up(w-\tilde w)dx + \int_\Omega u\tilde w(p-\tilde p)dx. \tag{2.11}
$$

Then (2.5) and (2.6) follows by applying (2.2) and (2.4) to (2.11) respectively.

4. Proof of (2.7). By Young's inequality and then by Hölder's inequality,

$$
\begin{aligned}
\int_\Omega uvwp\,dx &\le \frac{1}{2}\int_\Omega u^2v^2dx + \frac{1}{2}\int_\Omega w^2p^2dx \\
&\le \frac{1}{2}\|u\|_{L^2(\Omega)}\|u\|_{L^6(\Omega)}\|v\|^2_{L^6(\Omega)} + \|w\|^2_{L^6(\Omega)}\|p\|_{L^2(\Omega)}\|p\|_{L^6(\Omega)} \\
&\le \|u\|_{L^2(\Omega)}\|u\|_{H_0^1(\Omega)}\|v\|^2_{H_0^1(\Omega)} + \|w\|^2_{H_0^1(\Omega)}\|p\|_{L^2(\Omega)}\|p\|_{H_0^1(\Omega)},
\end{aligned}
\tag{2.12}
$$

by By Sobolev embedding theorem. (2.7) follows by applying Cauchy's inequalities with $\epsilon$ and $T^{1/2}$ to the first and second terms on the right side of (2.12) respectively.

5. Proof of (2.8).

$$
\begin{aligned}
&\int_\Omega u(vwp - \tilde{\bar{v}}\tilde{w}\tilde{p})dx \\
&= \int_\Omega uvw(p-\tilde{p})dx + \int_\Omega uv\tilde{p}(w-\tilde{w})dx + \int_\Omega u\tilde{p}\tilde{w}(v-\tilde{\bar{v}})dx\,.
\end{aligned}
\tag{2.13}
$$

Then (2.8) follows by applying (2.7) to the each term on the right side of (2.13). This concludes the proof of Lemma 2.2 ◻

**Lemma 2.3.** *Let* (1.1)-(1.4) *hold. Suppose* $\bar{v}_0, u_0, w_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, *then*

$$
\begin{aligned}
&\|\partial_t\bar{v}_0\|^2_{L^2(\Omega)} + \|\partial_t u_0\|^2_{L^2(\Omega)} + \|\partial_t w_0\|^2_{L^2(\Omega)} \\
&\le C(\|\nabla\bar{v}_0\|^1_H(\Omega)^2 + \|\nabla u_0\|^1_H(\Omega)^2 + \|\nabla w_0\|^1_H(\Omega)^2)(1+\|\bar{v}_0\|^2_{H_0^1(\Omega)}),
\end{aligned}
\tag{2.14}
$$

*where* $C = C(\nu, k, d, B, \rho, \Omega, Q)$

*Proof.* Taking (1.1) and (1.3) on $t=0$ and multiplying the corresponding equation to (1.3) by $\partial_t u_0$, we estimate

$$
\begin{aligned}
&\int_\Omega |\partial_t u_0|^2dx \\
&= -\int_\Omega \partial_t u_0.\bar{v}_0.\nabla u_0dx + k\int_\Omega \partial_t u_0\Delta u_0dx + Q\int_\Omega \partial_t u_0 w_0 f(u_0)dx \\
&\le 2\epsilon\int_\Omega |\partial_t u_0|^2dx + \frac{1}{4\epsilon}\Big(QB\int_0 |w_0|^2dx + k\int_\Omega |\Delta u_0|^2dx\Big) \\
&\quad \text{(Integrating by parts, using Cauchy's inequality with } \epsilon \text{ and (1.7))} \\
&\le 2\epsilon\int_\Omega |\partial_t\bar{v}_0|^2dx + \frac{C(Q,B,k,\Omega)}{\epsilon} \\
&\quad\times\Big(\|\bar{v}_0\|^2_{H_0^1(\Omega)}\|\nabla u_0\|^1_H(\Omega)^2 + \|w_0\|^2_{H_0^1(\Omega)} + \|\nabla^2u_0\|^2_{L^2(\Omega)}\Big),
\end{aligned}
\tag{2.15}
$$

by Hölder and Poincare's inequalities and using that $\|\Delta\bar{v}_0\|_{L^2(\Omega)} \le \|\nabla^2\bar{v}_0\|_{L^2(\Omega)}$. Choosing $\epsilon > 0$ sufficiently small and simplifying, we deduce

$$
\|\partial_t u_0\|^2_{L^2(\Omega)} \le C(Q,B,k,\Omega)\Big[(1+\|\bar{v}_0\|^2_{H_0^1(\Omega)})\|\nabla u_0\|^1_H(\Omega)^2 + \|w_0\|^2_{H_0^1(\Omega)}\Big] \tag{2.16}
$$

Evaluating (1.1), (1.2) and (1.4) at $t=0$, we obtain analogous estimates to (2.16), viz:

$$
\|\partial_t\bar{v}_0\|^2_{L^2(\Omega)} \le C(\nu,\Omega)(1+\|\bar{v}_0\|^2_{H_0^1(\Omega)})\|\nabla\bar{v}_0\|^1_H(\Omega)^2 \tag{2.17}
$$

$$
\|\partial_t w_0\|^2_{L^2(\Omega)} \le C(d,B,\Omega)\Big[(1+\|\bar{v}_0\|^2_{H_0^1(\Omega)})\|\nabla w_0\|^1_H(\Omega)^2 + \|w_0\|^2_{H_0^1(\Omega)}\Big] \tag{2.18}
$$

Combining (2.16), (2.17) and (2.18), we deduce (2.14). ◻

**Theorem 2.4.** *Let $\bar{v}_0, u_0, w_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. Suppose $(\bar{v}, u, w)$ is a strong solution of the system* (1.1)-(1.6)*. Then we have the estimate*

$$\begin{aligned}
&\sup_{[0,T]} \Big( \|\partial_t \bar{v}\|^2_{L^2(\Omega)} + \|\bar{v}\|^2_{H_0^1(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} + \|u\|^2_{H_0^1(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)} + \|w\|^2_{H_0^1(\Omega)} \Big) \\
&+ \|\nabla\left(\partial_t \bar{v}\right)\|^2_{L^2[0,T;L^2(\Omega)]} + \|\nabla(\partial_t u)\|^2_{L^2[0,T;L^2(\Omega)]} + \|\nabla\left(\partial_t w\right)\|^2_{L^2[0,T;L^2(\Omega)]} \\
&\le \frac{CT[(1+\|\bar{v}_0\|^2_{H_0^1(\Omega)})(G(\bar{v}_0,u_0,w_0)+1)]^3}{\{1-2CT[(1+\|\bar{v}_0\|^2_{H_0^1(\Omega)})(G(\bar{v}_0,u_0,w_0)+1)]^2\}^{3/2}} \\
&=: \Sigma = constant,
\end{aligned} \tag{2.19}$$

*for*

$$T < \{2C[(1+\|\bar{v}_0\|^2_{H_0^1(\Omega)})(\|\nabla \bar{v}_0\|^1_H(\Omega)^2 + \|\nabla u_0\|^1_H(\Omega)^2 + \|\nabla w_0\|^1_H(\Omega)^2) + 1]^2\}^{-1}, \tag{2.20}$$

*where*

$$G(\bar{v}_0, u_0, w_0) = \|\nabla \bar{v}_0\|^1_H(\Omega)^2 + \|\nabla u_0\|^1_H(\Omega)^2 + \|\nabla w_0\|^1_H(\Omega)^2$$

*and $C = C(\nu, k, d, Q, B, B', \Omega)$.*

We will use (1.3) and the corresponding conditions in (1.5) and (1.6) to obtain estimates for $u$; and thereafter, for brevity, state analogous estimates for $\bar{v}$ and $w$.

*Proof.* 1. Multiplying (1.3) by $\partial_t u$, integrating the ensuing equation by parts over $\Omega$ and using (1.1) and (1.5), we deduce

$$\begin{aligned}
&\int_\Omega |\partial_t u|^2 dx + \frac{k}{2}\frac{d}{dt}\Big(\int_\Omega |\nabla u|^2 dx\Big) \\
&= \int_\Omega \nabla(\partial_t u).(\bar{v}u) dx + Q\int_\Omega \partial_t u w f(u) dx \\
&\le \epsilon \|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + C(\Omega)\Big(\frac{1}{\epsilon}\|\bar{v}\|^1_H(\Omega)^2 \|u\|^1_H(\Omega)^2 + Q^2B^2\|w\|^2_{H_0^1(\Omega)}\Big) \\
&\quad + \|\partial_t u\|^2_{L^2(\Omega)},
\end{aligned} \tag{2.21}$$

using (2.2)) of Lemma 2.2. Simplifying, (2.21) yields

$$\begin{aligned}
&\frac{d}{dt}\big(\frac{k}{2}\|u\|^2_{H_0^1(\Omega)}\big) \\
&\le \epsilon \|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + C(\Omega)\Big(\frac{1}{\epsilon}\|\bar{v}\|^1_H(\Omega)^2 \|u\|^1_H(\Omega)^2 + Q^2B^2\|w\|^2_{H_0^1(\Omega)}\Big)
\end{aligned} \tag{2.22}$$

2. Differentiating (1.3) with respect to $t$ yields

$$\frac{\partial}{\partial t}(\partial_t u) - k\Delta(\partial_t u) = -\partial_t \bar{v}.\nabla u + \bar{v}.\nabla(\partial_t u) + Q\partial_t w.f(u) + Qw\partial_t u f'(u) \tag{2.23}$$

Multiply by $\partial_t u$ and integrating by parts over $\Omega$ and use (1.1), (1.5) to deduce:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|^2_{L^2(\Omega)}) + k\|\nabla(\partial_t u)\|^2_{L^2(\Omega)} \\
&= \int_\Omega \nabla(\partial_t u).\partial_t \bar{v}.u dx + \int_\Omega \nabla(\partial_t u).\bar{v}.\partial_t u dx \\
&\quad + Q\int_\Omega \partial_t u \partial_t u w f'(u)dx + Q\int_\Omega \partial_t u \partial_t w f(u)dx \\
&\le \epsilon\|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + \epsilon\|\nabla(\partial_t \bar{v})\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-3}\|\partial_t\bar{v}\|^2_{L^2(\Omega)}\|u\|^4_{H^1_0(\Omega)} \\
&\quad \epsilon\|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + \epsilon\|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-3}\|\partial_t u\|^2_{L^2(\Omega)}\|v\|^4_{H^1_0(\Omega)} \\
&\quad + QB'[\epsilon\|\partial_t u\|^2_{L^2(\Omega)} + \epsilon\|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + C(\Omega)\epsilon^{-3}\|\partial_t u\|^2_{L^2(\Omega)}\|w\|^4_{H^1_0(\Omega)}] \\
&\quad + QB(\|\partial_t u\|^2_{L^2(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)}),
\end{aligned}
\tag{2.24}
$$

using (2.3) of Lemma 2.2 and Young's inequality.

3. Combining (2.22) and (2.24) we deduce

$$
\begin{aligned}
&\frac{d}{dt}(\|\partial_t u\|^2_{L^2(\Omega)} + k\|u\|^2_{H^1_0(\Omega)}) + 2k\|\nabla(\partial_t u)\|^2_{L^2(\Omega)} \\
&\le C\Big[\epsilon\|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + \epsilon\|\nabla(\partial_t \bar{v})\|^2_{L^2(\Omega)} + \epsilon\|\partial_t u\|^2_{L^2(\Omega)} \\
&\quad + \big(1+\epsilon^{-1}+\epsilon^{-3}\big)\Big(1 + \|\partial_t\bar{v}\|^2_{L^2(\Omega)}\|\bar{v}\|^2_{H^1_0(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} \\
&\quad + \|u\|^2_{H^1_0(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)} + \|w\|^2_{H^1_0(\Omega)}\Big)^3\Big]
\end{aligned}
\tag{2.25}
$$

where $C = C(Q, B, B', \Omega)$.

4. Following steps 1-3 in respect of (1.1), (1.2), (1.4) and the corresponding conditions in (1.5), we obtain analogous estimates to (2.25):

$$
\begin{aligned}
&\frac{d}{dt}(\|\partial_t\bar{v}\|^2_{L^2(\Omega)} + \nu\|\bar{v}\|^2_{H^1_0(\Omega)}) + 2\nu\|\nabla(\partial_t\bar{v})\|^2_{L^2(\Omega)} \\
&\le C(\Omega)\Big[\epsilon\|\nabla(\partial_t\bar{v})\|^2_{L^2(\Omega)} + (\epsilon^{-1}+\epsilon^{-3})\Big(\|\partial_t\bar{v}\|^2_{L^2(\Omega)} + \|\bar{v}\|^2_{H^1_0(\Omega)}\Big)\Big],
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
&\frac{d}{dt}(\|\partial_t w\|^2_{L^2(\Omega)} + d\|w\|^2_{H^1_0(\Omega)}) + 2d\|\nabla(\partial_t w)\|^2_{L^2(\Omega)} \\
&\le C\Big[\epsilon\|\nabla(\partial_t w)\|^2_{L^2(\Omega)} + \epsilon\|\nabla(\partial_t\bar{v})\|^2_{L^2(\Omega)} + \epsilon\|\partial_t u\|^2_{L^2(\Omega)} \\
&\quad + (1+\epsilon^{-1}+\epsilon^{-3})\Big(1 + \|\partial_t\bar{v}\|^2_{L^2(\Omega)} + \|\bar{v}\|^2_{H^1_0(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} \\
&\quad + \|u\|^2_{H^1_0(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)} + \|w\|^2_{H^1_0(\Omega)}\Big)^3\Big],
\end{aligned}
\tag{2.27}
$$

where $C = C(B, B', \Omega)$.

5. Combining (2.25)-(2.27), choosing $\epsilon > 0$ sufficiently small and simplifying, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\partial_t\bar{v}\|^2_{L^2(\Omega)} + \|\bar{v}\|^2_{H^1_0(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} + \|u\|^2_{H^1_0(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)} \\
&+ \|w\|^2_{H^1_0(\Omega)}\Big) + \|\nabla(\partial_t\bar{v})\|^2_{L^2(\Omega)} + \|\nabla(\partial_t u)\|^2_{L^2(\Omega)} + \|\nabla(\partial_t w)\|^2_{L^2(\Omega)} \\
&\le C(\nu, k, d, Q, B, B', \Omega)\Big(1 + \|\partial_t\bar{v}\|^2_{L^2(\Omega)} + \|\bar{v}\|^2_{H^1_0(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)}
\end{aligned}
$$

$$+ \|u\|^2_{H^1_0(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)} + \|w\|^2_{H^1_0(\Omega)}\Big)^3$$

Solving the above equation, maximizing the left and right sides of the ensuing inequalities and using Lemma 2.3 concludes the proof of the Theorem 2.1. □

## 3. Existence of a Solution

We prove the existence of a unique local strong solution to the system (1.1)-(1.6), in a subset $K$ of the space $X^3$ equipped with the norm

$$\begin{aligned}
&\|(\eta,\xi,\zeta)\|_{X^3}\\
&\le \Big[\|\eta\|^2_{L^\infty[0,T;H^1_0(\Omega)]} + \|\partial_t\eta\|^2_{L^\infty[0,T;L^2(\Omega)]} + \|\xi\|^2_{L^\infty[0,T;H^1_0(\Omega)]}\\
&\quad + \|\partial_t\xi\|^2_{L^\infty[0,T;L^2(\Omega)]} + \|\zeta\|^2_{L^\infty[0,T;H^1_0(\Omega)]} + \|\partial_t\zeta\|^2_{L^\infty[0,T;L^2(\Omega)]}\\
&\quad + \|\nabla(\partial_t\eta)\|^2_{L^2[0,T;L^2(\Omega)]} + \|\nabla(\partial_t\xi)\|^2_{L^2[0,T;L^2(\Omega)]} + \|\nabla(\partial_t\zeta)\|^2_{L^2[0,T;L^2(\Omega)]}\Big]^{\frac12},
\end{aligned} \tag{3.1}$$

where $X$ is defined by (1.9).

**Theorem 3.1.** *Let $\bar v_0, u_0$ and $w_0 \in H^1_0(\Omega)\cap H^2(\Omega)$. Then there exists a unique local strong solution to the system* (1.1)-(1.6).

*Proof.* 1. The fixed point arguments for the system (1.1)-(1.6) are

$$\nabla.\bar Q = 0 \quad \text{in } \Omega_T \tag{3.2}$$

$$\frac{\partial \bar Q}{\partial t} - \nu\Delta\bar Q = -\nabla.(\bar v\otimes\bar v) - \frac{1}{\rho}\nabla Y \quad \text{in } \Omega_T \tag{3.3}$$

$$\frac{\partial R}{\partial t} - k\Delta R = -\nabla.(\bar v u) + Qwf(u) \quad \text{in } \Omega_T \tag{3.4}$$

$$\frac{\partial S}{\partial t} - d\Delta S = -\nabla.(\bar v w) - wf(u) \quad \text{in } \Omega_T \tag{3.5}$$

$$\bar Q = \bar 0, \quad R = S = 0 \quad \text{on } \partial\Omega\times[0,T) \tag{3.6}$$

$$\bar Q(x,0) = \bar{v_0}(x), \quad R(x,0) = u_0(x), \quad S(x,0) = w_0(x), \tag{3.7}$$

where $Y$ is the pressure distribution corresponding to the solution $(\bar Q, R, S)$.

2. We next define a mapping

$$\tau : X^3 \to X^3 \tag{3.8}$$

by setting $\tau[(\bar v,u,w)] = (\bar Q,R,S)$, whenever $(\bar Q,R,S)$ is derived from $(\bar v,u,w)$ via (3.2)-(3.7). We will prove that for sufficiently small $T>0$, $\tau$ is a contraction mapping. Choose $(\bar v,u,w),(\tilde{\bar v},\tilde u,\tilde w)\in X^3$ and define

$$\tau[(\bar v,u,w)] = (Q,R,S), \quad \tau[(\tilde{\bar v},\tilde u,\tilde w)] = (\tilde{\bar Q},\tilde R,\tilde S).$$

Thus, for two solutions $(Q,R,S)$, and $(\tilde{\bar Q},\tilde R,\tilde S)$ of the system (3.2)-(3.7), we have

$$\nabla.(\bar Q - \tilde{\bar Q}) = 0 \quad \text{in } \Omega_T \tag{3.9}$$

$$\frac{\partial}{\partial t}(\bar Q-\tilde{\bar Q}) - \nu\Delta(\bar Q-\tilde{\bar Q}) = -\nabla.(\bar v\otimes\bar v - \tilde{\bar v}\otimes\tilde{\bar v}) - \frac{1}{\rho}\nabla(Y-\tilde Y) \quad \text{in } \Omega_T \tag{3.10}$$

$$\frac{\partial}{\partial t}(R-\tilde R) - k\Delta(R-\tilde R) = -\nabla.(\bar v u - \tilde{\bar v}\tilde u) + Q(wf(u)-\tilde w f(\tilde u)) \quad \text{in } \Omega_T \tag{3.11}$$

$$\frac{\partial}{\partial t}(S-\tilde{S}) - d\Delta(S-\tilde{S}) = -\nabla.(\bar{v}w - \tilde{\bar{v}}\tilde{w}) - (wf(u) - \tilde{w}f(\tilde{u})) \quad \text{in } \Omega_T \tag{3.12}$$

$$\bar{Q} - \tilde{\bar{Q}} = \bar{0}, \quad R - \tilde{R} = S - \tilde{S} = 0 \quad \text{on } \partial\Omega \times [0,T) \tag{3.13}$$

$$(\bar{Q} - \tilde{\bar{Q}})(x,0) = \bar{0}, \quad (R-\tilde{R})(x,0) = 0, \quad (S-\tilde{S})(x,0) = 0 \tag{3.14}$$

3. Multiplying (3.11) by $\partial_t(R-\tilde{R})$, integrating the ensuing equation by parts over $\Omega$, using (3.13) and applying (2.5) of Lemma 2.2, we deduce

$$\begin{aligned}
&\|\partial_t(R-\tilde{R})\|^2_{L^2(\Omega)} + \frac{k}{2}\frac{d}{dt}\Big(\|\nabla(\partial_t(R-\tilde{R}))\|^2_{L^2(\Omega)}\Big) \\
&= \int_\Omega \nabla\Big(\partial_t(R-\tilde{R})\Big).(\bar{v}u - \tilde{\bar{v}}\tilde{u})dx + Q\int_\Omega \partial_t(R-\tilde{R})(wf(u) - \tilde{w}f(\tilde{u}))dx \\
&\le \epsilon\|\nabla(\partial_t(R-\tilde{R}))\|^2_{L^2(\Omega)} + \epsilon\|\partial_t(R-\tilde{R})\|^2_{L^2(\Omega)} \\
&\quad + C(\Omega)\epsilon^{-1}\Big(\|\bar{v}\|^2_{H_0^1(\Omega)}\|u-\tilde{u}\|^2_{H_0^1(\Omega)} + \|\tilde{u}\|^2_{H_0^1(\Omega)}\|v - \tilde{\bar{v}}\|^2_{H_0^1(\Omega)}\Big) \\
&\quad + C(Q,B,B',\Omega)\epsilon^{-1}\Big(\|w\|^2_{H_0^1(\Omega)}\|u-\tilde{u}\|^2_{H_0^1(\Omega)} + \|w-\tilde{w}\|^2_{H_0^1(\Omega)}\Big),
\end{aligned} \tag{3.15}$$

where we have used some bounds in (1.7) and (1.8).

4. Further, we differentiate (3.11) with respect $t$ to get

$$\begin{aligned}
&\frac{\partial}{\partial t}(\partial_t(R-\tilde{R})) - k\Delta(\partial_t(R-\tilde{R})) \\
&= -\nabla.\left(\partial_t\bar{v}u - \partial_t\tilde{\bar{v}}\tilde{u} + \bar{v}\partial_t u - \tilde{\bar{v}}\partial_t\tilde{u}\right) Q\big(\partial_t wf(u) \\
&\quad - \partial_t\tilde{w}f(\tilde{u}) + w\partial_t uf'(u) - \tilde{w}\partial_t\tilde{u}f'(\tilde{u})\big)
\end{aligned} \tag{3.16}$$

Multiplying (3.16) by $\partial_t(R-\tilde{R})$, integrating by parts over $\Omega$, and applying Young's inequality with $\epsilon$, (2.6) and (2.8) as appropriate, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial_t(R-\tilde{R})\|^2_{L^2(\Omega)}) + k\|\nabla(\partial_t(R-\tilde{R}))\|^2_{L^2(\Omega)} \\
&= \int_\Omega(\partial_t\bar{v}u - \partial_t\tilde{\bar{v}}\tilde{u} + \bar{v}\partial_t u - \tilde{\bar{v}}\partial_t\tilde{u}).\nabla(\partial_t(R-\tilde{R}))dx \\
&\quad + Q\int_\Omega \partial_t(R-\tilde{R}).(\partial_t wf(u) - \partial_t\tilde{w}f(\tilde{u}) + w\partial_t uf'(u) - \tilde{w}\partial_t\tilde{u}f'(\tilde{u}))dx \\
&\le 3\epsilon\|\nabla(\partial_t(R-\tilde{R}))\|^2_{L^2(\Omega)} + \Big(2\epsilon + C(\Omega)\epsilon^{-1}\|w\|^4_{H_0^1(\Omega)}\Big)\|\partial_t(R-\tilde{R})\|^2_{L^2(\Omega)} \\
&\quad + C(\Omega,B,B',Q,L)\epsilon^{-1}\Big\{\Big[T^{-1/2}\Big(\|\partial_t\tilde{\bar{v}}\|^2_{L^2(\Omega)} + \|\partial_t\tilde{u}\|^2_{L^2(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)} \\
&\quad + \|\partial_t u\|^2_{L^2(\Omega)}\Big) + T^{1/2}\Big(\|\nabla(\partial_t\tilde{\bar{v}})\|^2_{L^2(\Omega)} + \|\nabla(\partial_t\tilde{u})\|^2_{L^2(\Omega)} + \|\nabla(\partial_t w)\|^2_{L^2(\Omega)} \\
&\quad + \|\nabla(\partial_t u)\|^2_{L^2(\Omega)}\Big)\Big]\Big(\|u-\tilde{u}\|^2_{H_0^1(\Omega)} + \|\bar{v}-\tilde{\bar{v}}\|^2_{H_0^1(\Omega)} + \|w-\tilde{w}\|^2_{H_0^1(\Omega)}\Big) \\
&\quad + \Big[T^{-1/2}\Big(\|\partial_t(\bar{v}-\tilde{\bar{v}})\|^2_{L^2(\Omega)} + \|\partial_t(u-\tilde{u})\|^2_{L^2(\Omega)} + \|\partial_t(w-\tilde{w})\|^2_{L^2(\Omega)}\Big) \\
&\quad + T^{1/2}\Big(\|\nabla(\partial_t(\bar{v}-\tilde{\bar{v}}))\|^2_{L^2(\Omega)} + \|\nabla(\partial_t(u-\tilde{u}))\|^2_{L^2(\Omega)} \\
&\quad + \|\nabla(\partial_t(w-\tilde{w}))\|^2_{L^2(\Omega)}\Big)\Big]\Big(1 + \|u\|^2_{H_0^1(\Omega)} + \|\bar{v}\|^2_{H_0^1(\Omega)} + \|\tilde{w}\|^2_{H_0^1(\Omega)}\Big)\Big\}.
\end{aligned}$$

5. Combining the above inequality with (3.15), Choosing $\epsilon > 0$ sufficiently small, and simplifying, we deduce

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\partial_t(R-\tilde{R})\|^2_{L^2(\Omega)} + \|R-\tilde{R}\|^2_{H^1_0(\Omega)}\Big) + \|\nabla(\partial_t(R-\tilde{R}))\|^2_{L^2(\Omega)}\\
&\le C\Big\{(1+\|w\|^2_{H^1_0(\Omega)})^2\Big(\|\partial_t(R-\tilde{R})\|^2_{L^2(\Omega)} + \|R-\tilde{R}\|^2_{H^1_0(\Omega)}\Big) + \Big[1+\|\bar{v}\|^2_{H^1_0(\Omega)}\\
&\quad + \|\tilde{u}\|^2_{H^1_0(\Omega)} + \|w\|^2_{H^1_0(\Omega)} + T^{-1/2}\Big(\|\partial_t\tilde{\bar{v}}\|^2_{L^2(\Omega)} + \|\partial_t\tilde{u}\|^2_{L^2(\Omega)} + \|\partial_t w\|^2_{L^2(\Omega)}\\
&\quad + \|\partial_t u\|^2_{L^2(\Omega)}\Big) + T^{1/2}\Big(\|\nabla(\partial_t\tilde{\bar{v}})\|^2_{L^2(\Omega)} + \|\nabla(\partial_t\tilde{u})\|^2_{L^2(\Omega)} + \|\nabla(\partial_t w))\|^2_{L^2(\Omega)}\\
&\quad + \|\nabla(\partial_t u)\|^2_{L^2(\Omega)}\Big)\Big]\Big(\|u-\tilde{u}\|^2_{H^1_0(\Omega)} + \|\bar{v}-\tilde{\bar{v}}\|^2_{H^1_0(\Omega)} + \|w-\tilde{w}\|^2_{H^1_0(\Omega)}\Big)\\
&\quad + \Big[T^{-1/2}\Big(\|\partial_t(\bar{v}-\tilde{\bar{v}})\|^2_{L^2(\Omega)} + \|\partial_t(u-\tilde{u})\|^2_{L^2(\Omega)} + \|\partial_t(w-\tilde{w})\|^2_{L^2(\Omega)}\Big)\\
&\quad + T^{1/2}\Big(\|\nabla(\partial_t(\bar{v}-\tilde{\bar{v}}))\|^2_{L^2(\Omega)} + \|\nabla(\partial_t(u-\tilde{u}))\|^2_{L^2(\Omega)}\\
&\quad + \|\nabla(\partial_t(w-\tilde{w}))\|^2_{L^2(\Omega)}\Big)\Big]\Big(1+\|u\|^2_{H^1_0(\Omega)} + \|\bar{v}\|^2_{H^1_0(\Omega)} + \|\tilde{w}\|^2_{H^1_0(\Omega)}\Big)\Big\},
\end{aligned} \tag{3.17}$$

where $C = C(k, \Omega, B, B', Q, L)$.

6. There exist analogous estimates to (3.17) for $\bar{Q} - \tilde{\bar{Q}}$ and $S - \tilde{S}$, which for brevity, we do not render here. If we combine these estimates with (3.17), we deduce, after an application of the differential form of the Gronwall's inequality, the estimates:

$$\begin{aligned}
&\|(\bar{Q}, R, S) - (\tilde{\bar{Q}}, \tilde{R}, \tilde{S})\|_{X^3}\\
&= \|\tau[(\bar{v}, u, w)] - \tau[(\tilde{\bar{v}}, \tilde{u}, \tilde{w})]\|_{X^3}\\
&\le C\Big(T + T^{1/2}\Big)^{1/2}\exp\Big[2^{-1}T(1+\|w\|^2_{H^1_0(\Omega)})\Big]\\
&\quad\times\Big(1+\|(\bar{v}, u, w)\|^2_{X^3} + \|(\tilde{\bar{v}}, \tilde{u}, \tilde{w})\|^2_{X^3}\Big)^{1/2}\|(\bar{v}, u, w) - (\tilde{\bar{v}}, \tilde{u}, \tilde{w})\|_{X^3}
\end{aligned} \tag{3.18}$$

where $C = C(Q, B, B', \Omega, k, d, \nu, L)$.

7. Notice that the bound in (3.18) is not uniform. Thus we need to prove the existence of a unique solution in a subset of $X^3$. Define a convex set

$$K := \{(\bar{v}, u, w) | (\bar{v}, u, w) - (\bar{v}_0, u_0, w_0) \in X^3_0 \text{ and } \|(\bar{v}, u, w)\|_{X^3} \le 2\sqrt{\Sigma}\}, \tag{3.19}$$

where $X^3_0$ is the set where the initial and the boundary values are zero; and $\Sigma =$ constant is the bound in (2.19). We will show that, if $T > 0$ is sufficiently small, then

$$\tau[K] \subseteq K, \quad \|\tau[(\bar{v}, u, w)] - \tau[(\tilde{\bar{v}}, \tilde{u}, \tilde{w})]\|_{X^3} \le \gamma\|(\bar{v}, u, w) - (\tilde{\bar{v}}, \tilde{u}, \tilde{w})\|_{X^3} \tag{3.20}$$

for all $(\bar{v}, u, w), (\tilde{\bar{v}}, \tilde{u}, \tilde{w}) \in K$ and some $\gamma < 1$. Using (2.19) and (3.7), we have

$$\|\tau[(\bar{v}_0, u_0, w_0)]\|_{X^3} = \|(\bar{Q}(x, 0), R(x, 0), S(x, 0))\|_{X^3} = \|(\bar{v}_0, u_0, w_0)\|_{X^3} \le \sqrt{\Sigma} \tag{3.21}$$

Therefore, for $(\bar{v}, u, w) \in K$, using (3.18) and (3.21),

$$\begin{aligned} &\|\tau[(\bar{v}, u, w)]\|_{X^3} \\ &\le \|\tau[(\bar{v}_0, u_0, w_0)]\|_{X^3} + \|\tau[(\bar{v}, u, w)] - \tau[(\bar{v}_0, u_0, w_0)]\|_{X^3} \\ &\le \sqrt{\Sigma} + C\left(T + T^{1/2}\right)^{1/2} \exp\left[2^{-1}T(1 + \|w\|^2_{H^1_0(\Omega)})\right] \\ &\quad \times \left(1 + \|(\bar{v}, u, w)\|^2_{X^3} + \|(\bar{v}_0, u_0, w_0)\|^2_{X^3}\right)^{1/2} \|(\bar{v}, u, w) - (\bar{v}_0, u_0, w_0)\|_{X^3} \\ &\le \sqrt{\Sigma} + C\left(T + T^{1/2}\right)^{1/2} \exp\left[2^{-1}T(1 + 4\Sigma)\right](1 + 5\Sigma)^{1/2}(4\sqrt{\Sigma}) \\ &\le 2\sqrt{\Sigma}, \end{aligned} \tag{3.22}$$

for $T > 0$ sufficiently small such that

$$4C\big(T + T^{1/2}\big)^{1/2} \exp[2^{-1}T(1 + 4\Sigma)](1 + 5\Sigma)^{1/2} \le 1 \tag{3.23}$$

Thus $\tau[(\bar{v}, u, w)] \in K$, and hence $\tau(K) \subseteq K$ for $T > 0$ sufficiently small. Furthermore, if $T$ is chosen sufficiently small such that

$$C\big(T + T^{1/2}\big)^{1/2} \exp[2^{-1}T(1 + 4\Sigma)](1 + 5\Sigma)^{1/2} = \gamma < 1, \tag{3.24}$$

then, (3.18) implies

$$\|\tau[(\bar{v}, u, w)] - \tau[(\tilde{\bar{v}}, \tilde{u}, \tilde{w})]\|_{X^3} < \gamma\|(\bar{v}, u, w) - (\tilde{\bar{v}}, \tilde{u}, \tilde{w})\|_{X^3} \tag{3.25}$$

for all $(\bar{v}, u, w)$, $(\tilde{\bar{v}}, \tilde{u}, \tilde{w}) \in K$. Thus, the mapping $\tau$ is a strict contraction for sufficiently small $T > 0$.

8. Given $(\bar{v}_k, u_k, w_k)$ $(k = 0, 1, 2, \dots)$, inductively define

$$(Q, R, S) := (\bar{v}_{k+1}, u_{k+1}, w_{k+1}) \in K$$

to be the unique weak solution of the linear initial boundary value problem

$$\nabla.\bar{v}_{k+1} = 0 \quad \text{in } \Omega_T \tag{3.26}$$

$$\frac{\partial \bar{v}_{k+1}}{\partial t} - \nu\Delta\bar{v}_{k+1} = -\nabla.(\bar{v}_k \otimes \bar{v}_k) - \frac{1}{\rho}\nabla p_{k+1} \quad \text{in } \Omega_T \tag{3.27}$$

$$\frac{\partial u_{k+1}}{\partial t} - k\Delta u_{k+1} = -\nabla.(\bar{v}_k u_k) + Q w_k f(u_k) \quad \text{in } \Omega_T \tag{3.28}$$

$$\frac{\partial w_{k+1}}{\partial t} - d\Delta w_{k+1} = -\nabla.(\bar{v}_k w_k) - w_k f(u_k) \quad \text{in } \Omega_T \tag{3.29}$$

$$\bar{v}_{k+1} = \bar{0}, \quad u_{k+1} = w_{k+1} = 0 \quad \text{on } \partial\Omega \times [0, T) \tag{3.30}$$

$$\bar{v}_{k+1}(x, 0) = \bar{v}_0(x), \quad u_{k+1}(x, 0) = u_0(x), \quad w_{k+1}(x, 0) = w_0(x), \tag{3.31}$$

where $Y := p_{k+1}$ is the pressure distribution corresponding to $(\bar{v}_{k+1}, u_{k+1}, w_{k+1})$.

By the definition of the mapping $\tau$, we have (for $k = 0, 1, 2, \dots$), using (3.26)-(3.31) that

$$(\bar{v}_{k+1}, u_{k+1}, w_{k+1}) = \tau[(\bar{v}_k, u_k, w_k)]. \tag{3.32}$$

Consider the series

$$(\bar{v}_1, u_1, w_1) + \sum_{r \ge 2}[(\bar{v}_r, u_r, w_r) - (\bar{v}_{r-1}, u_{r-1}, w_{r-1})] \tag{3.33}$$

The partial sum of the first $k+1$ terms of the series (3.33) is

$$(\bar{v}_1, u_1, w_1) + \sum_{r=2}^{k+1} [(\bar{v}_r, u_r, w_r) - (\bar{v}_{r-1}, u_{r-1}, w_{r-1})] = (\bar{v}_{k+1}, u_{k+1}, w_{k+1}) \tag{3.34}$$

Now, using (3.25), we have

$$\begin{aligned} \|(\bar{v}_2, u_2, w_2) - (\bar{v}_1, u_1, w_1)\|_{X^3} &= \|\tau[(\bar{v}_1, u_1, w_1)] - \tau[(\bar{v}_0, u_0, w_0)]\|_{X^3} \\ &< \gamma \|(\bar{v}_1, u_1, w_1) - (\bar{v}_0, u_0, w_0)\|_{X^3} \end{aligned} \tag{3.35}$$

$$\begin{aligned} \|(\bar{v}_3, u_3, w_3) - (\bar{v}_2, u_2, w_2)\|_{X^3} &= \|\tau[(\bar{v}_2, u_2, w_2)] - \tau[(\bar{v}_1, u_1, w_1)]\|_{X^3} \\ &< \gamma^2 \|(\bar{v}_1, u_1, w_1) - (\bar{v}_0, u_0, w_0)\|_{X^3} \end{aligned} \tag{3.36}$$

By induction,

$$\begin{aligned} \|(\bar{v}_{k+1}, u_{k+1}, w_{k+1}) - (\bar{v}_k, u_k, w_k)\|_{X^3} &< \gamma^k \|(\bar{v}_1, u_1, w_1) - (\bar{v}_0, u_0, w_0)\|_{X^3} \\ &< 4\gamma^k \sqrt{\Sigma}, \end{aligned} \tag{3.37}$$

since $(\bar{v}_1, u_1, w_1)$, $(\bar{v}_0, u_0, w_0)$ are in $K$, defined by (3.19). Hence the series (3.33) is absolutely convergent, since using (3.37), the series

$$\sum_{k=0} 4\gamma^k \sqrt{\Sigma}, \tag{3.38}$$

which converges, dominates

$$\|(\bar{v}_1, u_1, w_1)\|_{X^3} + \sum_{r \geq 2} \|(\bar{v}_r, u_r, w_r) - (\bar{v}_{r-1}, u_{r-1}, w_{r-1})\|_{X^3}. \tag{3.39}$$

This implies that the series (3.33) is convergent. Define

$$\lim_{k\to\infty} (\bar{v}_{k+1}, u_{k+1}, w_{k+1}) := (\bar{v}, u, w).$$

Thus $(\bar{v}_{k+1}, u_{k+1}, w_{k+1}) \to (\bar{v}, u, w)$ uniformly in $K$. Thus

$$\lim_{k\to\infty} (\bar{v}_{k+1}, u_{k+1}, w_{k+1}) = (\bar{v}, u, w) = \lim_{k\to\infty} \tau[(\bar{v}_k, u_k, w_k)] = \tau[(\bar{v}, u, w)] \tag{3.40}$$

By (3.40), $(\bar{v}, u, w) \in K$ is the unique fixed point of $\tau$.

9. As in [15], define

$$V := \text{The closure of } \{\bar{\zeta} \in C_c^\infty(\Omega) : \nabla.\bar{\zeta} = 0\} \text{ in } H_0^1(\Omega). \tag{3.41}$$

In view of the previous steps of this section, we are motivated to give the following definition.

**Definition 3.2.** The weak formulation of (1.1)-(1.6) is: For given $(\bar{v}_0, u_0, w_0) \in [H_0^1(\Omega) \cap H^2(\Omega)]^3$, find $(\bar{v}, u, w) \in K$ satisfying

$$\int_\Omega \partial_t \bar{v}.\bar{\zeta} dx + \nu \int_\Omega \nabla \bar{v} : \nabla \bar{\zeta} dx = -\int_\Omega \nabla.(\bar{v} \otimes \bar{v}).\bar{\zeta} dx \tag{3.42}$$

$$\int_\Omega \partial_t u \xi dx + k \int_\Omega \nabla u.\nabla \xi dx = -\int_\Omega \nabla.(\bar{v} u) \xi dx + Q \int_\Omega w f(u) \xi dx \tag{3.43}$$

$$\int_\Omega \partial_t w \xi dx + d \int_\Omega \nabla w.\nabla \xi dx = -\int_\Omega \nabla.(\bar{v} w) \xi dx - \int_\Omega w f(u) \xi dx \tag{3.44}$$

$$\bar{v}(x,0) = \bar{v}_0(x), \quad u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \tag{3.45}$$

for each $\zeta \in V$ and each $\xi \in H_0^1(\Omega)$.

10. Before verifying that $(\bar{v}, u, w)$ is weak solution of (1.1)-(1.6), we first prove the following Lemma.

**Lemma 3.3.** *If* $(\bar{v}_k, u_k, w_k) \in K$, $\xi \in H_0^1(\Omega)$ *and* $\bar{\zeta} \in V$, *then*

$$\int_\Omega \nabla.(\bar{v}_k \otimes \bar{v}_k).\bar{\zeta} dx \to \int_\Omega \nabla.(\bar{v} \otimes \bar{v}).\bar{\zeta} dx \tag{3.46}$$

$$\int_\Omega \nabla.(\bar{v}_k u_k)\xi dx \to \int_\Omega \nabla.(\bar{v}u)\xi dx \tag{3.47}$$

$$\int_\Omega \nabla.(\bar{v}_k w_k)\xi dx \to \int_\Omega \nabla.(\bar{v}w)\xi dx \tag{3.48}$$

$$f(u_k) \to f(u) \quad \text{in } L^2(\Omega) \tag{3.49}$$

$$\int_\Omega w_k f(u_k)\xi dx \to \int_\Omega w f(u)\xi dx \tag{3.50}$$

*Proof.* (i). Proof of (3.46). Integrating by parts, we have

$$|\int_\Omega \nabla.(\bar{v}_k \otimes \bar{v}_k).\bar{\zeta} dx| = |\int_\Omega \bar{v}_k \otimes \bar{v}_k : \nabla\bar{\zeta} dx| \le \|\zeta\|_{H_0^1(\Omega)} \|u_k\|^2_{H_0^1(\Omega)}, \tag{3.51}$$

by using (2.9) of Lemma 2.2. Equation (3.46) follows by taking limits on both sides of (3.51). Further, the proofs of (3.47) and (3.48) follow by similar calculations.

(ii). Proof of (3.49). We have

$$\begin{aligned} &\int_\Omega |f(u_k)|^2 dx = \int_\Omega \Big| \int_0^{u_k} f'(r)dr + f(0)\Big|^2 dx \le \int_\Omega \big|B'|u_k| + f(0)\big|^2 dx \\ &\le C_1(B', f(0)) \int_\Omega (|u_k|^2 + 2|u_k| + 1|)dx \le C_2(B', f(0), \Omega)(\|u_k\|_{L^2(\Omega)} + 1)^2 \end{aligned} \tag{3.52}$$

where we have used the first inequality in (1.8) and the estimate $\int_\Omega |u_k| dx \le |\Omega|^{\frac{1}{2}} \|u_k\|_{L^2(\Omega)})$. Then (3.49) follows by taking limits on both sides of (3.52).

(iii). Proof of (3.50). We estimate

$$|\int_\Omega w_k f(u_k)\xi dx| \le \|w_k\|_{H_0^1(\Omega)} \|f(u_k)\|_{L^2(\Omega)} \|\xi\|_{H_0^1(\Omega)}, \tag{3.53}$$

using (2.9) of Lemma 2.2. Hence, (3.50) follows by taking limits in (3.53).

11. We now verify that $(\bar{v}, u, w) \in K$ is a weak solution of (1.1). Fix $\zeta \in V$ and $\xi \in H_0^1(\Omega)$. Using (3.26)-(3.31), we have

$$\int_\Omega \partial_t \bar{v}_{k+1}.\bar{\zeta} dx + \nu \int_\Omega \nabla \bar{v}_{k+1} : \nabla\bar{\zeta} dx = -\int_\Omega \nabla.(\bar{v}_k \otimes \bar{v}_k).\bar{\zeta} dx \tag{3.54}$$

$$\begin{aligned} &\int_\Omega \partial_t u_{k+1}\xi dx + k\int_\Omega \nabla u_{k+1}.\nabla\xi dx \\ &= -\int_\Omega \nabla.(\bar{v}_k u_k)\xi dx + Q\int_\Omega w_k f(u_k)\xi dx \end{aligned} \tag{3.55}$$

$$\begin{aligned} &\int_\Omega \partial_t w_{k+1}\xi dx + d\int_\Omega \nabla w_{k+1}.\nabla\xi dx \\ &= -\int_\Omega \nabla.(\bar{v}_k w_k)\xi dx - \int_\Omega w_k f(u_k)\xi dx \end{aligned} \tag{3.56}$$

$$\bar{v}_{k+1}(x,0) = \bar{v}_0(x), \quad u_{k+1}(x,0) = u_0(x), \quad w_{k+1}(x,0) = w_0(x)\,. \tag{3.57}$$

Letting $k \to \infty$ in (3.54)-(3.57) and using Lemma 2.3 to handle the nonlinear terms yield (3.42)-(3.45) as desired. □

12. We next demonstrate how to obtain the pressure $Y = p_{k+1}$. First, we obtain the boundary condition on pressure by taking (1.2) on the boundary and using (1.5) to deduce

$$\frac{1}{\rho}\nabla p = \nu\Delta\bar{v} \quad \text{on } \partial\Omega \tag{3.58}$$

Following the steps in [12], we express (3.58) in terms of the standard normal derivatives as

$$\frac{1}{\rho}\frac{\partial p}{\partial n} = \nu\hat{n}.\frac{\partial^2\bar{v}}{\partial n^2} \tag{3.59}$$

where $\hat{n}(x)$ is the inward normal at $x$ on $\partial\Omega$.

Taking the divergence of (3.27) yields the equation satisfied by $Y = p_{k+1}$ as

$$\Delta p_{k+1} = \rho\nabla.[\nabla.(\bar{v}_k \otimes \bar{v}_k)], \tag{3.60}$$

which is a form of Poisson's equation. Further, in sympathy with the boundary condition (3.59), we impose the the boundary condition on the pressure $p_{k+1}$ as

$$\frac{1}{\rho}\frac{\partial p_{k+1}}{\partial n} = \nu\hat{n}.\frac{\partial^2\bar{v}}{\partial n^2} \tag{3.61}$$

Hence, the formal solution of (3.60) subject to the condition (3.61) is

$$\begin{aligned} p_{k+1}(x,t) = &-\rho\int_\Omega G(x,y)\nabla.[\nabla.(\bar{v}_k(y,t)\otimes\bar{v}_k(y,t))]dy \\ &+ \rho\nu\int_{\partial\Omega} G(x,y)\hat{n}.\frac{\partial^2\bar{v}(y,t)}{\partial n^2}dS(y) \end{aligned} \tag{3.62}$$

where $G(x,y)$ is the Green's function satisfying the Laplace's equation in the form

$$\Delta G(x,y) = \delta(x-y) \tag{3.63}$$

with the condition

$$\frac{\partial G(x,y)}{\partial n} = 0 \quad (x \text{ on } \partial\Omega). \tag{3.64}$$

where $\delta$ is the Dirac delta function.

For $n = 3$, $G(x,y) = \frac{1}{|x-y|}$, $x, y \in \Omega$, we define

$$\begin{aligned} p^\epsilon_{k+1}(x,t) := &-\rho\int_\Omega \frac{1}{|x-y|+\epsilon}\nabla.[\nabla.(\bar{v}_k(y,t)\otimes\bar{v}_k(y,t))]dy \\ &+ \rho\nu\int_{\partial\Omega}\frac{1}{|x-y|+\epsilon}\hat{n}.\frac{\partial^2\bar{v}(y,t)}{\partial n^2}dS(y), \end{aligned} \tag{3.65}$$

$$\begin{aligned} p^\epsilon(x,t) := &-\rho\int_\Omega \frac{1}{|x-y|+\epsilon}\nabla.[\nabla.(\bar{v}(y,t)\otimes\bar{v}(y,t))]dy \\ &+ \rho\nu\int_{\partial\Omega}\frac{1}{|x-y|+\epsilon}\hat{n}.\frac{\partial^2\bar{v}(y,t)}{\partial n^2}dS(y) \end{aligned} \tag{3.66}$$

$$\begin{aligned} p(x,t) := &-\rho\int_\Omega \frac{1}{|x-y|}\nabla.[\nabla.(\bar{v}(y,t)\otimes\bar{v}(y,t))]dy \\ &+ \rho\nu\int_{\partial\Omega}\frac{1}{|x-y|}\hat{n}.\frac{\partial^2\bar{v}(y,t)}{\partial n^2}dS(y) \end{aligned} \tag{3.67}$$

where $\epsilon > 0$. Notice that

$$\lim_{\epsilon\to 0} p^{\epsilon}_{k+1}(x,t) = p_{k+1}(x,t), \quad \lim_{\epsilon\to 0} p^{\epsilon}(x,t) = p(x,t)$$

Hence, integrating twice by parts, using (1.1) and (1.5), we have

$$\begin{aligned} &|p^{\epsilon}_{k+1}(x,t) - p^{\epsilon}(x,t)| \\ &= |\rho \int_{\Omega} \frac{1}{|x-y|+\epsilon} \nabla.\{\nabla.[\bar{v}_k(y,t)\otimes \bar{v}_k(y,t) - \bar{v}(y,t)\otimes \bar{v}(y,t)]\}dy| \\ &= |\rho \int_{\Omega} \nabla\big\{\nabla[\frac{1}{|x-y|+\epsilon}]\big\} : [\bar{v}_k(y,t)\otimes \bar{v}_k(y,t) - \bar{v}(y,t)\otimes \bar{v}(y,t)]dy| \end{aligned} \tag{3.68}$$

which tends to 0 as $k \to \infty$. Therefore

$$\lim_{k\to\infty} p^{\epsilon}_{k+1}(x,t) = p^{\epsilon}(x,t), \tag{3.69}$$

From whence sending $\epsilon$ to 0, we obtain

$$\lim_{k\to \infty} p_{k+1}(x,t) = p(x,t), \tag{3.70}$$

where, $p(x,t)$ given by (3.67), is the pressure corresponding to the solution $(\bar{v}, u, w)$. Indeed, (3.67) is the formal solution for the pressure $p(x,t)$ satisfying

$$\Delta p = \rho\nabla.[\nabla.(\bar{v}\otimes\bar{v})], \tag{3.71}$$

in terms of $G(x,y)$, as obtained in [12]. □

## 4. Regularity

The Analysis so far carried out requires no smoothness assumption on the boundary. However, for smooth solution up to the boundary, one requires the boundary $\partial\Omega$ to be $C^{\infty}$. The lengthy proofs of the associated regularity theorems are currently being established by an analysis of certain difference quotients in another paper.

**Acknowledgments.** The author would like to thank the anonymous referee whose thoughtful comments improved the original version of this manuscript.

## References

[1] Avrin, J. D.; *Asymptotic behavior of one-step combustion models with multiple reactants on bounded domains*, SIAM J. Math. Anal., 24 (1993), 290-298.
[2] Avrin, J. D.; *Qualitative theory of the Cauchy problem for a one-step reaction model on bounded domains*, SIAM J. Math. Anal., 22 (1991), 379-391.
[3] Buckmaster J.; *The Mathematics of Combustion*, SIAM, Philadelphia (1985).
[4] Buckmaster, J.; Ludford, G.; *Theory of Laminar Flames*, Cambridge University Press, Cambridge (1982).
[5] Daddiouaissa, E.; *Existence of global solutions for a system of equations having a triangular matrix*, Electron. Journal of Diff. Equations, 2009(2009), No. 09, 1-7.
[6] De Oliveira, L. F.; Pereira, A. L.; Pereira, M. C.; *Continuity of attractors for a reaction-diffusion problem with respect to variations of the domain*, Electron. Journal of Diff. Equations, 2005(2005), No. 100, 1-18.
[7] Evans, L. C.; *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island,(1998).
[8] Fitzgibbon, W. E.; Martin, C. B.; *The longtime behavior of solutions to a quasilinear combustion model*, Nonlinear Analysis, Theory, Methods & Applications, 19(1992), No. 10, 947-961.
[9] Frank-Kamenetski, D.; *Diffusion and Heat Transfer in Chemical Kinetics.* Plenum Press, New York (1969).

[10] Henry, D.; *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math. 840, Springer Verlag, Berlin, New York, 1981.

[11] Kouach, S., *Existence of global solutions to reaction-diffusion systems with nonhomogeneous boundary conditions via Lyapunov functional*, Electron. Journal of Diff. Equations, 2002(2002), No. 88, 1-13.

[12] McCOMB, W. D.; *The Physics of Fluid Turbulence*, Clarendon Press, Oxford, (1990).

[13] Sanni, S. A.; *A Combustion model with unbounded thermal conductivity and reactant diffusivity in non-smooth domains*, Electron. Journal of Diff. Equations, 2009(2009), No. 60, 1-14.

[14] Sattinger, D.; *A nonlinear parabolic system in the theory of combustion*, Quart. Appl. math., 33(1975), 47-61.

[15] Temam, R.; *Navier-Stokes Equations*, North-Holland Publishing Company, Oxford, (1977).

Sikiru Adigun Sanni

Department of Mathematics, Statistics and Computer Science, University of Uyo, Uyo, Akwa Ibom State, Nigeria

*E-mail address*: sikirusanni@yahoo.com