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LYAPUNOV STABILITY OF CLOSED SETS IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

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Abstract. In this article, we consider impulsive semidynamical systems, defined in a metric space, with impulse effects at variable times. Converse-type theorems are included in our results giving necessary and sufficient conditions for various types of stability of closed subsets of the metric space. These results are achieved by means of Lyapunov functionals which indicate how the solutions behave when entering a "stable" set.

1. Introduction

Impulsive semidynamical systems present interesting and important phenomena such as "beating", "dying", "merging", "noncontinuation of solutions", etc. These systems present a more complex structure than the non-impulsive systems because of their irregularity. In recent years, the theory of such systems has been studied and developed intensively. See for instance [2]-[12].

Lyapunov stability theory has been studied by several authors in investigations of continuous dynamical systems and impulsive dynamical systems. The majority of these papers in the impulsive case deal with systems with impulse effects at preassigned times. In [11], the author considers a more general case where the impulsive semidynamical system admits impulse effects at variable times. He considers an impulsive semidynamical system $(\Omega, \widetilde{\pi})$, where $\Omega \subset X$ is an open set in a metric space X and the continuous impulsive function I is defined from $\partial\Omega$ to X ($\partial\Omega$ is the boundary of Ω in X). He introduces a continuous Lyapunov function in $(\Omega, \widetilde{\pi})$ denoted by $V : \overline{G} \to \mathbb{R}$, where $G \subset \Omega$ is a positively invariant closed set and $\overline{G}$ denotes the closure of a set G in X. The derivative of the function V is defined by

$$\dot{V}(x) = \lim_{t \to 0^+} \frac{V(\widetilde{\pi}(x,t)) - V(x)}{t}$$

and the set E by $\{x \in G : \dot{V}(x) = 0\}$. Considering $A \subset E$ as being the largest invariant set under $\widetilde{\pi}$, Kaul proved that A is asymptotically stable, provided $A \subset intG$ and $V(A) = a$ for some $a \in \mathbb{R}$. The converse result is given as follows: if A is asymptotically stable, then there exists a positively invariant set G in Ω containing

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A which admits a Lyapunov function $V : G \to \mathbb{R}_+$ satisfying some properties. We can observe that the set A does not contain points of $\partial\Omega$ where the discontinuities of the impulsive system occur.

In the present paper, we extend the results from [1] for the impulsive case. In [1] the authors present several results which give necessary and sufficient conditions for the stability of closed sets in non-impulsive dynamical systems defined in a metric space. We consider impulsive semidynamical systems of type $(X, \pi; M, I)$ subject to impulse action which varies in time, where X is a metric space, (X, π) is a semidynamical system, M is a non-empty closed subset of X that denotes the impulsive set and $I : M \to X$ is the impulse function. We give necessary and sufficient conditions for various types of stability of closed sets of X. In other words, we establish necessary and sufficient conditions so that the solutions of the impulsive system become "stable" in some sense after entering a closed subset of X. Converse-type results are included in the main theorems. In contrast to the paper [11], the set which we prove to be "stable" can contain points of M where the discontinuities of the impulsive system occur.

In the first part of this article, we present the basis of the theory of impulsive semidynamical systems. We present basic definitions and notations and then we discuss the continuity of a function which describes the times of reaching the impulsive set. We also present additional useful definitions.

The second part of the paper concerns the main results. We introduce two new concepts of stability of sets in impulsive semidynamical systems and we relate these concepts of stability to other known concepts. We give necessary and sufficient conditions for the various types of stability of closed sets of X. We prove that there exists a functional which plays the role of a Lyapunov functional indicating how the solutions behave when entering a "stable" closed set provided this set is "stable" and we also state the reciprocal of this fact. In addition, we show that this Lyapunov functional is continuous when the impulsive set is contained in the closed set. Finally we present two examples to show how the theory can be employed.

2. Preliminaries

In this section we present the basic definitions and notation of the theory of impulsive semidynamical systems. We also include some fundamental results which are necessary for understanding the basis of the theory.

2.1. **Basic definitions and terminology.**

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in X$ and $t, s \in \mathbb{R}_+$. We denote such system by $(X, \pi, \mathbb{R}_+)$ or simply (X, π). When $\mathbb{R}_+$ is replaced by $\mathbb{R}$ in the definition above, the triple $(X, \pi, \mathbb{R})$ is a *dynamical system*. For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\mathrm{C}^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$ which we also denote by $\pi^+(x)$. For $t \geq 0$ and $x \in X$, we define $F(x, t) = \{y \in X : \pi(y, t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset X$, we define

$$F(D, \Delta) = \cup\{F(x, t) : x \in D \text{ and } t \in \Delta\}.$$

Then a point $x \in X$ is called an *initial point*, if $F(x, t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(X, \pi; M, I)$ consists of a semidynamical system, (X, π), a nonempty closed subset M of X such that for every $x \in M$, there exists $\varepsilon_x > 0$ such that

$$F(x, (0, \varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x, (0, \varepsilon_x)) \cap M = \emptyset,$$

and a continuous function $I : M \to X$ whose action we explain below in the description of the impulsive trajectory of an impulsive semidynamical system. The points of M are isolated in every trajectory of system (X, π). The set M is called the *impulsive set*, the function I is called *impulse function* and we write $N = I(M)$. We also define

$$M^+(x) = (\pi^+(x) \cap M) \setminus \{x\}.$$

Another property of the impulsive set M is that M is a meager set in X as shown by the next lemma.

Lemma 2.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. The impulsive set M is a meager set in X.*

Proof. The proof is immediate because the points of M are isolated in every trajectory of the system (X, π). Therefore $\operatorname{int}(\overline{M}) = \emptyset$ in X and the result follows. $\square$

Given an impulsive semidynamical systems $(X, \pi; M, I)$ and $x \in X$ such that $M^+(x) \neq \emptyset$, it is always possible to find a smallest number s such that the trajectory $\pi_x(t)$ for $0 < t < s$ does not intercept the set M. This result is stated next and a proof of it can be found in [2].

Lemma 2.2. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Then for every $x \in X$, there is a positive number s, $0 < s \leq +\infty$, such that $\pi(x, t) \notin M$, whenever $0 < t < s$, and $\pi(x, s) \in M$ if $M^+(x) \neq \emptyset$.*

Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$. By means of Lemma 2.2, it is possible to define a function $\phi : X \to (0, +\infty]$ in the following manner

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x, s) \in M \text{ and } \pi(x, t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(X, \pi; M, I)$ is an $X-$valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x, t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $M^+(x) \neq \emptyset$, it follows from Lemma 2.2 that there is a smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in M$ and $\pi(x, t) \notin M$, for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I(x_1)$ and $\phi(x) = s_0$.

Since $s_0 < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\widetilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, for $s_0 \leq t < +\infty$, and $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.2 that there is a smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in M$ and $\pi(x_1^+, t-s_0) \notin M$, for $s_0 < t < s_0+s_1$. Then we define $\widetilde{\pi}_x$ on $[s_0, s_0+s_1]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t-s_0), & s_0 \le t < s_0+s_1 \\ x_2^+, & t = s_0+s_1, \end{cases}$$

where $x_2^+ = I(x_2)$ and $\phi(x_1^+) = s_1$, and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\widetilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $M^+(x_n^+) \neq \emptyset$, $n = 1, 2, 3, \dots$, and if $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Also given $x \in X$, one of the three properties hold:

i) $M^+(x) = \emptyset$ and hence the trajectory of x has no discontinuities.
ii) For some $n \ge 1$, each x_k^+, $k = 1, 2, \dots, n$, is defined and $M^+(x_n^+) = \emptyset$. In this case, the trajectory of x has a finite number of discontinuities.
iii) For all $k \ge 1$, x_k^+ is defined and $M^+(x_k^+) \neq \emptyset$. In this case, the trajectory of x has infinitely many discontinuities.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Given $x \in X$, the *impulsive positive orbit* of x is defined by the set

$$\widetilde{\mathrm{C}}^+(x) = \{\widetilde{\pi}(x,t) : t \in \mathbb{R}_+\},$$

which we also denote by $\widetilde{\pi}^+(x)$. We denote the closure of $\widetilde{\mathrm{C}}^+(x)$ in X by $\widetilde{K}^+(x)$.

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies standard properties which follow straightforwardly from the definition. See the next proposition and [3] for a proof of it.

Proposition 2.3. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$. The following properties hold:*

i) $\widetilde{\pi}(x, 0) = x$,
ii) $\widetilde{\pi}(\widetilde{\pi}(x,t), s) = \widetilde{\pi}(x, t+s)$, *for all* $t, s \in [0, T(x))$ *such that* $t+s \in [0, T(x))$.

2.2. **Semicontinuity and continuity of ϕ.** The result of this section is borrowed from [7]. It concerns the function ϕ defined previously which indicates the moments of impulse action of a trajectory in an impulsive system. Such result is applied sometimes intrinsically in the proofs of the main theorems of the next section.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \le \mu < \nu \le 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC. See [7], Theorem 3.8.

Theorem 2.4. *Consider an impulsive semidynamical system $(X, \pi; M, I)$. Assume that no initial point in (X, π) belongs to the impulsive set M and that each element of M satisfies the condition (TC). Then ϕ is continuous at x if and only if $x \notin M$.*

Remark 2.5. *Suppose the conditions of Theorem 2.4 are true. Although the function $\widetilde{\pi}$ is not continuous, by the continuity of the impulse function $I : M \to I(M)$ and function ϕ, we can obtain the following result: Suppose $x \in X \setminus M$. Given $\varepsilon > 0$, for each $k = 0, 1, 2, \dots$ and $t \in [0, \phi(x_k^+)]$, there is a $\delta_k > 0$ such that $\rho(\pi(x_k^+, t), \pi(y_k^+, t)) < \varepsilon$ whenever $\rho(y_k^+, x_k^+) < \delta_k$ (ρ is a metric in X and $x_0^+ = x$). This result is applied in the proofs of the main theorems of the next section.*

2.3. **Additional definitions.** Let us consider a metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in X$ and ratio δ. Let $B(A, \delta) = \{x \in X : \rho_A(x) < \delta\}$ and $B[A, \delta] = \{x \in X : \rho_A(x) \leq \delta\}$, where $\rho_A(x) = \inf\{\rho(x, y) : y \in A\}$. Throughout this paper, we use the notation ∂A, $\mathrm{int}(A)$ and $\overline{A}$ to denote respectively the boundary, interior and closure of A in X.

In what follows, $(X, \pi; M, I)$ is an impulsive semidynamical system and $x \in X$.

We define the *prolongation set* of x in $(X, \pi; M, I)$ by

$$\widetilde{D}^+(x) = \{y \in X : \widetilde{\pi}(x_n, t_n) \overset{n \to +\infty}{\to} y, \text{ for some } x_n \overset{n \to +\infty}{\to} x \text{ and } t_n \in [0, +\infty)\}.$$

For a set $A \subset X$ we consider $\widetilde{D}^+(A) = \cup\{\widetilde{D}^+(x) : x \in A\}$.

If $\widetilde{\pi}^+(A) \subset A$, we say that A is $\widetilde{\pi}$*-invariant*.

A point $x \in X$ is called *stationary* or *rest point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for all $t \geq 0$, it is a *periodic point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for some $t > 0$ and x is not stationary, and it is a *regular point* if it is neither a rest point nor a periodic point.

Let $A \subset X$. If for every $\varepsilon > 0$ and every $x \in A$, there is $\delta = \delta(x, \varepsilon) > 0$ such that $\widetilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(A, \varepsilon)$, then A is called $\widetilde{\pi}$-stable. The set A is orbitally $\widetilde{\pi}$-stable if for every neighborhood U of A, there is a positively $\widetilde{\pi}-$invariant neighborhood V of A, $V \subset U$. If for all $x \in A$ and all $y \notin A$, there exist a neighborhood V of x and a neighborhood W of y such that $W \cap \widetilde{\pi}(V, [0, +\infty)) = \emptyset$, we say that A is $\widetilde{\pi}$-stable according to Bhatia-Hajek [1]. We define the set

$$\begin{aligned} \widetilde{P}_W^+(A) = \big\{x \in X :\ & \text{for every neighborhood } U \text{ of } A, \text{ there is a sequence} \\ & \{t_n\} \subset \mathbb{R}_+, t_n \overset{n \to +\infty}{\to} +\infty \text{ such that } \widetilde{\pi}(x, t_n) \in U\}. \end{aligned}$$

The set $\widetilde{P}_W^+(A)$ is called *region of weak attraction* of A with respect to $\widetilde{\pi}$. If $x \in \widetilde{P}_W^+(A)$, then we say that x is $\widetilde{\pi}$*-weakly attracted* to A. A subset $A \subset X$ is called a *weak* $\widetilde{\pi}$*-attractor*, if $\widetilde{\mathrm{P}}_W^+(A)$ is a neighborhood of A. A set $A \subset X$ is called *asymptotically* $\widetilde{\pi}$*-stable*, if it is both a weak $\widetilde{\pi}$-attractor and orbitally $\widetilde{\pi}$-stable.

For results concerning the stability and invariancy of sets in an impulsive system, the reader may want to consult [2], [3], [8] and [11].

3. Main Results

We divide this section into two parts. The first part concerns the relations among some concepts of stability. In the second part, we discuss Lyapunov stability of closed sets in impulsive semidynamical systems where the results give necessary and sufficient conditions for various types of stability of closed sets.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system where X is a metric space. We assume the following additional hypotheses:

- No initial point in (X, π) belongs to the impulsive set M, that is, given $x \in M$ there are $y \in X$ and $t \in \mathbb{R}_+$ such that $\pi(y, t) = x$.
- Each element of M satisfies the condition (STC) (*consequently, ϕ is continuous on $X \setminus M$*).
- $M \cap \mathrm{I(M)} = \emptyset$.
- For all $x \in X$ and for all $k \geq 1$, x_k^+ is defined and $M^+(x_k^+) \neq \emptyset$, that is, the trajectory of $x \in X$ has infinitely many discontinuities. Consequently, $\phi(x) < +\infty$ for all $x \in X$.

3.1. Stability.

We introduce two new concepts of stability for impulsive semidynamical systems. Then, we relate these new concepts to known ones.

Definition 3.1. Let $(X, \pi; M, I)$ be an impulsive semidynamical system. A set $A \subset X$ is said to be

(a) equi $\widetilde{\pi}$-stable, if for each $x \notin A$, there is a $\delta = \delta(x) > 0$ such that

$$x \notin \overline{\widetilde{\pi}(B(A, \delta), [0, +\infty))}.$$

(b) uniformly $\widetilde{\pi}$-stable, if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \varepsilon).$$

The next result deals with the equivalence between equi $\widetilde{\pi}$-stability and uniform $\widetilde{\pi}$-stability of a compact subset $A \subset X$. This result is also valid when we replace the condition of equi $\widetilde{\pi}$-stability by $\widetilde{\pi}$-stability. The proof is similar to the continuous case, see [1].

Theorem 3.2. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system, X is locally compact and $A \subset X$ is compact. Then, A is equi $\widetilde{\pi}$-stable if and only if A is uniformly $\widetilde{\pi}$-stable. Replacing the hypotheses equi $\widetilde{\pi}-$stability by $\widetilde{\pi}-$stability, the result remains true.*

Remark 3.3. If the set $A \subset X$ is closed but not compact, then the sufficiency of the theorem does not necessarily hold. Indeed, consider the discontinuous flow shown in Figure 3.1, where $\widehat{M} = \{(-1, x_2) : x_2 \in \mathbb{R}\}$, $\widehat{N} = \{(2, x_2) : x_2 \in \mathbb{R}\}$, $p = (x_1', 0) \in \mathbb{R}^2$ and the impulsive function $I_1 : \widehat{M} \to \widehat{N}$ is given by $I_1(-1, x_2) = (2, x_2')$ such that $x_2' < x_2'' < x_2$, where x_2'' is such that for some unique $\lambda > 0$, $\pi((2, x_2'), \lambda) = (x_1', x_2'')$.

Note that the trajectories for $x_1 > x_1'$ are straight lines parallel to the axis $0x_1$. This discontinuous flow has the property that for all $x \in \mathbb{R}^2$, $\lim_{t \to -\infty} \widetilde{\pi}(x, t) = 0$. Now, consider the sets

$$M = \widehat{M} \cup_{n=1}^{+\infty} \{(x_1' + 3n, x_2) : x_2 \in \mathbb{R}\},$$

$$\mathrm{N} = \widehat{N} \cup_{n=1}^{+\infty} \{(x_1' + 3n + 1, x_2) : x_2 \in \mathbb{R}\}$$

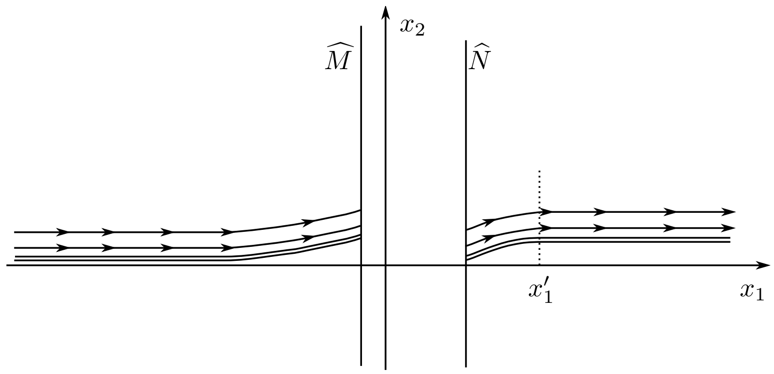

FIGURE 1. Discontinuous flow

and define the function $I : M \to N$ as follows

$$I(-1, x_2) = I_1(-1, x_2) \quad \text{for all } x_2 \in \mathbb{R}$$

and

$$I(x_1' + 3n, x_2) = (x_1' + 3n + 1, x_2) \quad \text{for all } x_2 \in \mathbb{R} \text{ and } n = 1, 2, \dots .$$

Then, the impulsive semidynamical system $(\mathbb{R}^2, \pi; M, I)$ has infinitely many discontinuities. Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Clearly A is $\widetilde{\pi}$-stable, but it is not uniformly $\widetilde{\pi}$-stable.

The next result shows the equivalence between the orbital stability and the uniform stability in impulsive semidynamical systems. The proof is similar to the continuous case, see [1].

Theorem 3.4. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Assume that X is locally compact and $A \subset X$ is compact. Then A is orbitally $\widetilde{\pi}$-stable if and only if A is uniformly $\widetilde{\pi}$-stable.*

By [8, Theorem 4.1] and Theorems 3.2 and 3.4 above, we have the following result which relates various concepts of stability.

Theorem 3.5. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Assume that X is locally compact and A is a compact subset of X. Then the following conditions are equivalent:*

(a) *A is $\widetilde{\pi}$-stable.*
(b) *A is orbitally $\widetilde{\pi}$-stable.*
(c) *A is $\widetilde{\pi}$-stable in the sense of Bhatia and Hajek.*
(d) *A is uniformly $\widetilde{\pi}$-stable.*
(e) *A is equi $\widetilde{\pi}$-stable.*
(f) *$\widetilde{\mathrm{D}}^+(A) = A$.*

The $\widetilde{\pi}$-stability of a closed subset A of X implies that $I(M) \subset A$, for $M \subset A$, as shown by the next lemma.

Lemma 3.6. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$ be closed. If A is $\widetilde{\pi}$-stable and $M \subset A$, then $I(M) \subset A$.*

Proof. Given $x \in A$ and $\varepsilon > 0$, there is a $\delta = \delta(x, \varepsilon) > 0$ such that

$$\widetilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(A, \varepsilon).$$

Since ε is arbitrary, we have $\widetilde{\pi}^+(x) \subset \overline{A} = A$. Therefore, $I(M) \subset A$ provided $M \subset A$. □

3.2. **Lyapunov Stability.** In this section, we shall present the results that concern the Lyapunov stability of certain closed sets of X. These results are achieved by means of functionals which play the role of a Lyapunov functional indicating how the solutions behave when entering a "stable" set. The results give necessary and sufficient conditions for the various types of stability of closed sets of X. We start by presenting a result on $\widetilde{\pi}$-stability.

Theorem 3.7. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$ be closed.*

(1) *If there exists a functional $\psi : X \to \mathbb{R}_+$ with the following properties:*
   (a) *ψ is continuous in $X \setminus (M \setminus A)$.*
   (b) *For every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \varepsilon$ and $x \notin M$, and for any sequence $\{w_n\}_{n\geq 1} \subset X$ such that $w_n \overset{n\to+\infty}{\longrightarrow} x \in A$ implies $\psi(w_n) \overset{n\to+\infty}{\longrightarrow} 0$.*
   (c) *$\psi(\pi(x, t)) \leq \psi(x)$ if $x \in X \setminus M$ and $0 \leq t \leq \phi(x)$, and $\psi(I(x)) \leq \psi(x)$ if $x \in M$.*

   Then A is $\widetilde{\pi}$-stable.
(2) *Reciprocally, if A is $\widetilde{\pi}$-stable, then there is a functional $\psi : X \to \mathbb{R}_+$ satisfying conditions a), b) and c) above.*

Proof. Let us prove the necessary condition. Given $\varepsilon > 0$ and $x \in A$, set $\mu = \inf\{\psi(w) : w \notin M \text{ and } \rho(w, A) \geq \frac{\varepsilon}{2}\}$. Note that $\mu > 0$, because by item (b), there is a $\delta > 0$ such that $\psi(a) \geq \delta$ whenever $\rho(a, A) \geq \frac{\varepsilon}{2}$ and $a \notin M$. We have two cases to consider: when $x \in int(A)$ and when $x \in \partial A$.

First, suppose $x \in int(A)$. Then, by the second part of item (b) and by the continuity of ψ in A, there is a $\delta_1 > 0$ such that

$$\psi(y) < \mu \quad \text{for all } y \in B(x, \delta_1) \subset A. \tag{3.1}$$

We suppose by contradiction that $\widetilde{\pi}(B(x, \delta_1), [0, +\infty))$ is not contained in $B(A, \varepsilon)$. Thus, there are $z \in B(x, \delta_1)$ and $t_1 \in (0, +\infty)$ such that

$$\widetilde{\pi}(z, t_1) \notin B(A, \varepsilon). \tag{3.2}$$

Note that $\widetilde{\pi}(z, t_1) \notin M$ because $M \cap I(M) = \emptyset$. By equation (3.2), $\rho(\widetilde{\pi}(z, t_1), A) \geq \varepsilon$ and this implies

$$\psi(\widetilde{\pi}(z, t_1)) \geq \inf\big\{\psi(w) : w \notin M \text{ and } \rho(w, A) \geq \frac{\varepsilon}{2}\big\} = \mu. \tag{3.3}$$

We have two cases to consider: when $z \in M$ and when $z \notin M$. First suppose that $z \notin M$. Note that as $z \in B(x, \delta_1)$, then $\psi(z) < \mu$ by (3.1). Hence for $0 \leq t < \phi(z)$, we have

$$\psi(\widetilde{\pi}(z, t)) = \psi(\pi(z, t)) \overset{(c)}{\leq} \psi(z) < \mu.$$

If $t = \phi(z)$ and remembering from the definition of $\widetilde{\pi}$ that $z_1 = \pi(z, \phi(z))$, then

$$\psi(\widetilde{\pi}(z, t)) = \psi(\widetilde{\pi}(z, \phi(z))) = \psi(I(z_1)) \overset{(c)}{\leq} \psi(z_1) = \psi(\pi(z, \phi(z))) \overset{(c)}{\leq} \psi(z) < \mu. \tag{3.4}$$

Now if $\phi(z) < t < \phi(z) + \phi(z_1^+)$, then

$$\psi(\widetilde{\pi}(z,t)) = \psi(\pi(z_1^+, t-\phi(z))) \overset{(c)}{\leq} \psi(z_1^+) = \psi(\widetilde{\pi}(z,\phi(z))) \overset{(3.4)}{<} \mu.$$

Repeating this argument, we get $\psi(\widetilde{\pi}(z,t)) < \mu$ for all $t \geq 0$. In particular for $t = t_1$, $\psi(\widetilde{\pi}(z,t_1)) < \mu$ which is a contradiction by (3.3). Hence, $\widetilde{\pi}(B(x,\delta_1),[0,+\infty)) \subset B(A,\varepsilon)$. Now, suppose $z \in M$. Take $\nu > 0$, $\nu < t_1$, such that $\widetilde{\pi}(z,\nu) = \pi(z,\nu) \in B(x,\delta_1)\setminus M$. By the same argument used above for $z \notin M$, we get $\psi(\widetilde{\pi}(\widetilde{\pi}(z,\nu),t)) < \mu$ for all $t \geq 0$. In particular for $t = t_1 - \nu$, $\psi(\widetilde{\pi}(z,t_1)) = \psi(\widetilde{\pi}(\widetilde{\pi}(z,\nu),t_1-\nu)) < \mu$ which is a contradiction by (3.3). Therefore, we get again $\widetilde{\pi}(B(x,\delta_1),[0,+\infty)) \subset B(A,\varepsilon)$.

Now we assume that $x \in \partial A$. Since ψ is continuous in $X \setminus (M \setminus A)$, M is a meager set in X and by the second part of item (b), there is a $\delta_2 > 0$, $\delta_2 < \varepsilon$, such that $\psi(y) < \mu$ for all $y \in B(x,\delta_2) \setminus M$. Supposing that $\widetilde{\pi}(B(x,\delta_2),[0,+\infty))$ is not contained in $B(A,\varepsilon)$, there are $z \in B(x,\delta_2)$ and $t_2 \in (0,+\infty)$ such that $\widetilde{\pi}(z,t_2) \notin B(A,\varepsilon)$. Thus $\rho(\widetilde{\pi}(z,t_2),A) \geq \varepsilon$, $\widetilde{\pi}(z,t_2) \notin M$ because $M \cap I(M) = \emptyset$ and therefore

$$\psi(\widetilde{\pi}(z,t_2)) \geq \inf\big\{\psi(w) : w \notin M \text{ and } \rho(w,A) \geq \frac{\varepsilon}{2}\big\} = \mu. \tag{3.5}$$

If $z \in B(x,\delta_2) \setminus M$, then it can be shown that $\psi(\widetilde{\pi}(z,t)) < \mu$ for all $t \geq 0$ as we did before. Hence, $\psi(\widetilde{\pi}(z,t_2)) < \mu$ which is a contradiction by (3.5). Also, if $z \in B(x,\delta_2) \cap M$, then z is an initial point for the impulsive system and there is a time $\tau > 0$ such that $\widetilde{\pi}(z,(0,\tau)) = \pi(z,(0,\tau)) \subset B(x,\delta_2) \setminus M$. Taking t^*, $0 < t^* < \tau$. By the previous case, $\psi(\widetilde{\pi}(\pi(z,t^*),t)) < \mu$ for all $t \geq 0$. As a result, $\psi(\widetilde{\pi}(z,t_2)) = \psi(\widetilde{\pi}(\pi(z,t^*),t_2-t^*)) < \mu$ and this is a contradiction by (3.5). Therefore, $\widetilde{\pi}(B(x,\delta_2),[0,+\infty)) \subset B(A,\varepsilon)$. Consequently, A is $\widetilde{\pi}$-stable.

Let us prove the sufficient condition. Define the function $\psi : X \to \mathbb{R}_+$ by

$$\psi(x) = \begin{cases} \sup_{k\geq 0}\Big(\sup_{0\leq t\leq \phi(x_k^+)} \frac{\rho(\pi(x_k^+,t),A)}{1+\rho(\pi(x_k^+,t),A)}\Big), & \text{if } x \in X \setminus M,\\ \psi(I(x)), & \text{if } x \in M, \end{cases}$$

where $x_0^+ = x$. We shall verify that ψ satisfies conditions (a), (b) and (c).

(a) Take $x \in X \setminus M$. Since $\{x\}$ is compact and M is closed, there is an $\eta > 0$, such that $B(x,\eta) \cap M = \emptyset$. Given a sequence $\{w_n\}_{n\geq 1} \subset X$ such that $w_n \overset{n\to+\infty}{\to} x$, there is an integer $n_0 > 0$ such that $w_n \in B(x,\eta)$ for $n > n_0$. Since I is a continuous function and ϕ is continuous on $X \setminus M$ we have

$$(w_n)_1^+ = I(\pi(w_n,\phi(w_n))) \overset{n\to+\infty}{\to} I(\pi(x,\phi(x))) = x_1^+.$$

Note that $x_1^+ \notin M$ because $M \cap I(M) = \emptyset$. But $\{x_1^+\}$ is compact and M is closed, then there is an $\eta_1 > 0$ such that $B(x_1^+,\eta_1) \cap M = \emptyset$. As $(w_n)_1^+ \overset{n\to+\infty}{\to} x_1^+$, there is an integer $n_0^1 > 0$ such that $(w_n)_1^+ \in B(x_1^+,\eta_1)$ for $n > n_0^1$. By the continuity of ϕ on $X \setminus M$ we have

$$\phi((w_n)_1^+) \overset{n\to+\infty}{\to} \phi(x_1^+).$$

Then

$$\sup_{0\leq t\leq \phi((w_n)_1^+)} \frac{\rho(\pi((w_n)_1^+,t),A)}{1+\rho(\pi((w_n)_1^+,t),A)} \overset{n\to+\infty}{\to} \sup_{0\leq t\leq \phi(x_1^+)} \frac{\rho(\pi(x_1^+,t),A)}{1+\rho(\pi(x_1^+,t),A)}.$$

Analogously, since $(w_n)_1^+ \overset{n\to+\infty}{\longrightarrow} x_1^+$ and $\phi((w_n)_1^+) \overset{n\to+\infty}{\longrightarrow} \phi(x_1^+)$, it follows

$$(w_n)_2^+ = I(\pi((w_n)_1^+, \phi((w_n)_1^+))) \overset{n\to+\infty}{\longrightarrow} I(\pi(x_1^+, \phi(x_1^+))) = x_2^+$$

and

$$\sup_{0\le t\le \phi((w_n)_2^+)} \frac{\rho(\pi((w_n)_2^+, t), A)}{1+\rho(\pi((w_n)_2^+, t), A)} \overset{n\to+\infty}{\longrightarrow} \sup_{0\le t\le \phi(x_2^+)} \frac{\rho(\pi(x_2^+, t), A)}{1+\rho(\pi(x_2^+, t), A)}.$$

We can continue with this process because $x_k^+ \notin M$ for all $k \ge 1$. Thus, for each integer $k \ge 1$ we obtain

$$(w_n)_k^+ = I(\pi((w_n)_{k-1}^+, \phi((w_n)_{k-1}^+))) \overset{n\to+\infty}{\longrightarrow} I(\pi(x_{k-1}^+, \phi(x_{k-1}^+))) = x_k^+$$

and

$$\sup_{0\le t\le \phi((w_n)_k^+)} \frac{\rho(\pi((w_n)_k^+, t), A)}{1+\rho(\pi((w_n)_k^+, t), A)} \overset{n\to+\infty}{\longrightarrow} \sup_{0\le t\le \phi(x_k^+)} \frac{\rho(\pi(x_k^+, t), A)}{1+\rho(\pi(x_k^+, t), A)}.$$

Therefore,

$$\begin{aligned} &\sup_{k\ge 0}\Big(\sup_{0\le t\le \phi((w_n)_k^+)} \frac{\rho(\pi((w_n)_k^+, t), A)}{1+\rho(\pi((w_n)_k^+, t), A)}\Big) \\ &\overset{n\to+\infty}{\longrightarrow} \sup_{k\ge 0}\Big(\sup_{0\le t\le \phi(x_k^+)} \frac{\rho(\pi(x_k^+, t), A)}{1+\rho(\pi(x_k^+, t), A)}\Big). \end{aligned}$$

In conclusion, $\psi(w_n) \overset{n\to+\infty}{\longrightarrow} \psi(x)$ and ψ is continuous on $X \setminus M$.

Since we want to prove that ψ is continuous on $X \setminus (M \setminus A)$, it is enough to show that ψ is continuous on $A \cap M$. Assume that $x \in A \cap M$. Since A is $\widetilde{\pi}$-stable, given $\varepsilon > 0$, there is a $\delta = \delta(x, \varepsilon) > 0$ such that $\widetilde{\pi}(B(x,\delta), [0, +\infty)) \subset B(A, \varepsilon)$. Since ε is arbitrary, we have $\widetilde{\pi}^+(x) \subset \overline{A} = A$. Thus, $\pi(x_k^+, t) \subset A$ for all $0 \le t \le \phi(x_k^+)$, $k \ge 0$ ($x_0^+ = x$). Consequently $\rho(\pi(x_k^+, t), A) = 0$ for all $0 \le t \le \phi(x_k^+)$, $k \ge 0$. Hence $\psi(x) = \psi(I(x)) = \psi(x_1^+) = 0$.

Considering $\widetilde{\pi}(B(x,\delta), [0, +\infty)) \subset B(A, \varepsilon)$ from the $\widetilde{\pi}$-stability of $\widetilde{\pi}$, if $\{z_n\}_{n\ge 1}$ is a sequence in X such that $z_n \overset{n\to+\infty}{\longrightarrow} x$, then there is a positive integer $n_0 > 0$ such that $z_n \in B(x, \delta)$ for $n > n_0$. Consequently, $\widetilde{\pi}(z_n, [0, +\infty)) \subset B(A, \varepsilon)$ for all $n > n_0$, that is, $\pi((z_n)_k^+, t)) \subset B(A, \varepsilon)$ for $0 \le t \le \phi((z_n)_k^+)$, $k = 0, 1, 2, \dots$ and $n > n_0$ ($(z_n)_0^+ = z_n$). Then $\psi(z_n) < \varepsilon$ for all $n > n_0$, that is, $\psi(z_n) \overset{n\to+\infty}{\longrightarrow} 0 = \psi(x)$. Therefore, ψ is continuous on $X \setminus (M \setminus A)$.

(b) Consider $x \in X \setminus M$. Given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{1+\varepsilon}$. Thus, if $\rho(x, A) \ge \varepsilon$ then $\frac{\rho(x,A)}{1+\rho(x,A)} \ge \delta$. Therefore, $\psi(x) \ge \delta$.

For the second part of item (b), let us assume that $x \in A$. If $x \notin M$, as M is closed and $\{x\}$ is compact, there is a $\delta > 0$ such that $B(x, \delta) \cap M = \emptyset$. Thus, if $\{w_n\}_{n\ge 1}$ is any sequence in X such that $w_n \overset{n\to+\infty}{\longrightarrow} x$, there exists a positive integer $N > 0$ such that $w_n \in B(x, \delta)$ for $n > N$, by continuity of ψ in $X \setminus (M \setminus A)$,

$$\psi(w_n) \overset{n\to+\infty}{\longrightarrow} \psi(x).$$

Now, suppose $x \in M$. First of all, we should note that if $\{z_n\}_{n\ge 1} \subset X \setminus M$ and $z_n \overset{n\to+\infty}{\longrightarrow} x$, then the continuity of ψ in $X \setminus (M \setminus A)$ implies

$$\psi(z_n) \overset{n\to+\infty}{\longrightarrow} \psi(x).$$

Also, if $\{z_n\}_{n\geq 1}\subset M$ and $z_n \overset{n\to+\infty}{\to} x$, since the impulsive operator I is continuous, we have

$$I(z_n) \overset{n\to+\infty}{\to} I(x).$$

Thus, since $I(z_n)\notin M$ for all $n\in\mathbb{N}$, $I(x)\notin M$ and ψ is continuous in $X\setminus(M\setminus A)$, it follows that

$$\psi(I(z_n)) \overset{n\to+\infty}{\to} \psi(I(x)),$$

then by the definition of ψ,

$$\psi(z_n) \overset{n\to+\infty}{\to} \psi(x).$$

Consequently, if $\{\overline{x}_n\}_{n\geq 1}\subset X$ is any sequence such that $\overline{x}_n\to x$ as $n\to+\infty$, then $\psi(\overline{x}_n)\to\psi(x)$ as $n\to+\infty$. Since $x\in A$, we have $\psi(x)=0$ (as shown above) and therefore, $\psi(\overline{x}_n)\to 0$ as $n\to+\infty$.

(c) Let $x\in X\setminus M$ and $0\leq s<\phi(x)$. Let $y=\pi(x,s)$. Since $0\leq s<\phi(x)$, we have $y_0^+=\pi(x,s)$ and $y_k^+=x_k^+$ for all integers $k\geq 1$. Then

$$\begin{aligned}\psi(\pi(x,s))=\psi(y)&=\sup_{k\geq 0}\Big(\sup_{0\leq t\leq\phi(y_k^+)}\frac{\rho(\pi(y_k^+,t),A)}{1+\rho(\pi(y_k^+,t),t),A)}\Big)\\&\leq\sup_{k\geq 0}\Big(\sup_{0\leq t\leq\phi(x_k^+)}\frac{\rho(\pi(x_k^+,t),A)}{1+\rho(\pi(x_k^+,t),A)}\Big)=\psi(x).\end{aligned}$$

Now, we shall prove that $\psi(\pi(x,\phi(x)))\leq\psi(x)$. Since $\pi(x,\phi(x))=x_1\in M$, we have

$$\begin{aligned}\psi(\pi(x,\phi(x)))&=\psi(x_1)=\psi(I(x_1))=\psi(x_1^+)\\&=\sup_{k\geq 1}\Big(\sup_{0\leq t\leq\phi(x_k^+)}\frac{\rho(\pi(x_k^+,t),A)}{1+\rho(\pi(x_k^+,t),A)}\Big)\\&\leq\sup_{k\geq 0}\Big(\sup_{0\leq t\leq\phi(x_k^+)}\frac{\rho(\pi(x_k^+,t),A)}{1+\rho(\pi(x_k^+,t),A)}\Big)=\psi(x).\end{aligned}$$

Consequently, $\psi(\pi(x,t))\leq\psi(x)$ for $0\leq t\leq\phi(x)$. Now we will prove the second part of c). Let $x\in M$. Then by the definition of ψ, $\psi(I(x))=\psi(x)$ and the theorem is proved. □

The next result is a corollary of Theorem 3.7. It says that if $M\subset A$, then we get the continuity of the function ψ.

Corollary 3.8. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system. A closed subset $A\subset X$ such that $M\subset A$ is $\widetilde{\pi}$-stable if and only if there exists a functional $\psi:X\to\mathbb{R}_+$, with the following properties:*

(a) *ψ is continuous.*
(b) *For every $\varepsilon>0$, there is a $\delta>0$ such that $\psi(x)\geq\delta$ whenever $\rho(x,A)\geq\varepsilon$, and for any sequence $\{x_n\}_{n\geq 1}\subset X$ such that $x_n \overset{n\to+\infty}{\to} x\in A$ implies $\psi(x_n) \overset{n\to+\infty}{\to} 0$.*
(c) *$\psi(\pi(x,t))\leq\psi(x)$ if $x\in X\setminus M$ and $t\geq 0$, and, $\psi(I(x))\leq\psi(x)$ if $x\in M$.*

Theorem 3.9 below deals with the equi $\widetilde{\pi}$-stability of a closed set of X.

Theorem 3.9. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system, $A\subset X$ be closed and $I(M\setminus A)\subset(X\setminus A)\setminus M$.*

(1) *If there exists a functional $\psi:X\to\mathbb{R}_+$ with the following properties:*

(a) *ψ is continuous in $X \setminus (M \setminus A)$.*
(b) *$\psi(x) = 0$ for $x \in A$, $\psi(x) > 0$ for $x \notin A \cup M$.*
(c) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \le \varepsilon$ whenever $\rho(x, A) \le \delta$.*
(d) *$\psi(\pi(x,t)) \le \psi(x)$ for $x \in X \setminus M$ and $0 \le t \le \phi(x)$, and $\psi(I(x)) \le \psi(x)$ for $x \in M$.*

Then A is equi $\widetilde{\pi}$-stable.

(2) *Reciprocally, if A is equi $\widetilde{\pi}$-stable, then there is a functional $\psi : X \to \mathbb{R}_+$ satisfying conditions (a), (b), (c) and (d) above.*

Proof. (1) Suppose $x \notin A$. We have two cases to consider: when $x \in M$ and otherwise $x \notin M$.

Suppose $x \notin M$. Set $\rho(x, A) = \varepsilon > 0$. Since $x \notin M$, then $\psi(x) > 0$. Let $\psi(x) = \mu$. The condition (c) says that there is an $\eta > 0$ such that

$$\psi(y) \le \frac{\mu}{2} \tag{3.6}$$

whenever $\rho(y, A) \le \eta$.

Let $\delta < \min\{\eta, \varepsilon\}$. We assert that $x \notin \overline{\widetilde{\pi}(B(A,\delta), [0, +\infty))}$. Indeed. Suppose the contrary. Then there are sequences $\{y_n\}_{n\ge1} \subseteq B(A,\delta)$ and $\{T_n\}_{n\ge1} \subseteq [0, +\infty)$ such that

$$\widetilde{\pi}(y_n, T_n) \overset{n\to+\infty}{\longrightarrow} x.$$

Since $x \notin M$, $\{x\}$ is compact and M is closed, there is a $\varrho > 0$ such that $B(x,\varrho) \cap M = \emptyset$. Moreover, there is an integer $n_0 > 0$ such that $\widetilde{\pi}(y_n, T_n) \in B(x, \varrho)$ for all $n > n_0$. Since ψ is a continuous function on $X \setminus (M \setminus A)$, there exists an integer $n_1 > n_0$ such that

$$|\psi(\widetilde{\pi}(y_n, T_n)) - \psi(x)| < \frac{\mu}{3}$$

for all $n \ge n_1$. As $\psi(x) = \mu$, we have

$$\frac{2\mu}{3} < \psi(\widetilde{\pi}(y_{n_1}, T_{n_1})) < \frac{4\mu}{3}. \tag{3.7}$$

Now note that $\psi(\widetilde{\pi}(w,t)) \le \psi(w)$ for all $t \ge 0$ and $w \in X \setminus M$. In fact, given $w \in X \setminus M$ we have $\psi(\pi(w,t)) \le \psi(w)$ for $0 \le t \le \phi(w)$ and $\psi(w_1^+) = \psi(I(w_1)) \le \psi(w_1) = \psi(\pi(w, \phi(w))) \le \psi(w)$. If $\phi(w) < t < \phi(w) + \phi(w_1^+)$, it follows that

$$\psi(\widetilde{\pi}(w,t)) = \psi(\pi(w_1^+, t - \phi(w))) \le \psi(w_1^+) = \psi(\widetilde{\pi}(w, \phi(w))) \le \psi(w).$$

For $t = \phi(w) + \phi(w_1^+)$,

$$\psi(\widetilde{\pi}(x,t)) = \psi(w_2^+) = \psi(I(w_2)) \le \psi(w_2) = \psi(\pi(w_1^+, \phi(w_1^+))) \le \psi(w_1^+) \le \psi(w),$$

and so on. Thus, $\psi(\widetilde{\pi}(w,t)) \le \psi(w)$ for all $t \ge 0$ and $w \in X \setminus M$. Using this fact and (3.6) we have

$$\psi(\widetilde{\pi}(y_{n_1}, T_{n_1})) \le \psi(y_{n_1}) \le \frac{\mu}{2}$$

which contradicts (3.7). Hence, $x \notin \overline{\widetilde{\pi}(B(A,\delta), [0, +\infty))}$.

Now we assume that $x \in M$. Suppose $x \in \overline{\widetilde{\pi}(B(A,\delta), [0, +\infty))}$ for every $\delta > 0$. Then, there are sequences $\{w_n^\delta\}_{n\ge1} \subset B(A,\delta)$ and $\{t_n^\delta\}_{n\ge1} \subset [0, +\infty)$ such that

$$\widetilde{\pi}(w_n^\delta, t_n^\delta) \overset{n\to+\infty}{\longrightarrow} x,$$

for each $\delta > 0$. Since each element of M satisfies the condition STC, we have two cases to consider.

Case 1: There are countably many elements of $\{\widetilde{\pi}(w_n^\delta, t_n^\delta)\}_{n\geq 1}$, denoted by $\{\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta)\}_{k\geq 1}$, such that

$$\pi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta), \phi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta))) \overset{k\to+\infty}{\longrightarrow} x.$$

In this case, since $\pi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta), \phi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta))) \in M$ and I is continuous, we have

$$\widetilde{\pi}(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta), \phi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta))) \overset{k\to+\infty}{\longrightarrow} I(x);$$

that is,

$$\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta + \phi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta))) \overset{k\to+\infty}{\longrightarrow} I(x).$$

But this is a contradiction, because $I(x) \notin A \cup M$ and by the previous case, there is a $\overline{\delta} > 0$ such that $I(x) \notin \overline{\widetilde{\pi}(B(A, \overline{\delta}), [0, +\infty))}$.

Case 2: There are countably many elements of $\{\widetilde{\pi}(w_n^\delta, t_n^\delta)\}_{n\geq 1}$, denoted by $\{\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta)\}_{k\geq 1}$, such that $\phi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta)) \overset{k\to+\infty}{\longrightarrow} \phi(x)$. In this case,

$$\pi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta), t) \overset{k\to+\infty}{\longrightarrow} \pi(x, t),$$

for all $0 \leq t < \phi(x)$. Let $0 < t_0 < \phi(x)$. Thus, $\pi(x, t_0) \notin M$ and $\pi(\widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta), t_0) = \widetilde{\pi}(w_{n_k}^\delta, t_{n_k}^\delta + t_0)$. This is a contradiction.

Therefore, if $x \in M \setminus A$, there is a $\delta > 0$ such that $x \notin \overline{\widetilde{\pi}(B(A, \delta), [0, +\infty))}$ and A is equi $\widetilde{\pi}$-stable.

(2) Consider the function $\psi(x)$ defined in Theorem 3.7. The result follows similarly as in Theorem 3.7 . □

We have the following corollary where we obtain the continuity of the function ψ, provided $M \subset A$.

Corollary 3.10. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. A closed subset $A \subset X$ such that $M \subset A$ is equi $\widetilde{\pi}$-stable if and only if there exists a functional $\psi : X \to \mathbb{R}_+$, with the following properties:*

- (a) *ψ is continuous.*
- (b) *$\psi(x) = 0$ for $x \in A$, $\psi(x) > 0$ for $x \notin A$.*
- (c) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \leq \varepsilon$ whenever $\rho(x, A) \leq \delta$.*
- d) *$\psi(\pi(x, t)) \leq \psi(x)$ for $x \in X \setminus M$ and $t \geq 0$, and $\psi(I(x)) \leq \psi(x)$ for $x \in M$.*

Lemma 3.11 will be necessary to prove Theorem 3.12.

Lemma 3.11. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$ be closed. Let $\psi : X \to \mathbb{R}_+$ be a continuous function on $X \setminus (M \setminus A)$ satisfying:*

- (1) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \varepsilon$ and $x \notin M$.*
- (2) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \leq \varepsilon$ whenever $\rho(x, A) \leq \delta$.*

Suppose there is a $\widetilde{\delta} > 0$ such that $\psi(\widetilde{\pi}(w, t)) \leq \psi(w)$ for all $t \geq 0$ and $w \in \overline{B(A, \widetilde{\delta})} \setminus M$. Then, there is a $\overline{\delta} > 0$, $0 < \overline{\delta} \leq \widetilde{\delta}$, such that $\widetilde{\pi}(B(A, \overline{\delta}), [0, +\infty)) \subset B(A, \widetilde{\delta})$.

Proof. We shall suppose that for each $\delta_n = \frac{\widetilde{\delta}}{n} > 0$, $n \in \mathbb{N}$, there are $w_n \in B\big(A, \widetilde{\delta}/n\big)$ and $t_0^n \in (0, +\infty)$ such that

$$\widetilde{\pi}(w_n, t_0^n) \notin B\Big(A, \widetilde{\delta}\Big).$$

Take $\mu = \inf\{\psi(x) : x \notin M \text{ and } \rho(x, A) \geq \widetilde{\delta}\}$. Note that $\mu > 0$ by item (1). By item (2), there is an $\eta > 0$, $\eta < \widetilde{\delta}$, such that $\psi(y) \leq \frac{\mu}{2}$ whenever $\rho(y, A) \leq \eta$. Note that, there is a positive integer n_{k_0} such that $B(A, \delta_{n_{k_0}}) \subset B(A, \eta)$. Since $w_{n_{k_0}} \in B(A, \delta_{n_{k_0}})$ we have $\psi(w_{n_{k_0}}) \leq \frac{\mu}{2}$. We have two cases to consider: when $w_{n_{k_0}} \in M$ and otherwise $w_{n_{k_0}} \notin M$.

First, suppose $w_{n_{k_0}} \notin M$. Then, $w_{n_{k_0}} \in \overline{B(A, \widetilde{\delta})} \setminus M$. Thus, $\psi(\widetilde{\pi}(w_{n_{k_0}}, t_0^{n_{k_0}})) \leq \psi(w_{n_{k_0}}) \leq \mu/2$, which leads to a contradiction because $\rho(\widetilde{\pi}(w_{n_{k_0}}, t_0^{n_{k_0}}), A) \geq \widetilde{\delta}$ and $\widetilde{\pi}(w_{n_{k_0}}, t_0^{n_{k_0}}) \notin M$ ($M \cap I(M) = \emptyset$).

Now we assume that $w_{n_{k_0}} \in M$. Then there is an $0 < \epsilon_{k_0} < t_0^{n_{k_0}}$ such that

$$\widetilde{\pi}(w_{n_{k_0}}, \epsilon_{k_0}) = \pi(w_{n_{k_0}}, \epsilon_{k_0}) \in B\big(A, \widetilde{\delta}/n_{k_0}\big) \setminus M \subset B(A, \eta) \setminus M.$$

Thus,

$$\psi(\widetilde{\pi}(w_{n_{k_0}}, \epsilon_{k_0})) \leq \frac{\mu}{2}$$

and

$$\psi(\widetilde{\pi}(w_{n_{k_0}}, t_0^{n_{k_0}})) = \psi(\widetilde{\pi}(\widetilde{\pi}(w_{n_{k_0}}, \epsilon_{k_0}), t_0^{n_{k_0}} - \epsilon_{k_0})) \leq \psi(\widetilde{\pi}(w_{n_{k_0}}, \epsilon_{k_0})) \leq \frac{\mu}{2}.$$

But this is a contradiction. Hence, there is a $\overline{\delta} > 0$, $0 < \overline{\delta} \leq \widetilde{\delta}$, such that $\widetilde{\pi}(B(A, \overline{\delta}), [0, +\infty)) \subset B(A, \widetilde{\delta})$. □

For the case of uniformly $\widetilde{\pi}$-stability, we have the following result.

Theorem 3.12. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$ be closed.*

(1) *If there exists a functional $\psi : X \to \mathbb{R}_+$ with the following properties:*
   (a) *ψ is continuous in $X \setminus (M \setminus A)$.*
   (b) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \varepsilon$ and $x \notin M$.*
   (c) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \leq \varepsilon$ wherever $\rho(x, A) \leq \delta$.*
   (d) *$\psi(\pi(x, t)) \leq \psi(x)$ if $x \in X \setminus M$ and $0 \leq t \leq \phi(x)$, and $\psi(I(x)) \leq \psi(x)$ if $x \in M$.*

   Then A is uniformly $\widetilde{\pi}$-stable.

(2) *Reciprocally, if A is uniformly $\widetilde{\pi}$-stable, then there is a functional $\psi : X \to \mathbb{R}_+$ satisfying conditions (a), (b), (c) and (d) above.*

Proof. (1) Note that $\psi(\widetilde{\pi}(x, t)) \leq \psi(x)$ for all $t \geq 0$ and $x \in X \setminus M$ (the proof is the same as in Theorem 3.9 item (1). Given $\varepsilon > 0$, in particular we have $\psi(\widetilde{\pi}(x, t)) \leq \psi(x)$ for all $t \geq 0$ and $x \in \overline{B(A, \varepsilon)} \setminus M$. By Lemma 3.11 there exists a $\delta > 0$ such that $\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \varepsilon)$. Therefore, A is uniformly $\widetilde{\pi}$-stable.

To prove condition (2), note that A is uniformly $\widetilde{\pi}$-stable, then A is $\widetilde{\pi}$-stable. Thus, the proof follows as in Theorem 3.7. □

Corollary 3.13. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system. A closed subset $A \subset X$ such that $M \subset A$ is uniformly $\widetilde{\pi}$-stable if and only if there exists a functional $\psi : X \to \mathbb{R}_+$ with the following properties:*

(a) *ψ is continuous.*
(b) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \geq \delta$ whenever $\rho(x, A) \geq \varepsilon$.*
(c) *for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\psi(x) \leq \varepsilon$ wherever $\rho(x, A) \leq \delta$.*

(d) $\psi(\pi(x,t)) \leq \psi(x)$ *if* $x \in X \setminus M$ *and* $t \geq 0$*, and,* $\psi(I(x)) \leq \psi(x)$ *if* $x \in M$*.*

Now, we present the result concerning asymptotically $\widetilde{\pi}$-stability. In this case we consider the stability of a compact set.

Theorem 3.14. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system,* X *be locally compact and* $A \subset X$ *be compact.*

1. *If there exists a functional* $\psi : X \to \mathbb{R}_+$ *with the following properties:*
   - (a) ψ *is continuous in* $X \setminus (M \setminus A)$*.*
   - (b) *for every* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that* $\psi(x) \leq \varepsilon$ *wherever* $\rho(x, A) \leq \delta$*.*
   - (c) *for every* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that* $\psi(x) \geq \delta$ *whenever* $\rho(x, A) \geq \varepsilon$ *and* $x \notin M$*.*
   - (d) $\psi(\pi(x,t)) \leq \psi(x)$ *if* $x \in X \setminus M$ *and* $0 \leq t \leq \phi(x)$*, and* $\psi(I(x)) \leq \psi(x)$ *if* $x \in M$*.*
   - (e) *there is a* $\delta > 0$ *such that if* $x \in B(A, \delta) \setminus A$*, then* $\psi(\widetilde{\pi}(x,t)) \to 0$ *as* $t \to +\infty$*.*

   Then A *is asymptotically* $\widetilde{\pi}$*-stable.*
2. *Reciprocally, if* A *is asymptotically* $\widetilde{\pi}$*-stable, then there is a functional* $\psi : X \to \mathbb{R}_+$ *satisfying conditions (a), (b), (c) , (d) and (e) above.*

Proof. (1) By Theorem 3.12, A is uniformly $\widetilde{\pi}$-stable, and by Theorem 3.4 A is orbitally $\widetilde{\pi}$-stable. The condition e) says that $B(A, \delta) \subset \widetilde{P}^+_W(A)$, then A is a weak $\widetilde{\pi}-$attractor. Hence, A is asymptotically $\widetilde{\pi}$-stable.

(2) Clearly the functional ψ given by

$$\psi(x) = \begin{cases} \sup_{k\geq 0}\Big(\sup_{0\leq t\leq \phi(x_k^+)} \frac{\rho(\pi(x_k^+,t),A)}{1+\rho(\pi(x_k^+,t),A)}\Big), & \text{if } x \in X \setminus M, \\ \psi(I(x)), & \text{if } x \in M, \end{cases}$$

where $x_0^+ = x$, satisfies the conditions of the theorem. □

Corollary 3.15. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system,* X *be locally compact,* $A \subset X$ *be compact and* $M \subset A$*. Then,* A *is asymptotically* $\widetilde{\pi}$*-stable if and only if there exists a functional* $\psi : X \to \mathbb{R}_+$*, with the following properties:*

- (a) ψ *is continuous.*
- (b) *for every* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that* $\psi(x) \leq \varepsilon$ *wherever* $\rho(x, A) \leq \delta$*.*
- (c) *for every* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that* $\psi(x) \geq \delta$ *whenever* $\rho(x, A) \geq \varepsilon$*.*
- (d) $\psi(\pi(x,t)) \leq \psi(x)$ *if* $x \in X \setminus M$ *and* $t \geq 0$*, and,* $\psi(I(x)) \leq \psi(x)$ *if* $x \in M$*.*
- (e) *there is a* $\delta > 0$ *such that if* $x \in B(A, \delta) \setminus A$*, then* $\psi(\widetilde{\pi}(x,t)) \to 0$ *as* $t \to +\infty$*.*

3.3. **Examples.** We apply the results above to two examples presented in [8].

Example 3.16. Let $X = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$. Consider the planar dynamical system

$$\dot{x} = -y,$$
$$\dot{y} = x.$$

Let $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, $A_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$ and $A = A_1 \cup A_2$. Set $M = \{(x,y) \in \mathbb{R}^2 : x = 0, -2 \leq y \leq -1\}$ and consider the impulsive function I given by $I((0,y)) = (0, y+3)$ for $y \in [-2,-1]$. Now define the function

$$\psi(x,y) = \begin{cases} 0, & \text{if } \sqrt{x^2+y^2} = 1, \\ \frac{\sqrt{x^2+y^2}-1}{\sqrt{x^2+y^2}}, & \text{if } 1 < \sqrt{x^2+y^2} \le \frac{3}{2}, \\ \frac{2-\sqrt{x^2+y^2}}{3-\sqrt{x^2+y^2}}, & \text{if } \frac{3}{2} < \sqrt{x^2+y^2} < 2, \\ 0, & \text{if } \sqrt{x^2+y^2} = 2. \end{cases}$$

Let us show that the set A is $\widetilde{\pi}$-stable. In order to do this, we are going to show that the three conditions of item (1) from Theorem 3.7 are satisfied. In fact:

(a) ψ is continuous in X and in particular in $X \setminus (M \setminus A)$.

(b) Given $\epsilon > 0$, let $(x,y) \in X$ be such that $\rho((x,y),A) \ge \varepsilon$ and $(x,y) \notin M$. Taking $\delta = \frac{\epsilon}{1+\epsilon}$, we have two cases to consider:

- If $\rho((x,y),A) \ge \varepsilon$ and $1 < \sqrt{x^2+y^2} \le \frac{3}{2}$, then

$$\frac{\sqrt{x^2+y^2}-1}{\sqrt{x^2+y^2}} \ge \frac{\epsilon}{1+\epsilon} = \delta.$$

- If $\rho((x,y),A) \ge \varepsilon$ and $\frac{3}{2} < \sqrt{x^2+y^2} < 2$, then

$$\frac{2-\sqrt{x^2+y^2}}{3-\sqrt{x^2+y^2}} \ge \frac{\epsilon}{1+\epsilon} = \delta.$$

Therefore, given $\epsilon > 0$ there is a $\delta > 0$ such that $\psi(x,y) \ge \delta$ whenever $\rho((x,y),A) \ge \varepsilon$ and $(x,y) \notin M$. On the other hand, for any sequence $\{(w_n,k_n)\}_{n\ge1} \subset X$ such that $(w_n,y_n) \overset{n\to+\infty}{\longrightarrow} (x,y) \in A$, we have $\psi((w_n,k_n)) \overset{n\to+\infty}{\longrightarrow} 0$, because $w_n^2 + k_n^2 \overset{n\to+\infty}{\longrightarrow} 1$ if $(x,y) \in A_1$ and $w_n^2 + k_n^2 \overset{n\to+\infty}{\longrightarrow} 4$ if $(x,y) \in A_2$.

(c) If $1 < \sqrt{x^2+y^2} \le \frac{3}{2}$, we have

$$\dot{\psi}(x,y) = \frac{-xy}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}} = 0$$

and if $\frac{3}{2} < \sqrt{x^2+y^2} < 2$,

$$\dot{\psi}(x,y) = \frac{xy}{(x^2+y^2)[3-\sqrt{x^2+y^2}]^2} + \frac{-xy}{(x^2+y^2)[3-\sqrt{x^2+y^2}]^2} = 0.$$

Therefore, $\dot{\psi}(x,y) = 0$ for all $(x,y) \in X$. Then $\psi(\pi((x,y),t)) = \psi((x,y))$ if $(x,y) \in X \setminus M$ and $0 \le t \le \phi((x,y))$. Furthermore, $\psi(I(x,y)) = \psi(x,y)$ if $(x,y) \in M$. By Theorem 3.7, A is $\widetilde{\pi}$-stable.

Example 3.17. Consider the space $X = \mathbb{R}^2 \times \{0,1\}$ and the dynamical system

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= -y, \end{aligned} \tag{3.8}$$

on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$, independently. Now let $M_0 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1, z = 0\}$, $M_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1/4, z = 1\}$ and $M = M_0 \cup M_1$. We define $I(x,y,0) = (x,y,1)$ for $(x,y,0) \in M_0$ and $I(x,y,1) = (x,y,0)$ for $(x,y,1) \in M_1$. Take $A_0 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\} \times \{0\}$, $A_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\} \times \{1\}$

and $A = A_0 \cup A_1$. We claim that the set A is $\widetilde{\pi}-$asymptotically stable. In fact. Consider the function

$$\psi(x,y,z) = \begin{cases} \frac{\sqrt{x^2+y^2}-1}{\sqrt{x^2+y^2}}, & \text{if } \sqrt{x^2+y^2} > 1 \text{ and } z \in \{0,1\}, \\ 0, & \text{if } \sqrt{x^2+y^2} \le 1 \text{ and } z \in \{0,1\}. \end{cases}$$

We are going to use Corollary 3.15 to prove it, because $M \subset A$. Let us start by verifying the five conditions of Corollary 3.15:

(a) It is easy to see that the function ψ is continuous on X.

(b) Given $\epsilon > 0$, $\epsilon < 1$, if $\rho((x,y,0), A_0) < \frac{\epsilon}{1-\epsilon}$ then

$$\sqrt{x^2+y^2} - 1 \le \frac{\epsilon}{1-\epsilon} \Rightarrow 1 - \frac{1}{\sqrt{x^2+y^2}} \le \epsilon.$$

Thus, $\psi(x,y,0) < \epsilon$. Analogously, if $\rho((x,y,1), A_1) < \frac{\epsilon}{1-\epsilon}$ then $\psi(x,y,1) < \epsilon$. Hence, given $\epsilon > 0$, there is $\delta > 0$ such that $\psi(x,y,z) \le \epsilon$ whenever $\rho((x,y,z), A) \le \delta$.

(c) Given $\epsilon > 0$, there is a $\delta = \frac{\epsilon}{1+\epsilon}$ such that

$$\psi(x,y,z) \ge \frac{\epsilon}{1+\epsilon} := \delta,$$

whenever $\rho((x,y,z), A) \ge \epsilon$.

(d) Consider the two flows $\varphi_1((x_0,y_0,0),t) = (x(x_0,t), y(y_0,t), 0)$ and $\varphi_2((x_0,y_0,1),t) = (x(x_0,t), y(y_0,t), 1)$ such that $(x(t),y(t)) = (x(x_0,t), y(y_0,t))$ satisfies system (3.8) and $(x(0),y(0)) = (x_0,y_0)$. Let $z_0 = (x_0,y_0,0)$ and $w_0 = (x_0,y_0,1)$.

If $\sqrt{x^2+y^2} > 1$, we have

$$\dot{\psi}(\varphi_1(z_0,t)) = \frac{\partial\psi}{\partial x}\dot{x}(x_0,t) + \frac{\partial\psi}{\partial y}\dot{y}(y_0,t) + \frac{\partial\psi}{\partial z}\dot{z}(z_0,t) = -\frac{1}{\sqrt{x^2(x_0,t)+y^2(y_0,t)}} < 0$$

and

$$\dot{\psi}(\varphi_2(w_0,t)) = -\frac{x^2(x_0,t)+y^2(y_0,t)}{\sqrt{x^2(x_0,t)+y^2(y_0,t)}\left[\frac{1}{2}+\sqrt{x^2(x_0,t)+y^2(y_0,t)}\right]^2} < 0,$$

for $0 \le t < \phi(z_0)$. Hence, $\dot{\psi}(\varphi_1(z_0,t)) \le 0$ and $\dot{\psi}(\varphi_2(w_0,t)) \le 0$ whenever $(x_0,y_0) \in \mathbb{R}^2$ and $t \ge 0$. Then, $\psi(\varphi_1(z_0,t)) \le \psi(z_0)$ and $\psi(\varphi_2(w_0,t)) \le \psi(w_0)$ whenever $(x_0,y_0) \in \mathbb{R}^2$ and $t \ge 0$.

Since $\psi(x,y,z) = 0$ for each $(x,y,z) \in A$ and $I(x,y,z) \subset$ A for $(x,y,z) \in M$, we have $\psi(I(x,y,z)) = \psi(x,y,z) = 0$ if $(x,y,z) \in M$.

(e) Let φ_1 and φ_2 be flows in $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ respectively. Now, consider $\widetilde{\pi}$ an impulsive flow in X obtained from φ_1 and φ_2. By Corollary 3.8, A is $\widetilde{\pi}$-stable and by Theorem 3.5, A is uniformly $\widetilde{\pi}$-stable. Then, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\widetilde{\pi}(B(A,\delta)) \subset B(A,\epsilon).$$

Let $(x_0,y_0,0) \in B(A_1,\delta) \setminus A_1$. Since

$$\frac{d}{dt}\psi(\varphi_1((x_0,y_0,0),t)) \le 0$$

for $t \ge 0$, the limit $\lim_{t\to+\infty} \psi(\varphi_1((x_0,y_0,0),t))$ exists.

Suppose $\lim_{t\to+\infty}\psi(\varphi_1((x_0,y_0,0),t)) := \ell > 0$. Define $K = \{(x,y,z) \in \mathbb{R}^2 \times \{0\} : \rho((x,y,z),A_1) \le \epsilon$ and $\psi(x,y,z) \ge \ell\}$. It is clear that K is compact. Note $\varphi_1((x_0,y_0,0),t) \in K$ for all $t \ge 0$. Now, define

$$\eta := \min\{-\dot{\psi}(w) : w \in K\}.$$

Since A is not contained in K, then $\eta > 0$. Thus

$$-\dot{\psi}(\varphi_1((x_0,y_0,0),t)) \ge \eta$$

for all $t \ge 0$. Then, integrating the inequality above from 0 to t, we have

$$\psi(\varphi_1((x_0,y_0,0),t)) \le \psi(x_0,y_0,0) - \eta t,$$

for all $t \ge 0$, which is a contradiction since ψ is positive. Therefore, $\ell = 0$.

If $(x_0,y_0,0) \in B(A_2,\delta) \setminus A_2$, the result follows analogously. By Corollary 3.15, A is $\tilde{\pi}$-asymptotically stable.

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