Electronic Journal of Differential Equations, Vol. 2010(2010), No. 85, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SECOND-ORDER DIFFERENTIAL INCLUSIONS WITH LIPSCHITZ RIGHT-HAND SIDES

DALILA AZZAM-LAOUIR, FATIHA BOUNAMA

Abstract. We study the existence of solutions of a three-point boundary-value problem for a second-order differential inclusion,

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. } t \in [0,1], \\ u(0) = 0, \quad u(\theta) = u(1). \end{gathered}$$

Here F is a set-valued mapping from $[0,1] \times E \times E$ to E with nonempty closed values satisfying a standard Lipschitz condition, and E is a separable Banach space.

1. Introduction

We study the existence of solutions to the second-order differential inclusion

$$\begin{gathered} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. } t \in [0,1], \\ u(0) = 0, \quad u(\theta) = u(1), \end{gathered} \tag{1.1}$$

where $F : [0,1] \times E \times E \to E$ is a nonempty closed valued multifunction and θ is a given number in $[0,1]$. Existence of solutions for (1.1) has been investigated by many authors [2, 3, 4, 9] under the assumption that F is a convex bounded-valued multifunction upper semicontinuous on $E \times E$ and integrably compact.

The aim of this article is to provide existence of solutions for (1.1) under the standard Lipschitz condition for the multifunction F, when it is nonconvex.

After some preliminaries in section 3, we present our main result which is the existence of $\mathbf{W}^{2,1}_E([0,1])$-solutions for (1.1). We suppose that F is a closed valued multifunction satisfying the Lipschitz condition

$$\mathcal{H}(F(t, x_1, y_1), F(t, x_2, y_2)) \le k_1(t)\|x_1 - y_1\| + k_2(t)\|x_2 - y_2\|$$

where $\mathcal{H}(\cdot,\cdot)$ stands for the Hausdorff distance.

For first-order differential inclusions satisfying the standard Lipschitz condition we refer the reader to [5, 6, 7, 10] and the references therein.

2000 *Mathematics Subject Classification.* 34A60, 34B15, 47H10.
Key words and phrases. Differential inclusion; Lipschitz multifunction.

Submitted March 29, 2010. Published June 18, 2010.

2. Notation and preliminaries

In this article, $(E, \|\cdot\|)$ is a separable Banach space and E' is its topological dual, $\overline{\mathbf{B}}_E$ is the closed unit ball of E, $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue-measurable sets of $[0,1]$, $\lambda = dt$ is the Lebesgue measure on $[0,1]$, and $\mathcal{B}(E)$ is the σ-algebra of Borel subsets of E. By $L^1_E([0,1])$, we denote the space of all Lebesgue-Bochner integrable E-valued mappings defined on $[0,1]$.

Let $\mathbf{C}_E([0,1])$ be the Banach space of all continuous mappings $u : [0,1] \to E$, endowed with the supremum norm, and let $\mathbf{C}^1_E([0,1])$ be the Banach space of all continuous mappings $u : [0,1] \to E$ with continuous derivative, equipped with the norm

$$\|u\|_{\mathbf{C}^1} = \max\{\max_{t\in[0,1]} \|u(t)\|, \max_{t\in[0,1]} \|\dot{u}(t)\|\}.$$

Recall that a mapping $v : [0,1] \to E$ is said to be scalarly derivable when there exists some mapping $\dot{v} : [0,1] \to E$ (called the weak derivative of v) such that, for every $x' \in E'$, the scalar function $\langle x', v(\cdot)\rangle$ is derivable and its derivative is equal to $\langle x', \dot{v}(\cdot)\rangle$. The weak derivative $\ddot{v}$ of $\dot{v}$ when it exists is the weak second derivative.

By $\mathbf{W}^{2,1}_E([0,1])$ we denote the space of all continuous mappings $u \in \mathbf{C}_E([0,1])$ such that their first usual derivatives are continuous and scalarly derivable and such that $\ddot{u} \in L^1_E([0,1])$.

For closed subsets A and B of E, the Hausdorff distance between A and B is defined by

$$\mathcal{H}(A,B) = \sup(e(A,B)), e(B,A))$$

where

$$e(A,B) = \sup_{a\in A} d(a,B) = \sup_{a\in A}(\inf_{b\in B} \|a-b\|)$$

stands for the excess of A over B.

3. Existence results under Lipschitz condition

We begin with a proposition that summarizes some properties of some Green type function (see [1, 2, 8, 9]). It will use it in the study of our boundary value problems.

Proposition 3.1. *Let E be a separable Banach space and let $G : [0,1]\times[0,1] \to \mathbb{R}$ be the function defined by*

$$G(t,s) = \begin{cases} -s & \text{if } 0 \le s \le t,\\ -t & \text{if } t < s \le \theta,\\ t(s-1)/(1-\theta) & \text{if } \theta < s \le 1, \end{cases}$$

if $0 \le t < \theta$, and

$$G(t,s) = \begin{cases} -s & \text{if } 0 \le s < \theta,\\ (\theta(s-t)+s(t-1))/(1-\theta) & \text{if } \theta \le s \le t,\\ t(s-1)/(1-\theta) & \text{if } t < s \le 1, \end{cases}$$

if $\theta \le t \le 1$. Then the following assertions hold.

(1) *If $u \in \mathbf{W}^{2,1}_E([0,1])$ with $u(0) = 0$ and $u(\theta) = u(1)$, then*

$$u(t) = \int_0^1 G(t,s)\ddot{u}(s)ds, \quad \forall t \in [0,1].$$

(2) *$G(\cdot, s)$ is derivable on $[0,1]$ for every $s \in [0,1]$, and its derivative is given by*

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & \text{if } 0 \le s \le t, \\ -1 & \text{if } t < s \le \theta, \\ (s-1)/(1-\theta) & \text{if } \theta < s \le 1, \end{cases}$$

if $0 \le t < \theta$, and

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & \text{if } 0 \le s < \theta, \\ (s-\theta)/(1-\theta) & \text{if } \theta \le s \le t, \\ (s-1)/(1-\theta) & \text{if } t < s \le 1, \end{cases}$$

if $\theta \le t \le 1$.

(3) *$G(\cdot,\cdot)$, and $\frac{\partial G}{\partial t}(\cdot,\cdot)$ satisfy*

$$\sup_{t,s\in[0,1]} |G(t,s)| \le 1, \quad \sup_{t,s\in[0,1]} |\frac{\partial G}{\partial t}(t,s)| \le 1. \tag{3.1}$$

(4) *For $f \in L^1_E([0,1])$ and for the mapping $u_f : [0,1] \to E$ defined by*

$$u_f(t) = \int_0^1 G(t,s) f(s) ds, \quad \forall t \in [0,1], \tag{3.2}$$

one has $u_f(0) = 0$ and $u_f(\theta) = u_f(1)$. Furthermore, the mapping u_f is derivable, and its derivative $\dot{u}_f$ satisfies

$$\lim_{h\to 0} \frac{u_f(t+h) - u_f(t)}{h} = \dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s) ds \tag{3.3}$$

for all $t \in [0,1]$. Consequently, $\dot{u}_f$ is a continuous mapping from $[0,1]$ into the space E.

(5) *The mapping $\dot{u}_f$ is scalarly derivable; that is, there exists a mapping $\ddot{u}_f : [0,1] \to E$ such that, for every $x' \in E'$, the scalar function $\langle x', \dot{u}_f(\cdot)\rangle$ is derivable with $\frac{d}{dt}\langle x', \dot{u}_f(t)\rangle = \langle x', \ddot{u}_f(t)\rangle$; furthermore*

$$\ddot{u}_f = f \quad \text{a.e. on } [0,1]. \tag{3.4}$$

Let us mention a useful consequence of Proposition 3.1.

Proposition 3.2. *Let E be a separable Banach space and let $f : [0,1] \to E$ be a continuous mapping (respectively a mapping in $L^1_E([0,1])$). Then the mapping*

$$u_f(t) = \int_0^1 G(t,s) f(s) ds, \quad \forall t \in [0,1]$$

is the unique $\mathbf{C}^2_E([0,1])$-solution (respectively $\mathbf{W}^{2,1}_E([0,1])$-solution) to the differential equation

$$\ddot{u}(t) = f(t), \quad \forall t \in [0,1],$$
$$u(0) = 0, \quad u(\theta) = u(1).$$

Now we are able to state and prove our main result. The approach below used some techniques and arguments from [2, 6, 7].

Theorem 3.3. *Let E be a separable Banach space and let $F:[0,1]\times E\times E\to E$ be a measurable multifunction with nonempty closed values. Let $g\in L^1_E([0,1])$ and let $u_g:[0,1]\to E$ be the mapping defined by*

$$u_g(t)=\int_0^1 G(t,s)g(s)ds,\quad \forall t\in[0,1].$$

Assume that for some $r\in]0,+\infty]$ and

$$\mathbf{X}_r=\{(t,x,y)\in[0,1]\times E\times E:\|x-u_g(t)\|<r;\ \|y-\dot u_g(t)\|<r\},$$

the following conditions hold:

(i) *there exist two functions $k_1,k_2\in L^1_{\mathbb{R}}([0,1])$ with $k_1(t)\ge 0$ and $k_2(t)\ge 0$ satisfying $\|k_1+k_2\|_{L^1_{\mathbb{R}}}<1$ such that*

$$\mathcal{H}(F(t,x_1,y_1),F(t,x_2,y_2))\le k_1(t)\|x_1-x_2\|+k_2(t)\|y_1-y_2\|$$

for all (t,x_1,y_1), $(t,x_2,y_2)\in\mathbf{X}_r$;

(ii) *there is $\eta\in L^1_{\mathbb{R}}([0,1])$ satisfying $\|\eta\|_{L^1_{\mathbb{R}}}<[1-\|k_1+k_2\|_{L^1_{\mathbb{R}}}]r$, such that*

$$d(g(t),F(t,u_g(t),\dot u_g(t)))\le\eta(t),\quad\forall t\in[0,1].$$

Then the differential inclusion (1.1) has at least one solution $u\in\mathbf{W}^{2,1}_E([0,1])$, with

$$\|u(t)\|\le r+\|g(t)\|,\quad \|\dot u(t)\|\le r+\|g(t)\|,\quad\forall t\in[0,1].$$

Proof. Step 1. Since $\|k_1+k_2\|_{L^1_{\mathbb{R}}}<1$ and $\|\eta\|_{L^1_{\mathbb{R}}}<[1-\|k_1+k_2\|_{L^1_{\mathbb{R}}}]r$ we may choose some real number $\alpha>0$ satisfying

$$(1+\alpha)\|k_1+k_2\|_{L^1_{\mathbb{R}}}<1,\quad (1+\alpha)\|\eta\|_{L^1_{\mathbb{R}}}<[1-(1+\alpha)\|k_1+k_2\|_{L^1_{\mathbb{R}}}]r. \tag{3.5}$$

We will define a sequence of mappings f_n, $n\in\mathbb{N}$, of $L^1_E([0,1])$ such that the following conditions are fulfilled (see (3.2) for the definition of u_f).

$$f_n\in L^1_E([0,1]),\quad f_n(t)\in F(t,u_{f_{n-1}}(t),\dot u_{f_{n-1}}(t)),\quad \text{a.e. } t\in[0,1]; \tag{3.6}$$

$$\|f_n(t)-f_{n-1}(t)\|\le(1+\alpha)d(f_{n-1}(t),F(t,u_{f_{n-1}}(t),\dot u_{f_{n-1}}(t))),\quad\forall t\in[0,1]; \tag{3.7}$$

$$\operatorname{gph}(u_{f_n}(\cdot),\dot u_{f_n}(\cdot))=\{(u_{f_n}(t),\dot u_{f_n}(t)):\ t\in[0,1]\}\subset\mathbf{X}_r. \tag{3.8}$$

We put $f_0=g$ and $u_{f_0}(t)=\int_0^1 G(t,s)f_0(s)ds=u_g(t)$, for all $t\in[0,1]$. Let us consider the multifunction $H_0:[0,1]\to E$ defined by

$$H_0(t)=\{v\in F(t,u_{f_0}(t),\dot u_{f_0}(t)):\|v-f_0(t)\|\le(1+\alpha)d(f_0(t),F(t,u_{f_0}(t),\dot u_{f_0}(t)))\}.$$

Observe first that $H_0(t)\neq\emptyset$ for any $t\in[0,1]$.

Since $F(\cdot,u_{f_0}(\cdot),\dot u_{f_0}(\cdot))$ is measurable, the multifunction H_0 is also measurable with nonempty closed values. In view of the existence theorem of measurable selections (see [5]), there is a measurable mapping $f_1:[0,1]\to E$ such that $f_1(t)\in H_0(t)$, for all $t\in[0,1]$. This yields, for all $t\in[0,1]$, $f_1(t)\in F(t,u_{f_0}(t),\dot u_{f_0}(t))$ and $\|f_1(t)-f_0(t)\|\le(1+\alpha)d(f_0(t),F(t,u_{f_0}(t),\dot u_{f_0}(t)))$, and hence according to the assumption (ii),

$$\|f_1(t)-f_0(t)\|\le(1+\alpha)\eta(t).$$

So, we have

$$\|f_1(t)\|\le\|f_1(t)-f_0(t)\|+\|f_0(t)\|\le(1+\alpha)\eta(t)+\|f_0(t)\|. \tag{3.9}$$

Since $\eta \in L^1_{\mathbb{R}}([0,1])$ and $f_0 \in L^1_E([0,1])$, the last inequality shows that $f_1 \in L^1_E([0,1])$. Then we define the mapping $u_{f_1} : [0,1] \to E$ by

$$u_{f_1}(t) = \int_0^1 G(t,s) f_1(s) ds, \quad \forall t \in [0,1],$$

and by relation (3.3) in Proposition 3.1

$$\dot{u}_{f_1}(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f_1(s) ds, \quad \forall t \in [0,1].$$

On the other hand,

$$\begin{aligned}
\|u_{f_1}(t) - u_{f_0}(t)\| &= \| \int_0^1 G(t,s)(f_1(s) - f_0(s)) ds\| \\
&\le \int_0^1 \|f_1(s) - f_0(s)\| ds \\
&\le (1+\alpha) \int_0^1 d(f_0(s), F(t, u_{f_0}(s), \dot{u}_{f_0}(s))) ds \\
&\le (1+\alpha)\|\eta\|_{L^1_{\mathbb{R}}} \\
&< [1 - (1+\alpha)\|k_1 + k_2\|_{L^1_E}] r < r,
\end{aligned}$$

the first inequality being due to (3.1) and the fourth one to (3.5). Similarly we have

$$\|\dot{u}_{f_1}(t) - \dot{u}_{f_0}(t)\| = \| \int_0^1 \frac{\partial G}{\partial t}(t,s)(f_1(s) - f_0(s)) ds\| \le \int_0^1 \|f_1(s) - f_0(s)\| ds < r.$$

This shows that $\mathrm{gph}(u_{f_1}(\cdot), \dot{u}_{f_1}(\cdot)) \subset \mathbf{X}_r$.

Suppose that f_i and u_{f_i} have been defined on $[0,1]$ satisfying (3.6), (3.7) and (3.8) for $i = 0, 1, \dots, n$. Let us consider the multifunction $H_n : [0,1] \to E$ defined by

$$\begin{aligned}
H_n(t) = \big\{ v \in F(t, u_{f_n}(t), \dot{u}_{f_n}(t)) : \|v - f_n(t)\| \le (1+\alpha) d(f_n(t), \\
F(t, u_{f_n}(t), \dot{u}_{f_n}(t)) \big\}.
\end{aligned}$$

Observe first that $H_n(t) \neq \emptyset$ for any $t \in [0,1]$.

Since $F(\cdot, u_{f_n}(\cdot), \dot{u}_{f_n}(\cdot))$ is measurable, the multifunction H_n is also measurable with nonempty closed values. As above, in view of the existence theorem of measurable selections (see [5]), there is a measurable mapping $f_{n+1} : [0,1] \to E$ such that $f_{n+1}(t) \in H_n(t)$, for all $t \in [0,1]$. This yields for all $t \in [0,1]$, $f_{n+1}(t) \in F(t, u_{f_n}(t), \dot{u}_{f_n}(t))$ and $\|f_{n+1}(t) - f_n(t)\| \le (1+\alpha) d(f_n(t), F(t, u_{f_n}(t), \dot{u}_{f_n}(t))$. The second inequality implies

$$\begin{aligned}
&\|f_{n+1}(t) - f_n(t)\| \\
&\le (1+\alpha) d(f_n(t), F(t, u_{f_n}(t), \dot{u}_{f_n}(t))) \\
&\le (1+\alpha)\mathcal{H}(F(t, u_{f_{n-1}}(t), \dot{u}_{f_{n-1}}(t)), F(t, u_{f_n}(t), \dot{u}_{f_n}(t))) \\
&\le (1+\alpha)[k_1(t)\|u_{f_n}(t) - u_{f_{n-1}}(t)\| + k_2(t)\|\dot{u}_{f_n}(t) - \dot{u}_{f_{n-1}}(t)\|],
\end{aligned} \tag{3.10}$$

where the last inequality follows from assumption (i). However, the mapping f_{n-1} and f_n being integrable by the induction assumption, we have on the one hand

$$\begin{aligned}\|u_{f_n}(t) - u_{f_{n-1}}(t)\| &= \| \int_0^1 G(t,s) f_n(s) ds - \int_0^1 G(t,s) f_{n-1}(s) ds\| \\ &\le \int_0^1 |G(t,s)| \|f_n(s) - f_{n-1}(s)\| ds \\ &\le \|f_n - f_{n-1}\|_{L^1_E},\end{aligned}$$

where the last inequality follows from the first inequality in (3.1). On the other hand using the second inequality in (3.6), we may write

$$\begin{aligned}\|\dot{u}_{f_n}(t) - \dot{u}_{f_{n-1}}(t)\| &= \| \int_0^1 \frac{\partial G}{\partial t}(t,s) f_n(s) ds - \int_0^1 \frac{\partial G}{\partial t}(t,s) f_{n-1}(s) ds\| \\ &\le \int_0^1 |\frac{\partial G}{\partial t}(t,s)| \|f_n(s) - f_{n-1}(s)\| ds \\ &\le \|f_n - f_{n-1}\|_{L^1_E}.\end{aligned}$$

Combining those last inequalities and (3.10), we obtain

$$\|f_{n+1}(t) - f_n(t)\| \le (1+\alpha)(k_1(t) + k_2(t)) \|f_n - f_{n-1}\|_{L^1_E}. \tag{3.11}$$

Since k_1, $k_2 \in L^1_{\mathbb{R}}([0,1])$ and $f_n, f_{n-1} \in L^1_E([0,1])$, we see that $f_{n+1} \in L^1_E([0,1])$. We may then integrate (3.11),

$$\begin{aligned}\int_0^1 \|f_{n+1}(t) - f_n(t)\| dt &\le (1+\alpha) \int_0^1 (k_1(t) + k_2(t)) \|f_n - f_{n-1}\|_{L^1_E} dt \\ &= (1+\alpha) \|k_1 + k_2\|_{L^1_{\mathbb{R}}} \|f_n - f_{n-1}\|_{L^1_E};\end{aligned}$$

that is,

$$\|f_{n+1} - f_n\|_{L^1_E} \le (1+\alpha) \|k_1 + k_2\|_{L^1_{\mathbb{R}}} \|f_n - f_{n-1}\|_{L^1_E}. \tag{3.12}$$

Taking (3.2) into account, we define the mapping $u_{f_{n+1}} : [0,1] \to E$ by

$$u_{f_{n+1}}(t) = \int_0^1 G(t,s) f_{n+1}(s) ds, \quad \forall t \in [0,1],$$

and relation (3.3) in Proposition 3.1 says that $u_{f_{n+1}}$ is derivable with

$$\dot{u}_{f_{n+1}}(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f_{n+1}(s) ds, \quad \forall t \in [0,1].$$

Next, let us prove that the graph of $(u_{f_{n+1}}(\cdot), \dot{u}_{f_{n+1}}(\cdot))$ is contained in $\mathbf{X}_r$. Setting $\gamma = (1+\alpha)\|k_1 + k_2\|_{L^1_{\mathbb{R}}}$ and using successively relation (3.12), we obtain

$$\|f_{n+1} - f_n\|_{L^1_E} \le \gamma^n \|f_1 - f_0\| \le \gamma^n (1+\alpha) \|\eta\|_{L^1_{\mathbb{R}}} \tag{3.13}$$

with $\gamma < 1$, the last inequality being due to (3.9). On the other hand, since

$$\begin{aligned}\|u_{f_{n+1}}(t) - u_{f_n}(t)\| &\le \|f_{n+1} - f_n\|_{L^1_E}, \\ \|\dot{u}_{f_{n+1}}(t) - \dot{u}_{f_n}(t)\| &\le \|f_{n+1} - f_n\|_{L^1_E},\end{aligned}$$

(3.13) yields

$$\|u_{f_{n+1}} - u_{f_n}\|_{\mathbf{C}^1} \le \|f_{n+1} - f_n\|_{L^1_E} \le \gamma^n (1+\alpha) \|\eta\|_{L^1_{\mathbb{R}}}. \tag{3.14}$$

Writing,

$$\begin{aligned}\|u_{f_{n+1}}(t)-u_{f_0}(t)\| &\le \|u_{f_{n+1}}(t)-u_{f_n}(t)\|+\|u_{f_n}(t)-u_{f_0}(t)\|\\ &\le \gamma^n(1+\alpha)\|\eta\|_{L^1_{\mathbb{R}}}+\|u_{f_n}(t)-u_{f_0}(t)\|,\end{aligned}$$

and using successively this relation, we obtain thanks to the second inequality of (3.11),

$$\|u_{f_{n+1}}(t)-u_{f_0}(t)\| \le (\sum_{p=0}^{n}\gamma^p)(1+\alpha)\|\eta\|_{L^1_{\mathbb{R}}} \le \frac{1}{1-\gamma}(1+\alpha)\|\eta\|_{L^1_{\mathbb{R}}} < r. \tag{3.15}$$

Using again (3.14) to write

$$\begin{aligned}\|\dot u_{f_{n+1}}(t)-\dot u_{f_0}(t)\| &\le \|\dot u_{f_{n+1}}(t)-\dot u_{f_n}(t)\|+\|\dot u_{f_n}(t)-\dot u_{f_0}(t)\|\\ &\le \gamma^n(1+\alpha)\|\eta\|_{L^1_{\mathbb{R}}}+\|\dot u_{f_n}(t)-\dot u_{f_0}(t)\|,\end{aligned}$$

we obtain, in a similar way,

$$\|\dot u_{f_{n+1}}(t)-\dot u_{f_0}(t)\| < r. \tag{3.16}$$

Consequently the sequences (f_n) and (u_{f_n}) are well defined satisfying (3.6), (3.7) and (3.8).

Step 2. By (3.13) we see that (f_n) is a Cauchy sequence in $L^1_E([0,1])$, hence it converges to some mapping $f \in L^1_E([0,1])$. In the same way (3.14) shows that (u_{f_n}) is a Cauchy sequence in $\mathbf{C}^1_E([0,1])$, consequently it converges to some mapping $w \in \mathbf{C}^1_E([0,1])$. Observe that

$$\begin{aligned}\|u_{f_n}(t)-u_f(t)\| &= \|\int_0^1 G(t,s)f_n(s)ds-\int_0^1 G(t,s)f(s)ds\|\\ &\le \int_0^1\|f_n(s)-f(s)\|ds = \|f_n-f\|_{L^1_E},\end{aligned}$$

and

$$\begin{aligned}\|\dot u_{f_n}(t)-\dot u_f(t)\| &= \|\int_0^1 \frac{\partial G}{\partial t}(t,s)f_n(s)ds-\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds\|\\ &\le \int_0^1\|f_n(s)-f(s)\|ds = \|f_n-f\|_{L^1_E},\end{aligned}$$

which, according to the strong convergence in $L^1_E([0,1])$ of (f_n) to the mapping f means that (u_{f_n}) converges in $(\mathbf{C}^1_E([0,1]),\|\cdot\|_{\mathbf{C}^1})$ to u_f. Thus we get $w=u_f$, and by Proposition 3.1 (relations (3.2), (3.3) and (3.4)) we have $\ddot u_f = f$, with $u_f(0)=0$, $u_f(\theta)=u_f(1)$.

Let us prove now that u_f is a solution of the problem (1.1). For this purpose, let us prove that, for each $t\in[0,1]$, the graph of the multifunction $(x,y)\mapsto F(t,x,y)$ is closed relatively to $\mathbf{X}_r(t)\times E$ where

$$\mathbf{X}_r(t)=\{(x,y)\in E\times E:\ (t,x,y)\in\mathbf{X}_r\}.$$

Let $(x_n,y_n,v_n)_n$ be a sequence in $\mathrm{gph}(F(t,\cdot,\cdot))$ converging to $(x,y,v)\in\mathbf{X}_r(t)\times E$. For each integer n, $v_n\in F(t,x_n,y_n)$, and hence

$$\begin{aligned}d(v,F(t,x,y)) &\le \|v-v_n\|+d(v_n,F(t,x,y))\\ &\le \|v-v_n\|+\mathcal{H}(F(t,x_n,y_n),F(t,x,y))\\ &\le \|v-v_n\|+k_1(t)\|x_n-x\|+k_2(t)\|y_n-y\|.\end{aligned}$$

Since the last member goes to 0 as n tends to $+\infty$, this says that $v \in F(t, x, y)$ according to the closedness of this set. Consequently the graph of $F(t, \cdot, \cdot)$ is closed relatively to $\mathbf{X}_r(t) \times E$. Since (f_n) converges to f strongly in $L^1_E([0,1])$, by extracting a subsequence we may suppose that (f_n) converges to f almost everywhere on $[0,1]$. As $f_{n+1}(t) \in F(t, u_{f_n}(t), \dot{u}_{f_n}(t))$ and as (u_{f_n}) converges to u_f in $\mathbf{C}^1_E([0,1])$ and $(t, u_{f_n}(t), \dot{u}_{f_n}(t))$, $(t, u_f(t), \dot{u}_f(t)) \in \mathbf{X}_r$, we conclude that $f(t) \in F(t, u_f(t), \dot{u}_f(t))$, a.e., equivalently $\ddot{u}_f(t) \in F(t, u_f(t), \dot{u}_f(t))$, a.e., with $u_f(0) = 0$; $u_f(\theta) = u_f(1)$. Furthermore, the relations (3.15) and (3.16) show that

$$\|u_f(t)\| \le r + \|g(t)\|, \quad \|\dot{u}_f(t)\| \le r + \|g(t)\|, \quad \forall t \in [0,1].$$

This completes the proof of our theorem. $\square$

The following corollary translates the above result in a more amenable way.

Corollary 3.4. *Let E be a separable Banach space and $F : [0,1] \times E \times E \to E$ be a measurable multifunction with nonempty closed values such that*

(i) *there exist two functions k_1, $k_2 \in L^1_{\mathbb{R}}([0,1])$ with $k_1(t) \ge 0$ and $k_2(t) \ge 0$ satisfying $\|k_1 + k_2\|_{L^1_{\mathbb{R}}} < 1$ such that*

$$\mathcal{H}(F(t, x_1, y_1), F(t, x_2, y_2)) \le k_1(t)\|x_1 - x_2\| + k_2(t)\|y_1 - y_2\|$$

for all (t, x_1, y_1), $(t, x_2, y_2) \in [0,1] \times E \times E$;

(ii) *the function $t \mapsto d(0, F(t, 0, 0))$ is integrable.*

Then the differential inclusion (1.1) has at least a solution $u \in \mathbf{W}^{2,1}_E([0,1])$.

Proof. Taking $g \equiv 0$ and $r = +\infty$, we see in Theorem 3.3 that $\mathbf{X}_r = [0,1] \times E \times E$. Further putting $\eta(t) = d(0, F(t, 0, 0))$, the function η is integrable and satisfies the assumption (ii) of Theorem 3.3. We may then conclude that the corollary is a consequence of Theorem 3.3. $\square$

Acknowledgements. The authors would like to thank Professor L. Thibault for his important comments and suggestions.

References

[1] D. Azzam-Laouir.; Contribution à l'étude de problèmes d'évolution du second ordre. *Thèse de doctorat d'état, Université de Constantine.*, 2003.

[2] D. Azzam-Laouir, C. Castaing and L. Thibault; Three boundary value problems for second order differential inclusions in Banach spaces. *Control Cybernet, Vol.* 31 No. 3 (2002), pp. 659-693.

[3] D. Azzam-Laouir, S. Lounis; Existence of solutions for a class of second order differential inclusions. *J. Nonlinear Convex Anal., Vol* 6 no. 2(2005), 339-346.

[4] D. Azzam-Laouir, S. Lounis and L. Thibault; Existence solutions for second-order differential inclusions with nonconvex perturbations. *Applicable Analysis* Vol. 86, No. 10 (2007), 1199-1210.

[5] C. Castaing, M. Valadier; Convex analysis and measurable multifunctions. *Lectures Notes in Math.*, 580 springer-Verlag, Berlin, 1977.

[6] A. F. Filippov; Classical solutions of differential inclusions with multivalued right-hand sides, SIAM J. Control 5 (1967), 609-621.

[7] B. Gely; Solutions d'équations différentielles multivoques. *Séminaire d'analyse convexe* Montpellier, Exposé no 4, 1972.

[8] P. Hartman; Ordinary differential equations. *John Wiley and Sons Inc.,* New York, 1964,.

[9] A. G. Ibrahim and A. M. M. Gomaa; Existence theorems for functional multivalued three-point boundary value problem of second order. *J. Egypt. Math. Soc.* 8(2) (2000), 155-168.

[10] T. Wazewski; Sur une généralisation de la notion de solution d'une équation au contingent, *Bull. Pol. Ac. Sci.* 10 (1962), 11-15.

Dalila Azzam-Laouir
Laboratoire de Mathématiques Pures et Appliquées, Département de Mathématiques, Université de Jijel, Algérie
E-mail address: laouir.dalila@gmail.com

Fatiha Bounama
Laboratoire de Mathématiques Pures et Appliquées, Département de Mathématiques, Université de Jijel, Algérie
E-mail address: bounamaf@yahoo.fr