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# A CONTINUITY ARGUMENT FOR A SEMILINEAR SKYRME MODEL

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Abstract. We investigate a semilinear modification for the wave map problem proposed by Adkins and Nappi [1], and prove that in the equivariant case the solution remain continuous at the first possible singularity. This is usually one of the steps in proving existence of global smooth solutions for certain equations.

## 1. Introduction

Let $\phi : \mathbb{R}^{n+1} \to M$ be a map from the $n+1$ dimensional spacetime, with Lorentzian metric $g$ of signature $(1, n)$, to a Riemannian manifold $(M, h)$. The action of the wave map equation, or the nonlinear $\sigma$ model in physics terminology, is

$$S = \frac{1}{2} \int g^{\mu\nu} \, \partial_\mu \phi^i \, \partial_\nu \phi^j \, h_{ij}(\phi) \, dg \tag{1.1}$$

The initial value problem for the Euler-Lagrange equations associated with (1.1) has been intensely studied, especially the issues of global existence and regularity of its solutions. We mention here the pioneering works of Christodoulou - Tahvildar-Zadeh [5], Grillakis [8], Shatah - Tahvildar-Zadeh [14], and Struwe [16].

The particular case when $M = \mathbb{S}^3$ and $n = 3$ is of special interest in high energy physics. The nuclei of atoms are held together by forces mediated by the pi mesons. These are a set of three particles whose masses are small compared to the nuclei themselves, so to a first approximation they can be considered to be massless, i.e., travelling at the speed of light. If interactions among them are ignored, the pi mesons are described by a field $\phi : R^{3+1} \to R^3$ satisfying the wave equation. Interactions would add nonlinearities. A remarkable fact of physics is that the interactions among the pi mesons are described, to a good approximation (Gell-Mann - Levy [7], Gursey [9], [10], and Lee [11]), by considering the target manifold to be the sphere $\mathbb{S}^3$ and replacing the wave equation by the corresponding wave map equation. Physicists call this the nonlinear $\sigma$ model for historical reasons.

In order for the energy to be finite, the gradient of the field must vanish at infinity: it tends to the same value in all directions of spatial infinity. Thus, each instantaneous pi meson configuration corresponds to a map $\tilde{\phi} : \mathbb{S}^3 \to \mathbb{S}^3$ obtained by identifying the points at spatial infinity. A continuous map of the sphere to itself

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has an integer associated with it, the winding number or element of the homotopy group $\pi_3(\mathbb{S}^3)$. In terms of the original field, we can write this as

$$Q = c \int \epsilon_{ijk} \partial_a \phi^i \partial_b \phi^j \partial_c \phi^k \epsilon^{abc} \, dx \tag{1.2}$$

where $\epsilon$ is the Levi-Civita symbol and $c$ is a normalizing constant. Small perturbations from the constant configurations would have $Q = 0$. If time evolutions were continuous, $Q$ would be a conserved quantity, a 'topological charge'.

It was suggested by Skyrme [15] in the 1960s that this topological charge $Q$ is just the total number of neutrons and protons in a nucleus: the baryon number. Thus, a proton (hydrogen nucleus) would have $Q = 1$, a helium nucleus would have either $Q = 3$ or 4 (depending on the isotope), and so on. This has been confirmed by connecting with Quantum chromodynamics (Balachandran - Nair - Rajeev - Stern [2, 3], Witten [18, 19]), the fundamental theory of nuclear interactions. Numerical calculations (Battye - Sutcliffe [4]) also support this idea. Some essential theoretical puzzles had to be resolved before this rather strange idea of Skyrme could be established. These were resolved in the mid-80's; e.g. by using topological notions ('anomalies') [15, 2, 3, 18, 19].

One of these puzzles is that actually the wave map does *not* have continuous time evolution: it is supercritical. Shatah [12] has exhibited an example of a solution with smooth initial conditions that breaks down in finite time. Physically, this is because the forces among the pi mesons are mainly attractive; so a configuration of winding number one would shrink to a point, emitting its energy as radiation carried away to infinity. At the singularity, $Q$ would jump to zero. Since baryon number is strictly conserved by nuclear forces, this cannot be right.

Skyrme suggested a modification of the action of the wave map that could stabilize the configurations with $Q \neq 0$. The corresponding equations are quite intricate, being quasilinear. What we study here is a semilinear modification of the above action $S$, previously introduced by Adkins - Nappi [1], which is also expected, for physical reasons, to have regular solutions[1]. The idea is to add a short range repulsion among the pi mesons, created by their interactions with an omega meson. Geometrically, it describes the nonlinear coupling of the wave map with a gauge field, the source (charge density) of the gauge field being the density of the topological charge $Q$.

In detail, the action of the theory we investigate is

$$\tilde{S} = S + \frac{1}{4} \int F^{\mu\nu} F_{\mu\nu} \, dg - \int A_\mu j^\mu \, dx \tag{1.3}$$

where the omega meson is represented by the gauge potential $A = A_\mu dx^\mu$, the 2-form $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is its associated electromagnetic field, and

$$j^\mu = c \, \epsilon^{\mu\nu\rho\sigma} \partial_\nu \phi^i \partial_\rho \phi^j \partial_\sigma \phi^k \epsilon_{ijk}$$

is the flux or the baryonic current[2].

[1] We consider only the case when the masses of the mesons are set to zero, which is sufficient to understand short distance singularities.

[2] Note that $Q = \int j^0 \, dx$.

Here we study time dependent equivariant maps associated to $\tilde{S}$ with winding number 1, i.e.,

$$g = \operatorname{diag}(-1,1,1,1) \qquad M = \mathbb{S}^3$$
$$\phi(t,r,\psi,\theta) = (u(t,r),\psi,\theta) \qquad A(t,r,\psi,\theta) = (V(t,r),0,0,0)$$

with boundary conditions $u(t,0) = 0$ and $u(t,\infty) = \pi$.

Routine computations lead to

$$\frac{\tilde{S}}{2\pi} = \iint \{ r^2(-u_t^2 + u_r^2) + 2\sin^2 u + \frac{\alpha^2(u - \sin u\cos u)^2}{r^2} - \Big[ rV_r + \frac{\alpha(u - \sin u \cos u)}{r} \Big]^2 \} \, dr\, dt$$

where $\alpha$ is an appropriate constant. Eliminating $V$ using its variational equation and scaling the coordinates by a factor of $\alpha$, we obtain the main equation satisfied by $u$:

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{1}{r^2}\sin 2u + \frac{1}{r^4}(u - \sin u\cos u)(1 - \cos 2u) = 0 \tag{1.4}$$

Based on the physical intuition detailed above, **we conjecture that smooth finite energy initial data for this equation will evolve into globally regular solutions**. This is also supported by the existence of static numerical solutions of winding number 1 for (1.3), that were found in [1]. Regularity at the origin forces the initial data to vanish at $r = 0$, which is consistent with the previously imposed boundary condition, $u(t,0) = 0$.

One needs to compare (1.4) with the corresponding wave map equation

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{1}{r^2}\sin 2u = 0 \tag{1.5}$$

for which Shatah's solution (later found by Turok and Spergel [17] in close form)

$$u(t,r) = 2\arctan\frac{r}{T_0 - t}$$

provides an example of smooth initial data that develop singularities in finite time.

The goal of this paper is to take the first step in proving our conjecture, which is to show that a smooth solution for (1.4) remains continuous at the first possible singularity. Using the fact that (1.4) is semilinear and it is invariant under translations, we can assume, without any loss of generality, that our solution starts at time $t = -1$ and breaks down at the origin $(0,0)$. Our main result is

**Theorem 1.1.** *The solution $u$ for* (1.4) *is continuous at the origin and*

$$u(t,r) \to 0 \quad \text{as} \quad (t,r) \to (0,0)$$

The proof follows the lines of the one for $2+1$ dimensional equivariant wave maps (e.g., [13]); however, because we are in $3+1$ dimensions, we lose a critical estimate which is in turn bypassed by a new argument, using the sign of the last term in (1.4):

$$u \cdot \frac{1}{r^4}(u - \sin u\cos u)(1 - \cos 2u) \geq 0 \tag{1.6}$$

In all of these problems (i.e., wave maps and Skyrme model) the initial configuration shrinks as time evolves, causing energy and winding number to accumulate at the origin. For the $2+1$ dimensional wave map equation, enough energy is radiated away so that the energy density at the origin is finite. In $3+1$ dimensions, the energy density diverges at the origin although the total energy tends to zero. For the Adkins - Nappi version of the Skyrme model, there is a repulsive part in the energy density. Our new argument shows that the energy corresponding to this part does not concentrate, allowing us to prove the continuity of the field.

## 2. Main argument

We will be working with backward truncated cones, their mantels and bases, denoted as

$$\begin{aligned}
K_T^S &= \{(t,r) : T \le t \le S,\, 0 \le r \le |t|\}\\
C_T^S &= \{(t,r) : T \le t \le S,\, r = |t|\}\\
B_T &= \{(t,r) : t = T,\, 0 \le r \le |t|\}
\end{aligned}$$

where $-1 \le T \le S \le 0$. For narrower bases, we use the notation

$$B_T(\lambda) = \{(t,r) : t = T,\, -\lambda t \le r \le -t\}$$

where $0 \le \lambda \le 1$.

Multiplying (1.4) by $r^2u_t$, $r^2u_r$, $r^3u_r$, $r^2u$, respectively $r^2tu_t$, we obtain the following differential identities:

**Lemma 2.1.** *A classical solution for* (1.4) *satisfies:*

$$\partial_t\left(\frac{r^2}{2}(u_t^2+u_r^2)+\sin^2 u+\frac{1}{2r^2}(u-\sin u\cos u)^2\right)-\partial_r(r^2u_tu_r)=0, \tag{2.1}$$

$$\begin{aligned}
&\partial_t(r^2u_tu_r)-\partial_r\left(\frac{r^2}{2}(u_t^2+u_r^2)-\sin^2 u-\frac{1}{2r^2}(u-\sin u\cos u)^2\right)\\
&=r(u_r^2-u_t^2)-\frac{1}{r^3}(u-\sin u\cos u)^2,
\end{aligned}\tag{2.2}$$

$$\begin{aligned}
&\partial_t(r^3u_tu_r)-\partial_r\left(\frac{r^3}{2}(u_t^2+u_r^2)-r\sin^2 u-\frac{1}{2r}(u-\sin u\cos u)^2\right)\\
&=\frac{r^2}{2}(u_r^2-3u_t^2)+\sin^2 u-\frac{1}{2r^2}(u-\sin u\cos u)^2,
\end{aligned}\tag{2.3}$$

$$\begin{aligned}
&\partial_t(r^2uu_t)-\partial_r(r^2uu_r)\\
&=r^2(u_t^2-u_r^2)-u\left(\sin 2u+\frac{1}{r^2}(u-\sin u\cos u)(1-\cos 2u)\right),
\end{aligned}\tag{2.4}$$

$$\begin{aligned}
&\partial_t\left(\frac{tr^2}{2}(u_t^2+u_r^2)+t\sin^2 u+\frac{t}{2r^2}(u-\sin u\cos u)^2\right)-\partial_r(tr^2u_tu_r)\\
&=\frac{r^2}{2}(u_t^2+u_r^2)+\sin^2 u+\frac{1}{2r^2}(u-\sin u\cos u)^2
\end{aligned}\tag{2.5}$$

**Remark 2.2.** The argument for $2+1$ dimensional equivariant wave maps relies on the counterparts of all five identities. For our analysis we will not use (2.2).

Next, we define the local energy and the flux associated to our problem.

**Definition 2.3.** The energy of the time slice $t = T$ is defined as

$$E(T)=\int_{B_T}\frac{1}{2}(u_t^2+u_r^2)+\frac{1}{r^2}\sin^2 u+\frac{1}{2r^4}(u-\sin u\cos u)^2$$

while the flux between the time slices $t = T$ and $t = S$ is given by

$$F(T,S) = \frac{1}{\sqrt{2}} \int_{C_T^S} \frac{1}{2}(u_t - u_r)^2 + \frac{1}{r^2} \sin^2 u + \frac{1}{2r^4}(u - \sin u \cos u)^2$$

The classical energy estimate, obtained by integrating (2.1) over $K_T^S$,

$$E(T) - E(S) = F(T,S)$$

implies that the local energy is decreasing and the flux decays to 0, i.e.,

$$E(S) \le E(T) \quad \text{for} \quad -1 \le T \le S, \tag{2.6}$$

$$\lim_{T \to 0-} F(T,0) = 0 \tag{2.7}$$

These two facts allow us to go further and show that:

**Proposition 2.4.** *The solution $u$ is bounded, with*

$$\|u\|_{L^\infty} \le C(E(-1)), \tag{2.8}$$

$$\lim_{t \to 0-} u(t,-t) = 0 \tag{2.9}$$

*Proof.* For

$$I(z) = \int_0^z (w - \sin w \cos w)\, dw = \frac{z^2 - \sin^2 z}{2}$$

we have

$$I(0) = 0, \quad I(z) > 0 \ (z \ne 0), \quad \lim_{|z| \to \infty} I(z) = \infty$$

Based on $u(t,0) = 0$, we can write

$$I(u(t,r)) = \int_0^r (u - \sin u \cos u) u_r \, dr$$

which implies

$$I(u(t,r)) \lesssim \Big( \int_{B_t} u_r^2 \Big)^{1/2} \Big( \int_{B_t} \frac{(u - \sin u \cos u)^2}{r^4} \Big)^{1/2} \lesssim E(t)$$

The monotonicity of the energy, (2.6), then settles the first claim.

Next, denoting $v(t) = u(t,-t)$, we use (2.7) to infer

$$\lim_{T \to 0-} \int_T^0 \frac{(v - \sin v \cos v)^2}{t^2} \, dt = 0$$

which leads to the existence of a sequence $t_n \to 0$ for which $v(t_n) \to 0$. For fixed $T$ and large $n$ we obtain

$$|I(v(T)) - I(v(t_n))| \lesssim \int_T^{t_n} |(v - \sin v \cos v) \cdot v_s| \, ds \lesssim F(T,0)$$

which proves the last claim. $\square$

We use these results to reduce the proof of Theorem 1.1 to the one of a local energy estimate.

**Theorem 2.5.** *If the solution $u$ is smooth on $K_{-1}^t$ for all $-1 < t < 0$ and*

$$\lim_{T\to 0-}\int_{B_T}\frac{(u-\sin u\cos u)^2}{r^4} = 0 \tag{2.10}$$

*then $u$ is continuous at the origin and*

$$u(t,r)\to 0 \quad as \quad (t,r)\to(0,0)$$

*Proof.* We argue as in Proposition 2.4 to deduce

$$\begin{aligned}I(u(t,r)) &= I(u(t,-t)) + \int_{-t}^{r}(u-\sin u\cos u)(t,r')\cdot u_r(t,r')\,dr'\\ &\lesssim I(u(t,-t)) + E(t)^{1/2}\cdot\Big(\int_{B_t}\frac{(u-\sin u\cos u)^2}{r^4}\Big)^{1/2}\end{aligned}$$

which obviously provides the desired conclusion based on (2.6) and (2.9). □

**Remark 2.6.** Theorem 2.5 is the point where our argument leaves the approach for equivariant wave maps. There, one relies on the weaker bound

$$I(u(t,r)) \lesssim I(u(t,-t)) + E(t)$$

and proves nonconcentration of the energy (i.e., $E(t)\to 0$ as $t\to 0$). The crucial ingredient for that analysis is that the annular energy doesn't concentrate,

$$E_\lambda(t) = \int_{B_t(\lambda)} e \to 0$$

where $e$ is the energy density and $0\le\lambda\le 1$. This is in turn obtained from

$$((rm)_t - (re)_r)^2 \lesssim (e-m)(e+m) \tag{2.11}$$

where $m$ is the momentum density. We refer the interested reader to [5] or [13] for more details.

In our case, the corresponding estimate should read

$$((r^2m)_t - (r^2e)_r)^2 \lesssim r^2(e-m)(e+m) \tag{2.12}$$

where

$$e = \frac{1}{2}(u_t^2+u_r^2) + \frac{1}{r^2}\sin^2 u + \frac{1}{2r^4}(u-\sin u\cos u)^2, \quad m = u_r u_t$$

The left-hand side in (2.12), which can be obtained through (2.2), takes the form:

$$\begin{aligned}(r^2m)_t - (r^2e)_r = r(u_r^2-u_t^2) + \frac{(u-\sin u\cos u)^2}{r^3} - 2\sin 2u\cdot u_r\\ - \frac{2(u-\sin u\cos u)(1-\cos 2u)}{r^2}\,u_r\end{aligned}$$

Precisely the last term above fails to obey the bound in (2.12). This is the reason why we do not use (2.2) in our argument.

## 3. Local energy estimates

These estimates are deduced by integrating the differential identities (2.3)-(2.5) on the backward cone $K_T^0$ and then allow for $T \to 0$. First, we notice that for $f \in L^\infty$ one obtains

$$\lim_{T\to 0-} \frac{1}{|T|} \int_{K_T^0} \frac{f}{r^2} = \lim_{T\to 0-} \int_{B_T} \frac{f}{r^2} = 0 \tag{3.1}$$

which allows us to ignore certain terms in the analysis.

**Lemma 3.1.** *For $u$ solution of (1.4), smooth on $K_{-1}^t$ for all $-1 < t < 0$, the following estimates hold:*

$$\lim_{T\to 0-} \frac{1}{|T|} \int_{K_T^0} \big[\frac{1}{2}(3u_t^2 - u_r^2) + \frac{1}{2r^4}(u - \sin u \cos u)^2\big] - \frac{1}{|T|} \int_{B_T} r u_r u_t = 0 \tag{3.2}$$

$$\lim_{T\to 0-} \frac{1}{|T|} \int_{K_T^0} \big[u_r^2 - u_t^2 + u \cdot \frac{1}{r^4}(u - \sin u \cos u)(1 - \cos 2u)\big] = 0 \tag{3.3}$$

$$\lim_{T\to 0-} \frac{1}{|T|} \int_{K_T^0} \big[-\frac{1}{2}(u_t^2 + u_r^2) - \frac{1}{2r^4}(u - \sin u \cos u)^2\big] + E(T) = 0 \tag{3.4}$$

*Proof.* We prove only the first estimate, the other two being treated similarly. As mentioned above, we integrate (2.3) on $K_T^0$ to infer

$$\begin{aligned}
&\int_{K_T^0} \big[\frac{1}{2}(3u_t^2 - u_r^2) - \frac{1}{r^2}\sin^2 u + \frac{1}{2r^4}(u - \sin u \cos u)^2\big] \\
&= \int_{B_T} r u_r u_t + \int_{C_T^0} \Big[r \cdot \Big(\frac{1}{2}(u_t - u_r)^2 + \frac{1}{r^2}\sin^2 u + \frac{1}{2r^4}(u - \sin u \cos u)^2\Big)\Big]
\end{aligned}$$

Using (3.1) we deduce that

$$\lim_{T\to 0-} \frac{1}{|T|} \int_{K_T^0} \frac{1}{r^2}\sin^2 u = 0$$

(3.2) follows immediately as

$$\lim_{T\to 0-} \frac{1}{|T|} \int_{C_T^0} \Big[r \cdot \Big(\frac{1}{2}(u_t - u_r)^2 + \frac{1}{r^2}\sin^2 u + \frac{1}{2r^4}(u - \sin u \cos u)^2\Big)\Big] = 0$$

due to (2.7) and that $r \le |T|$ in $C_T^0$. $\square$

Finally, combining (3.2)-(3.4) and relying on (1.6), we obtain:

$$\lim_{T\to 0-} E(T) - \frac{1}{|T|} \int_{B_T} r u_r u_t = 0 \tag{3.5}$$

Coupling this with

$$\begin{aligned}
E(T) - \frac{1}{|T|} \int_{B_T} r u_r u_t &= \int_{B_T} \big[\frac{1}{4}(1 - \frac{r}{|T|})(u_t - u_r)^2 + \frac{1}{4}(1 + \frac{r}{|T|})(u_t + u_r)^2\big] \\
&\quad + \int_{B_T} \big[\frac{1}{r^2}\sin^2 u + \frac{1}{2r^4}(u - \sin u \cos u)^2\big]
\end{aligned}$$

we deduce the sufficient condition (2.10) from Theorem 2.5, which finishes the argument.

It is worth noting that (3.5) yields also

$$\lim_{T\to 0-}\int_{B_T}\Big[(1-\frac{r}{|T|})(u_t-u_r)^2+(u_t+u_r)^2+\frac{1}{r^2}\sin^2 u\Big]=0$$

Thus we obtain that the entire energy does not concentrate, except maybe for

$$\int_{B_T}\frac{r}{|T|}(u_t-u_r)^2$$

This question is addressed in an upcoming article [6].

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