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INITIAL-BOUNDARY VALUE PROBLEMS IN A PLANE CORNER FOR THE HEAT EQUATION

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Abstract. We study the Dirichlet initial problem for the heat equation by the Fourier-Bessel method in a plane corner. We prove classical solvability for the problem in weighted Hölder spaces.

1. Introduction

There are various approaches in investigations of initial boundary value problems for parabolic equations in domains with singularities. In the works of Grisvard [9], Solonnikov [13], Amann [1], Garroni, Solonnikov and Vivaldi [7], Frolova [6], the existence of solutions and qualitative properties of solutions are described in the terms of Sobolev or weighted Sobolev spaces. The similar results in Hölder classes are represented in works Guidetti [10], Colombo, Guidetti, and Lorenzi [5], Solonnikov [13] (see also references in these works).

Note that in the pointed out works the method of the Green function or the theory of analytic semigroups were used to construct some explicit representation of a solution and to obtain the optimal estimates.

In the present paper we use the classical Fourier method to get a solution in the form of Fourier-Bessel series in an angular domain. Then we apply some results on trigonometric series theory, in particular, Bernstein theorem and Jackson's construction of approximating trigonometric polynomials to obtain estimates of the higher derivatives of the solution to the Dirichlet initial problem for the heat equation in weighted Hölder spaces.

Sometimes a classical solution of the initial value problem for a parabolic equation is defined as a function, that has required higher derivatives in any internal subdomain of a cylindrical domain. In the one dimensional case a similar result was published by Chernyatin [4] where a solution was represented as the sum of the trigonometric series. But to the best of our knowledge, we have not found similar results concerning with two-dimensional case.

The paper is organized as follows: in Section 2, we formulate the problem, introduce the appropriate functional space, and show the formal solution to the Dirichlet initial problem for the heat equation in the form of the sum of the trigonometric

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series. Section 3 contains some auxiliary estimates. In Section 4, we recall some results from the trigonometric series theory, and in Section 5 we show that the trigonometric series representing the formal solution converge together with its higher order derivatives. Section 6 consists of some final remarks to the existence and uniqueness theorem and some results concerning the parabolic equation with singular coefficients.

2. The statement of the problem and main results

We use the polar coordinate system (r,φ) on a plane $\mathbb{R}^2$. Let

$$G=\{(r,\varphi): r>0,\ 0<\varphi<\theta\},\ \theta\in(0,2\pi),$$

be an infinite angle on $\mathbb{R}^2$, $G_T=G\times[0,T]$, $T\in\mathbb{R}_+$, and its boundary be

$$g=g_0\cup g_1, g_0=\{(r,\varphi): r>0,\varphi=0\},$$
$$g_1=\{(r,\varphi): r>0,\varphi=\theta\}, g_{iT}=g_i\times[0,T],\quad i=0,1.$$

Let $\alpha\in(0,1)$, l be an integer. We use the weighted Hölder space $P_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$ of functions $u(r,\varphi,t)$ with the finite norm

$$\begin{aligned}
&\|u\|_{P_s^{l+\alpha,(l+\alpha)/2}(\overline{G}_T)}\equiv|u|^{l+\alpha}_{s,G_T}\\
&=\sum_{0\le\beta_1+\beta_2+2a\le l}\ \sup_{(r,\varphi,t)\in\overline{G}_T} r^{-s+\beta_1+2a}|D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u|\\
&\quad+\sum_{0<l+\alpha-(\beta_1+\beta_2+2a)<2}\Big\{\langle D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u\rangle^{(\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}_{t;s-\beta_1-2a-\alpha,G_T}\\
&\quad+[D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u]^{(\alpha,\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}_{r,t;s-\beta_1-2a-2\alpha,G_T}+[D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u]^{(\alpha,\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}_{\varphi,t;s-\beta_1-2a-\alpha,G_T}\Big\}\\
&\quad+\sum_{\beta_1+\beta_2+2a=l}\Big\{\langle D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u\rangle^{(\alpha)}_{r;s-\beta_1-2a-\alpha,G_T}\\
&\quad+\langle D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u\rangle^{(\alpha)}_{\varphi;s-\beta_1-2a,G_T}\Big\},
\end{aligned}$$

with the seminorms defined as follows, $\overline{r}=\min(\rho,r)$, $\alpha,\gamma\in(0,1)$:

$$\langle v\rangle^{(\alpha)}_{r;\mu,G_T}=\sup_{(\rho,\varphi,t),(r,\varphi,t)\in\overline{G}_T,\ |\rho-r|\le\overline{r}/2}\overline{r}^{-\mu}\frac{|v(\rho,\varphi,t)-v(r,\varphi,t)|}{|\rho-r|^\alpha},$$
$$\langle v\rangle^{(\alpha)}_{\varphi;\mu,G_T}=\sup_{(r,\varphi,t),(r,\psi,t)\in\overline{G}_T} r^{-\mu}\frac{|v(r,\varphi,t)-v(r,\psi,t)|}{|\varphi-\psi|^\alpha},$$
$$\langle v\rangle^{(\gamma)}_{t;\mu,G_T}=\sup_{(\rho,\varphi,t),(r,\varphi,\tau)\in\overline{G}_T} r^{-\mu}\frac{|v(r,\varphi,t)-v(r,\varphi,\tau)|}{|t-\tau|^\gamma},$$
$$[v]^{(\alpha,\gamma)}_{r,t;\mu,G_T}=\sup_{\substack{(\rho,\varphi,t),(r,\varphi,t),\\(r,\varphi,\tau),(\rho,\varphi,\tau)\in\overline{G}_T\\|\rho-r|\le\overline{r}/2}}\overline{r}^{-\mu}\frac{|v(r,\varphi,t)-v(r,\varphi,\tau)-v(\rho,\varphi,t)+v(\rho,\varphi,\tau)|}{|\rho-r|^\alpha|t-\tau|^\gamma},$$
$$[v]^{(\alpha,\gamma)}_{\varphi,t;\mu,G_T}=\sup_{\substack{(r,\varphi,t),(r,\psi,t),\\(r,\varphi,\tau),(r,\psi,\tau)\in\overline{G}_T}} r^{-\mu}\frac{|v(r,\varphi,t)-v(r,\varphi,\tau)-v(r,\psi,t)+v(r,\psi,\tau)|}{|\varphi-\psi|^\alpha|t-\tau|^\gamma}.$$

The seminorms $[\cdot]^{(\alpha,\gamma)}$ were introduced in [14]. In a similar way we introduce the space $P_s^{l+\alpha}(\overline{G})$ of the functions $u(r,\varphi)$ on G. Hereinafter we will use the subspace $\widehat{P}_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$ of the space $P_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$ which is introduced as follows. Let

$$\mathbb{R}_T=\{(r,t):r>0,t\in(0,T)\},$$

and function $v(r,\varphi,t)\in P_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$ be such that

$$v(r,\varphi,t)=\sum_{k=1}^{\infty}v_k(r,t)\sin\lambda_k\varphi,\quad \lambda_k=\frac{\pi k}{\theta},$$

where

$$v_k(r,t)=\frac{2}{\theta}\int_0^{\theta}v(r,\psi,t)\sin(\lambda_k\psi)d\psi.$$

We will say the function $v(r,\varphi,t)\in \widehat{P}_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$ if $v(r,\varphi,t)\in P_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$ and the following inequality holds:

$$\begin{aligned}
S(v)_{\widehat{P}_s^{l+\alpha,(l+\alpha)/2}(\overline{G}_T)} :=\sum_{k=1}^{\infty}\Big(&\sum_{0\le\beta_1+\beta_2+2a\le l}\ \sup_{(r,t)\in\overline{R}_T} r^{-s+\beta_1+2a}\lambda_k^{\beta_2}|D_r^{\beta_1}D_t^a v_k(r,t)|\\
&+\sum_{0<l+\alpha-(\beta_1+\beta_2+2a)<2}\Big\{\langle D_r^{\beta_1}D_t^a v_k\rangle_{t;s-\beta_1-2a-\alpha,\mathbb{R}_T}^{(\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}\\
&+[D_r^{\beta_1}D_t^a v_k]_{r,t;s-\beta_1-2a-2\alpha,\mathbb{R}_T}^{(\alpha,\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}\Big\}\lambda_k^{\beta_2}\\
&+\sum_{\beta_1+\beta_2+2a=l}\lambda_k^{\beta_2}\langle D_r^{\beta_1}D_t^a v_k\rangle_{r;s-\beta_1-2a-\alpha,\mathbb{R}_T}^{(\alpha)}\Big)<\infty.
\end{aligned}$$

The subspace $\widehat{P}_s^{l+\alpha}(\overline{G})$ of the function $v(r,\varphi)$ on G is introduced similarly. One can easily check that if $v(r,\varphi,t)\in \widehat{P}_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$ or $v(r,\varphi)\in\widehat{P}_s^{l+\alpha}(\overline{G})$ then there are the constants c_i or $\tilde c_i$, $i=1,2$, such that

$$\begin{aligned}
&c_1\|v\|_{P_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)}\\
&\le S(v)_{\widehat{P}_s^{l+\alpha,(l+\alpha)/2}(\overline{G}_T)}+\sum_{\beta_1+\beta_2+2a=l}\langle D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a v\rangle_{\varphi;s-\beta_1-2a,G_T}^{(\alpha)}\\
&\quad+\sum_{0<l+\alpha-(\beta_1+\beta_2+2a)<2}[D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a v]_{\varphi,t;s-\beta_1-2a-\alpha,G_T}^{(\alpha,\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}\\
&\le c_2\|v\|_{P_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)};
\end{aligned}$$

$$\tilde c_1\|v\|_{P_s^{l+\alpha}(\overline{G})}\le S(v)_{\widehat{P}_s^{l+\alpha}(\overline{G})}+\sum_{\beta_1+\beta_2=l}\langle D_r^{\beta_1}D_\varphi^{\beta_2}v\rangle_{\varphi;s-\beta_1,G}^{(\alpha)}\le \tilde c_2\|v\|_{P_s^{l+\alpha}(\overline{G})}. \tag{2.1}$$

Along with the spaces $P_s^{l+\alpha,\frac{l+\alpha}{2}}(\overline{G}_T)$, we will use the usual Hölder classes $C_{x,t}^{\alpha,\beta}:=C_{x,t}^{\alpha,\beta}(\overline{\Omega}_T)$ where $\beta\in(0,1)$, $x\in\Omega$, $t\in(0,T)$, and

$$\|u\|_{C_{x,t}^{\alpha,\beta}(\overline{\Omega}_T)}=\sup_{(x,t)\in\overline{\Omega}_T}|u(x,t)|+\langle u\rangle_x^{(\alpha)}+\langle u\rangle_t^{(\beta)},$$

$$\langle u\rangle_x^{(\alpha)}=\sup_{(x,t),(y,t)\in\overline{\Omega}_T}\frac{|u(x,t)-u(y,t)|}{|x-y|^{\alpha}},$$

$$\langle u\rangle_t^{(\beta)} = \sup_{(x,t),(x,\tau)\in\overline{\Omega}_T} \frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^\beta}.$$

We are looking for a solution of the Dirichlet initial problem

$$\begin{gathered} \frac{\partial u}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = f(r,\varphi,t), \quad (r,\varphi,t)\in G_T, \\ u|_{g_{iT}} = 0, \quad u|_{t=0} = u_0(r,\varphi), \quad (r,\varphi)\in G. \end{gathered} \tag{2.2}$$

We suppose that

$$f(r,0,t) = f(r,\theta,t) = 0. \tag{2.3}$$

As can be seen from further arguments, conditions (2.3) are, at least formally, necessary to get a solution to problem (2.2) in the form of the Fourier-Bessel series in the space $P_{s+2}^{2+\alpha,(2+\alpha)/2}(\overline{G}_T)$. Note that, due to the presence of the seminorms $[\cdot]^{(\alpha,\alpha/2)}$ in the definition of the norm in the space $P_{s+2}^{2+\alpha,(2+\alpha)/2}(\overline{G}_T)$, the series solution is more smooth than the solution from the ordinary weighted Hölder spaces.

Theorem 2.1. *Let equality* (2.3) *and the consistency conditions of the first order in problem* (2.2) *be fulfilled. The functions* $f \in \widehat{P}_s^{\alpha,\alpha/2}(\overline{G}_T)$ *and* $u_0 \in \widehat{P}_{s+2}^{2+\alpha}(\overline{G})$. *Then there exists a unique solution* $u \in \widehat{P}_{s+2}^{2+\alpha,(2+\alpha)/2}(\overline{G}_T)$ *with*

$$\begin{aligned} &\|u\|_{P_{s+2}^{2+\alpha,(2+\alpha)/2}(\overline{G}_T)} + S(u)_{\widehat{P}_{s+2}^{2+\alpha,(2+\alpha)/2}(\overline{G}_T)} \\ &\le const.(\|f\|_{P_s^{\alpha,\alpha/2}(\overline{G}_T)} + \|u_0\|_{P_{s+2}^{2+\alpha}(\overline{G})} + S(f)_{\widehat{P}_s^{\alpha,\alpha/2}(\overline{G}_T)} + S(u_0)_{\widehat{P}_{s+2}^{2+\alpha}(\overline{G})}), \end{aligned} \tag{2.4}$$

where the constant in (2.4) *is independent of* u, $-\pi/\theta+\alpha < s+2 < \pi/\theta$, $\alpha\in(0,1)$.

Under the proof of Theorem 2.1, we will omit the subindex G_T in the notations of the seminorms if it is clearly from the context. We will assume that the function $f(r,\varphi,0) = 0$ and, hence, can be extended by zero onto $t<0$ with the same norm.

It can be easily seen that one of the factor in the eigenfunctions to problem (2.2) is $\sin(\lambda_k\phi)$, $\lambda_k = \frac{\pi k}{\theta}$, $k = 1,2,\dots$. So, after the standard procedure of separation of variables (see Appendix 7.1), we get the series representation of the solution:

$$u(r,\varphi,t) = R_1(r,\varphi,t) + R_2(r,\varphi,t), \tag{2.5}$$

$$\begin{aligned} R_1(r,\varphi,t) &= \sum_k \sin(\lambda_k\varphi)\int_{-\infty}^t d\tau\int_0^\infty d\rho\frac{\rho}{2(t-\tau)}e^{-\frac{\rho^2+r^2}{4(t-\tau)}} I_{\lambda_k}(\frac{\rho r}{2(t-\tau)})b_k(\rho,\tau) \\ &\equiv \sum_k R_{1,k}(r,t)\sin(\lambda_k\varphi), \end{aligned} \tag{2.6}$$

$$R_2(r,\varphi,t) = \sum_k \sin(\lambda_k\varphi)\int_0^\infty d\rho\frac{\rho}{2t}e^{-\frac{\rho^2+r^2}{4t}} I_{\lambda_k}(\frac{\rho r}{2t})u_{0k}(\rho), \tag{2.7}$$

$$u_{0k}(r) = \frac{2}{\theta}\int_0^\theta u_0(r,\psi)\sin(\lambda_k\psi)d\psi, \quad b_k(r,t) = \frac{2}{\theta}\int_0^\theta f(r,\psi,t)\sin(\lambda_k\psi)d\psi, \tag{2.8}$$

where $I_\mu(z)$ is a modified Bessel function. Equality (2.5) means that the desired solution is the sum of the volume potential $R_1(r,\phi,t)$ and the potential of the initial data $R_2(r,\phi,t)$.

The general case of $f(r,\varphi,t)$; i.e., $f(r,\varphi,0)\neq 0$, can be reduced to mention above with the following procedure. Let in problem (2.2), (2.3), $f(r,\varphi,0)\in \widehat{P}_s^\alpha(\overline{G})$, and the function $\widehat{w}(r,\varphi)$ be a solution of the problem

$$\Delta\widehat{w} = -f(r,\varphi,0) \quad G, \quad \widehat{w}|_g = 0,$$

and $\widehat{w}(r,\varphi) \in \widehat{P}^{2+\alpha}_{s+2}(\overline{G})$ (see [16]). Then we consider the functions $w(r,\varphi,t) = \widehat{w}(r,\varphi)\cos t$ and $v(r,\varphi,t) = u(r,\varphi,t) - w(r,\varphi,t)$ such that

$$\begin{gathered}
\begin{aligned}
\frac{\partial v}{\partial t} - \Delta v &= \frac{\partial u}{\partial t} - \Delta u + \widehat{w}\sin t + \Delta\widehat{w}\cos t \\
&= f(r,\varphi,t) + \widehat{w}(r,\varphi)\sin t - f(r,\varphi,0)\cos t \equiv F(r,\varphi,t) \quad \text{in } G_T,
\end{aligned} \\
v|_{g_{iT}} = 0, \quad v|_{t=0} = u_0(r,\varphi) - \widehat{w}(r,\varphi).
\end{gathered} \tag{2.9}$$

One can see that the consistency conditions of the first order are fulfilled in problem (2.9); the function $F(r,\varphi,t) \in \widehat{P}^{\alpha,\alpha/2}_s(\overline{G}_T)$, satisfies condition (2.3), and $F(r,\varphi,0) = 0$.

Thus, the investigation of problem (2.2) can be reduced to study of problem (2.9) with the homogeneous right part in the equation if $t \le 0$.

Note that, to prove inequality (2.4) and $u \in \widehat{P}^{2+\alpha,(2+\alpha)/2}_{s+2}(\overline{G}_T)$ in Theorem 2.1, it is sufficient to show the following estimate (due to the first of inequalities in (2.1))

$$\begin{aligned}
&S(u)_{\widehat{P}^{2+\alpha,(2+\alpha)/2}_{s+2}(\overline{G}_T)} + \sum_{\beta_1+\beta_2+2a=2} \langle D^{\beta_1}_r D^{\beta_2}_\varphi D^a_t u\rangle^{(\alpha)}_{\varphi;s+2-\beta_1-2a,G_T} \\
&+ \sum_{0<2+\alpha-(\beta_1+\beta_2+2a)<2} [D^{\beta_1}_r D^{\beta_2}_\varphi D^a_t u]^{(\alpha,\frac{2+\alpha-\beta_1-\beta_2-2a}{2})}_{\varphi,t;s+2-\beta_1-2a-\alpha,G_T} \\
&\le \text{const.}\big(\|f\|_{P^{\alpha,\alpha/2}_s(\overline{G}_T)} + \|u_0\|_{P^{2+\alpha}_{s+2}(\overline{G})}\big).
\end{aligned} \tag{2.10}$$

In the all following inequalities, the constants do not depend on k.

3. Convergence of series (2.6) and (2.7)

Let us denote as

$$\Delta_s = \int_0^t d\tau \int_0^\infty d\rho \frac{\rho^{1+s}}{2\tau} e^{-\frac{\rho^2+r^2}{4\tau}} I_{\lambda_k}(\frac{\rho r}{2\tau}). \tag{3.1}$$

Lemma 3.1. *The following estimate holds*

$$\Delta_s \le \frac{r^{2+s}}{\lambda_k^2 - (s+2)^2}, \quad \text{if } s+2 < \pi/\theta. \tag{3.2}$$

Proof. The successive changes of variables: $\frac{\rho r}{2\tau} = x$ and $x^2\tau/r^2 = z$ leads to

$$\begin{aligned}
\Delta_s &= r^s \int_0^t d\tau \int_0^\infty 2^{1+s}(\frac{x\tau}{r^2})^{1+s} e^{-\frac{r^2}{4\tau} - \tau\frac{x^2}{r^2}} I_{\lambda_k}(x)dx \\
&= 2^{1+s} \int_0^\infty \frac{r^{2+s}}{x^{3+s}} I_{\lambda_k}(x)dx \int_0^{t\frac{x^2}{r^2}} z^{1+s} e^{-z-\frac{x^2}{4z}} dz \\
&\le 2^{1+s} \int_0^\infty \frac{r^{2+s}}{x^{3+s}} I_{\lambda_k}(x)dx \int_0^\infty z^{1+s} e^{-z-\frac{x^2}{4z}} dz.
\end{aligned}$$

The internal integral in the above inequality can be calculated, [8, 3.471 (9)],

$$\int_0^\infty z^{1+s} e^{-z-\frac{x^2}{4z}} dz = 2(\frac{x^2}{4})^{1+s/2} K_{2+s}(x)$$

where $K_\mu(z)$ is a modified Bessel function of the second kind. Hence,

$$\Delta_s \le \int_0^\infty \frac{r^{2+s}}{x} I_{\lambda_k}(x) K_{2+s}(x) dx. \tag{3.3}$$

The condition $s+2<\pi/\theta$ is sufficient to obtain the boundedness of the right part in (3.3) for all k. Really, we take into account the tabular integral in the right part of (3.3) [8, 6.576(5)], so that

$$\begin{aligned}\int_0^\infty \frac{1}{x} I_{\lambda_k}(x) K_{2+s}(x) dx &= \frac{1}{4}\frac{\Gamma((\lambda_k+s+2)/2)\Gamma((\lambda_k-s-2)/2)}{\Gamma(\lambda_k+1)}\\ &\quad \times F((\lambda_k+s+2)/2,(\lambda_k-s-2)/2;\lambda_k+1;1)\\ &= \frac{1}{4}\frac{1}{\frac{\lambda_k^2}{4}-(1+\frac{s}{2})^2},\end{aligned} \tag{3.4}$$

here we employed the definition of the function $F(\alpha,\beta;\gamma;z)$ [8, 9.111]. Inequality (3.3) together with (3.4) complete the proof of Lemma 3.1. □

Similar arguments lead to forllowing remark.

Remark 3.2. If $s+2<\pi/\theta$, then

$$\int_0^\infty d\tau \int_0^\infty d\rho \frac{\rho^{1+s}}{2\tau} e^{-\frac{\rho^2+r^2}{4\tau}} I_{\lambda_k}(\frac{\rho r}{2\tau}) = \text{const.}\frac{r^{2+s}}{\lambda_k^2-(s+2)^2}\,.$$

Note that we take advantage of some tabular integrals in order to obtain the sharp estimates of the weight in the statement of Lemma 3.1. It is possible to apply simpler arguments to derive only the asymptotic Δ_s with respect to λ_k.

Hereinafter we will use the following properties of the Bessel functions

$$\begin{gathered} I_\mu(z) \sim \text{const.}\frac{(z/2)^\mu}{\Gamma(\mu+1)}, \quad \text{for small values of } z,\\ I_\mu(z) \sim e^z/\sqrt{2\pi z} + \frac{C(\mu)}{z^{3/2}}, \quad \text{for large values of } z \end{gathered} \tag{3.5}$$

where $C(\mu)$ is some function,

$$\begin{gathered} K_\mu(z) \sim \text{const.} z^{-\mu} \quad \text{for } |\mu| \le \text{const. and small values of } z,\\ K_\mu(z) \sim e^{-z}/\sqrt{2\pi z} \quad \text{for } |\mu| \le \text{const. and large values of } z. \end{gathered} \tag{3.6}$$

Lemma 3.3. *The following estimate holds for $s<\pi/\theta$:*

$$D_s := D_s(r,t) = \int_0^\infty d\rho \frac{\rho^{1+s}}{2t} e^{-\frac{\rho^2+r^2}{4t}} I_{\lambda_k}\big(\frac{\rho r}{2t}\big) \le const. r^s. \tag{3.7}$$

Proof. Let us consider the problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \Delta u = 0 \quad in \quad G_T,\\ u|_{t=0} = r^s \sin\lambda_k\varphi, \quad u|_{g_{iT}} = 0.\end{gathered}$$

Denote by $w(r,\varphi)=r^s\sin\lambda_k\varphi$ and introduce the function $v(r,\varphi,t)=u(r,\varphi,t)-w(r,\varphi)$. The function $w(r,\varphi)$ satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2} = r^{s-2}(s^2-\lambda_k^2)\sin\lambda_k\varphi,$$

and $v(r,\varphi,t)$ is a solution of the problem

$$\frac{\partial v}{\partial t}-\Delta v=r^{s-2}(s^2-\lambda_k^2)\sin\lambda_k\varphi \quad \text{in} \quad G_T,$$
$$v|_{t=0}=0,\quad v|_{g_{iT}}=0.$$

Hence, by (2.6)

$$v(r,\varphi,t)=(s^2-\lambda_k^2)\sin(\lambda_k\varphi)\int_0^t d\tau\int_0^\infty d\rho\frac{\rho^{1+s-2}}{2(t-\tau)}e^{-\frac{\rho^2+r^2}{4(t-\tau)}}I_{\lambda_k}(\frac{\rho r}{2(t-\tau)}),$$

and due to Lemma 3.1 $|v(r,\varphi,t)|\le \text{const}.r^s$, so that

$$|u(r,\varphi,t)|\le \text{const}.r^s.$$

On the other hand the solution $u(r,\varphi,t)$ of the initial problem can be represented by using (2.7)

$$u(r,\varphi,t)=\sin(\lambda_k\varphi)\int_0^\infty d\rho\frac{\rho^{1+s}}{2t}e^{-\frac{\rho^2+r^2}{4t}}I_{\lambda_k}(\frac{\rho r}{2t}).$$

If we take here $\varphi=\frac{\theta}{2k}$, we will obtain inequality (3.7). □

Corollary 3.4. *The inequality*

$$e^{-z}z^{1/2}I_{\lambda_k}(z)\le \mathit{const.},\quad z\in(0,\infty)$$

is valid for any k with a constant is independent of k.

The proof of this Corollary is given in Appendix (see subsection 7.2).

Lemma 3.5. *The following equality holds if $s=0$,*

$$\lim_{t\to 0}D_0=1 \tag{3.8}$$

for every λ_k.

Proof. First of all we will prove the following fact. Let

$$\Delta_{-2,s}:=\int_0^t d\tau\int_0^\infty\frac{1}{2\tau\rho^{1-s}}e^{-\frac{r^2+\rho^2}{4\tau}}I_{\lambda_k}(\frac{r\rho}{2\tau})d\rho$$

where s will be chosen below. We show that $\lim_{t\to 0}\Delta_{-2,s}=0$. In fact, using the changes of variables $\frac{\rho r}{2\tau}=x$ and $\tau\frac{x^2}{r^2}=z$,

$$\begin{aligned}
\Delta_{-2,s}&=\int_0^\infty\Big(\frac{1}{2x}\Big)^{1-s}r^{-s}I_{\lambda_k}(x)dx\int_0^t\frac{1}{\tau^{1-s}}e^{-\frac{r^2}{4\tau}-\tau\frac{x^2}{r^2}}d\tau\\
&=\Big(\frac{1}{2}\Big)^{1-s}\int_0^\infty\frac{r^s}{x^{1+s}}I_{\lambda_k}(x)dx\int_0^{t\frac{x^2}{r^2}}z^{-1+s}e^{-z-\frac{x^2}{4z}}dz\\
&\le \text{const.}\int_0^\infty\frac{r^s}{x^{1+s}}I_{\lambda_k}(x)\Big(t\frac{x^2}{r^2}\Big)^\alpha dx\int_0^\infty z^{-1+s-\alpha}e^{-z-\frac{x^2}{4z}}dz\\
&\le \text{const.}t^\alpha r^{s-2\alpha}\int_0^\infty\frac{1}{x^{1-\alpha}}I_{\lambda_k}(x)K_{-\alpha+s}(x)dx.
\end{aligned}$$

To estimate the inner integral in the next to last inequality, we used the integral representation of the function $K_\nu(y)$ [8, 8.432(6)]. The convergence of the integral in the right part as $x\to 0$ is ensured (see (3.5),(3.6)) if $-1+2\alpha+\lambda_k-s>-1$, i.e. for $s < 2\alpha + \lambda_k \le 2\alpha + \pi/\theta$. The convergence of the integral as $x \to \infty$ follows from the second expressions in (3.5) and (3.6). That is why

$$\lim_{t\to 0} \Delta_{-2,s} = 0. \tag{3.9}$$

The function $u(r,\varphi) = \sin(\overline{\lambda_k}\varphi)$ where $\overline{\lambda_k}$ is some fixed number from the set $\{\lambda_k\}$ is the solution of the problem

$$\begin{gathered}
\frac{\partial u}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \frac{\overline{\lambda_k}^2}{r^2}\sin(\overline{\lambda_k}\varphi),\\
u|_{t=0} = \sin(\overline{\lambda_k}\varphi), \quad u|_{\varphi=0,\theta} = 0,
\end{gathered}$$

and, hence, there is in view of (2.5)-(2.7) for the solution

$$\begin{aligned}
&\sin(\overline{\lambda_k}\varphi)\\
&= \sin(\overline{\lambda_k}\varphi)\int_0^t d\tau \int_0^\infty d\rho \frac{\overline{\lambda_k}^2}{2\tau\rho} e^{-\frac{\rho^2+r^2}{4\tau}} I_{\overline{\lambda_k}}(\frac{\rho r}{2\tau}) + \sin(\overline{\lambda_k}\varphi)\int_0^\infty d\rho \frac{\rho}{2t} e^{-\frac{\rho^2+r^2}{4t}} I_{\overline{\lambda_k}}(\frac{\rho r}{2t}).
\end{aligned}$$

After that, (3.8) follows from (3.9). □

As an application of Lemma 3.5 is the next result.

Lemma 3.6. *The equality*

$$\lim_{t\to 0} R_2(r,\varphi,t) = u_0(r,\varphi) \tag{3.10}$$

is true for the function $R_2(r,\varphi,t)$ from (2.5).

Proof. Let us denote

$$L_k(\rho,r,t) = \frac{\rho}{2t} e^{-\frac{\rho^2+r^2}{4t}} I_{\lambda_k}(\frac{\rho r}{2t}).$$

To prove the lemma, it suffices to show that the first term in the right part of the following equality (which follows from Lemma 3.5)

$$\lim_{t\to 0}\int_0^\infty L_k(\rho,r,t)u_{0k}(\rho)d\rho = \lim_{t\to 0}\int_0^\infty L_k(\rho,r,t)[u_{0k}(\rho) - u_{0k}(r)]d\rho + u_{0k}(r) = 0.$$

Let

$$\int_0^\infty L_k(\rho,r,t)[u_{0k}(\rho) - u_{0k}(r)]d\rho \equiv d_k.$$

We apply the mean value theorem, Corollary 3.4, and take into account that $u_0(r,\varphi) \in P^{2+\alpha}_{s+2}(G)$. We have

$$u_{0k}(\rho) - u_{0k}(r) = (\rho - r)\frac{du_{0k}}{d\rho}(\overline{r}), \quad \overline{r} \in [r,\rho],$$

so that

$$\begin{aligned}
|d_k| &\le \text{const.}\overline{r}^{s+1}\int_0^\infty L_k(\rho,r,t)|\rho - r| \max_{\overline{r}} \overline{r}^{-s-1}|\frac{du_{0k}(\overline{r})}{d\overline{r}}|d\rho\\
&\le \text{const.}\overline{r}^{s+1}\max_r r^{-s-1}|\frac{du_{0k}(r)}{dr}|\int_0^\infty \frac{\rho}{2t} e^{-\frac{\rho^2+r^2}{4t}+\frac{\rho r}{2t}}(\frac{2t}{\rho r})^{1/2}|\rho - r|d\rho\\
&\le \text{const.}\frac{\overline{r}^{s+1}}{r^{1/2}}\max_r r^{-s-1}|\frac{du_{0k}(r)}{dr}|\int_0^\infty \frac{\rho^{1/2}}{t^{1/2}} e^{-\frac{(\rho-r)^2}{4t}}|\rho - r|d\rho.
\end{aligned}$$

Denote $(\rho - r)/2\sqrt{t} = z$ then

$$
\begin{aligned}
|d_k| &\le \text{const.}\frac{\overline{r}^{s+1}}{r^{1/2}} \max_r r^{-s-1} \Big|\frac{du_{0k}(r)}{dr}\Big| \int_{-\infty}^{\infty} \frac{|zt^{1/2}+r|^{1/2}}{t^{1/2}} e^{-z^2} zt\,dz \\
&\le \text{const.} t^{1/2}(t^{1/4}+r^{1/2}) \frac{\overline{r}^{s+1}}{r^{1/2}} \max_r r^{-s-1} \Big|\frac{du_{0k}(r)}{dr}\Big|.
\end{aligned}
$$

Thus, $\lim_{t\to 0} d_k = 0$ for every fixed r and all k.

Due to $u_0(r,\varphi) \in P^{2+\alpha}_{s+2}(G)$, we have $r^{-s-2}u_{0k}(r) \sim \frac{1}{k^{2+\alpha}}$ and $r^{-s-1}\frac{du_{0k}(r)}{dr} \sim \frac{1}{k^{1+\alpha}}$, and the all written above gives

$$
\sum_k \lim_{t\to 0} d_k = 0. \tag{3.11}
$$

Let us represent $R_2(r,\varphi,t)$ as

$$
\begin{aligned}
&R_2(r,\varphi,t) \\
&= \sum_k \sin(\lambda_k\varphi) \int_0^\infty L_k(\rho,r,t)[u_{0k}(\rho) - u_{0k}(r)]d\rho + \sum_k \sin(\lambda_k\varphi) u_{0k}(r) D_0(r,t)
\end{aligned}
$$

where $D_0(r,t)$ was introduced in Lemma 3.3. After passing on to the limit in this representation and taking into account (3.8) and (3.11), we obtain

$$
\begin{aligned}
\lim_{t\to 0} R_2(r,\varphi,t) &= \lim_{t\to 0} \sum_k \sin(\lambda_k\varphi) \int_0^\infty L_k(\rho,r,t)[u_{0k}(\rho) - u_{0k}(r)]d\rho \\
&\quad + \lim_{t\to 0} \sum_k \sin(\lambda_k\varphi) u_{0k}(r) D_0(r,t) = u_0(r,\varphi).
\end{aligned}
$$

□

As a some preliminary result we note that Lemma 3.1 gives the order of the decreasing to the coefficients of the trigonometric series for $r^{-s-2}R_1(r,\varphi,t)$. If one takes into account that the Fourier coefficients of functions from Hölder classes C^α have the order $1/k^\alpha$, Lemma 3.1 will lead the Fourier coefficients of $r^{-s-2}R_1(r,\varphi,t)$ have the order $1/k^{2+\alpha}$. Therefore, the function $r^{-s-2}R_1(r,\varphi,t)$ can be differentiated with respect to φ in the case r and t are fixed. We will show that the function will be differentiated twice with respect to φ. If the function $r^{-s-2}u_0(r,\varphi)$ from $R_2(r,\varphi,t)$ has the second derivative with respect to φ for the fixed r which belongs to classes C^α, Lemma 3.3 asserts that the Fourier coefficients of the function $r^{-s-2}R_2(r,\varphi,t)$ have also the order $1/k^{2+\alpha}$.

4. Some facts from the trigonometric series theory

Let $f(x)$ be a 2π-periodic function with the corresponding trigonometric series

$$
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = S[f].
$$

Note that the series $S[f]$ converges to $f(x)$ in the point x due to Dini's test for $f \in C^\alpha$. Let $f(x)$ be a continuous function, and $T_n(x)$ be any trigonometric polynomial of the order not higher then n,

$$
\Delta(T_n) := \max_{x\in[0,2\pi]} |f(x) - T_n(x)|, \quad E_n(f) := \inf \Delta(T_n)
$$

where the infimum is considered throughout the set of the polynomials $T_n(x)$. The value of $E_n(f)$ is called the best approximation of the order n to the function $f(x)$ (see [17, Ch.3, n.13]).

Theorem 4.1 (Bernstein's Theorem [2, Appendix to Ch.4, n.7])**.** *)*

$$E_n(f) = O(1/n^\alpha) \tag{4.1}$$

if and only if $f(x) \in C^\alpha$, $\alpha \in (0,1)$. *Moreover, if*

$$E_n(f) \le A\frac{1}{n^\alpha},$$

then $\langle f\rangle_x^{(\alpha)} \le const.A$.

The proof can be found in [11, Ch.4, n.2]. The next theorem contains the method of the building of the approximating trigonometric polynomial (Jackson's construction [11, Ch.4, n.2]).

Theorem 4.2. *Let a* 2π*-periodic function* $f(x) \in C^\alpha([0,2\pi])$ *and have the module of continuity* $\omega(\delta)$. *Define*

$$u_n(x) = c(n)\int_{-\pi}^{\pi} f(l)K(l-x)dl, \quad c(n) = \frac{3}{2\pi n(2n^2+1)}, \quad K(z) = \Big(\frac{\sin(nz/2)}{\sin(z/2)}\Big)^4.$$

Then the following statements hold

(1) *The function* $u_n(x)$ *has the form*

$$u_n(x) = A + \sum_{k=1}^{2n-2}(a_k\cos kx + b_k\sin kx);$$

i.e., $u_n(x)$ *is a trigonometric polynomial of the* $(2n-2)$ *order.*

(2) *If* $\int_{-\pi}^{\pi} f(x)dx = 0$, *then* $A = 0$.

(3) *The following estimates holds for all* x

$$|u_n(x) - f(x)| \le 6\omega(1/n). \tag{4.2}$$

We apply these theorems in the following case. Let one have the function $f(x,q) = \sum_k b_k(q)\sin kx$ where $q \in \Omega \subset \mathbb{R}^1$, and $b_k(q) = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x,q)\sin kx dx$, $f(x,q)$ is continuous with respect to x and q, $f(x,q) \in C_x^\alpha([0,2\pi])$ with $\alpha \in (0,1)$, uniformly with respect to q, and $\omega(\delta)$ be the module of continuity to the function $f(x,q)$ with respect to x which is uniform with respect to q,

$$\sum_k \max_q |b_k(q)| < \infty.$$

This inequality implies

$$\max_{x,q}|f| \ge \text{const.}\sum_k |b_k(q)|. \tag{4.3}$$

Indeed, because the series $\sum_k \max_q |b_k(q)|$ converges it is possible to choose N so as

$$\sum_{k=N+1}^{\infty} \max_q |b_k(q)| \le \frac{1}{2}\max_{x,q}|f|.$$

On the other hand the suitable constant can be searched such that

$$\frac{1}{2}\max_{x,q}|f| \geq \text{const.}\sum_{k=1}^{N}\max_{q}|b_k(q)|,$$

that completes the proof of (4.3).

Let us introduce the linear operator $A: C \to C$ which acts by the following rule

$$v(x,q) = Af(x,q) = \sum_{k}\mu_k(b_k(q))\sin kx, \quad |\mu_k(b_k(q))| \leq M\max_{q}|b_k(q)|, \tag{4.4}$$

where M is an independent constant of k and q. The question arises, does $v(x,q)$ have the same module of continuity as $f(x,q)$. In accordance with Bernstein's theorem this question can be reformulated as is it possible to construct the approximating polynomial to $v(x,q)$ with the same approximation like (4.1). Denote $T_n(x,q) = Au_n(x,q)$ where $u_n(x,q)$ is the approximating trigonometric polynomial to the function $f(x,q)$, then

$$v(x,q) - T_n(x,q) = Af(x,q) - Au_n(x,q).$$

Following the proof of Theorem 4.2, we can write

$$\begin{aligned} Au_n(x,q) &= A\big\{c(n)\int_0^{\pi/2}[f(x+2z,q)+f(x-2z,q)]K(2z)dz\big\} \\ &= c(n)\int_0^{\pi/2}A\big\{f(x+2z,q)+f(x-2z,q)\big\}K(2z)dz. \end{aligned}$$

After that we use the equality

$$2c(n)\int_0^{\pi/2}\Big(\frac{\sin(nz)}{\sin(z)}\Big)^4 dz = 2c(n)\int_0^{\pi/2}K(2z)dz = 1,$$

and obtain

$$T_n(x,q)-v(x,q) = c(n)\int_0^{\pi/2}A\big\{f(x+2z,q)+f(x-2z,q)-2f(x,q)\big\}K(2z)dz. \tag{4.5}$$

The definition of the operator A together with the properties of the function $f(x,q)$ lead to the estimates:

$$\max_{x,q}|A(f(x,q))| \leq M\sum_{k}\max_{q}|b_k(q)| \leq \text{const.}\max_{x,q}|f(x,q)|,$$

and that is why

$$\max_{x,q}|A\{f(x+2z,q)+f(x-2z,q)-2f(x,q)\}| \leq \text{const.}\omega(2z).$$

After that the estimate of the right part can be finished like the proof of [11, Theorem 4.2]. We have

$$|T_n(x,q) - v(x,q)| \leq \text{const.}\omega(1/n) \leq \text{const.}\frac{1}{n^{\alpha}}.$$

Bernstein's theorem leads to $Af(x,q) \in C^{\alpha}([0,2\pi])$. Moreover, the estimate

$$\langle Af(x,q)\rangle_x^{(\alpha)} \leq \text{const.}\langle f(x,q)\rangle_x^{(\alpha)} \tag{4.6}$$

follows from the proof of Bernstein's theorem. Thus we obtained the following fact.

Lemma 4.3. *Let $f(x,q)$ be continuous with respect to x,q and $f(x,q) \in C_x^\alpha$ uniformly with respect to q and $\alpha \in (0,1)$,*

$$\sum_k \max_q |b_k(q)| < \infty,$$

then $Af \in C_x^\alpha$ and estimate (4.6) holds.

Assume that $f(x,t)$ is a 2π-periodical function with respect to x and $f(x,t) \in C_{x,t}^{\alpha,\beta}$, $\alpha,\beta \in (0,1)$, and

$$\sum_k (\max_t |b_k(t)| + \langle b_k \rangle_t^\beta) < \infty. \tag{4.7}$$

Let

$$u_n(x,t) = c(n)\int_{-\pi}^{\pi} f(s,t)K(x-s)ds$$

be the trigonometric polynomial from Theorem 4.2 which approximates the function $f(x,t)$. We have

$$u_n(x,t_1) - u_n(x,t_2) = c(n)\int_{-\pi}^{\pi} [f(s,t_1) - f(s,t_2)]K(x-s)ds.$$

The properties of the kernel $K(x-s)$ ensure the inequality

$$\max_{x,t_1,t_2} \frac{|u_n(x,t_1) - u_n(x,t_2)|}{|t_1 - t_2|^\beta} \le \text{const.}\langle f(x,t)\rangle_t^{(\beta)};$$

i.e., the trigonometric polynomial approximating $f(x,t)$ has a uniformly bounded Hölder constant with respect to t. Let, as before, $T_n(x,t) = Au_n(x,t)$. Then

$$\begin{aligned} |T_n(x,t_1) - T_n(x,t_2)| &= |c(n)\int_{-\pi}^{\pi} A\{f(s,t_1) - f(s,t_2)\}K(x-s)ds| \\ &\le \text{const.} \max_{x,t_1,t_2} |f(x,t_1) - f(x,t_2)|. \end{aligned}$$

It leads to

$$\max_{x,t_1,t_2} \frac{|T_n(x,t_1) - T_n(x,t_2)|}{|t_1 - t_2|^\beta} \le \text{const.} \max_{x,t_1,t_2} \frac{|f(x,t_1) - f(x,t_2)|}{|t_1 - t_2|^\beta}. \tag{4.8}$$

If one passes to a limit in (4.8) as $n \to \infty$ (here we keep in mind that $T_n(x,t_k) \to Af(x,t_k), \quad k = 1,2$) then

$$\max_{x,t_1,t_2} \frac{|Af(x,t_1) - Af(x,t_2)|}{|t_1 - t_2|^\beta} \le \text{const.}\langle f(x,t)\rangle_t^{(\beta)}.$$

Lemma 4.4. *Let the function $f(x,t)$ be a 2π-periodical function with respect to x, and $f(x,t) \in C_{x,t}^{\alpha,\beta}$, $\alpha,\beta \in (0,1)$ and (4.7) holds. Then $Af(x,t) \in C_{x,t}^{\alpha,\beta}$ and*

$$\langle Af\rangle_x^{(\alpha)} \le const.\langle f(x,t)\rangle_x^{(\alpha)}, \quad \langle Af\rangle_t^{(\beta)} \le const.\langle f(x,t)\rangle_t^{(\beta)}. \tag{4.9}$$

Remark 4.5. Lemmas 4.3 and 4.4 will hold if we change the functions $f(x,q) \in C_x^\alpha$ and $f(x,t) \in C_{x,t}^{\alpha,\beta}$ onto $f(x,q_1,\dots q_n) \in C_x^\alpha$ uniformly with respect to $q_1 \dots q_n$ in Lemma 4.3, and $f(x,t_1,\dots t_n) \in C_{x,t_1,\dots t_n}^{\alpha,\beta_1,\dots\beta_n}$ with $0 < \beta_i < 1$, $i = \overline{1,n}$ in Lemma 4.4, correspondingly, and the inequality like (4.7) holds.

5. Estimates of the higher seminorms of the solution

5.1. Estimate for $\frac{\partial^2 R_1}{\partial\varphi^2}$.

Lemma 5.1. *The function $\frac{\partial^2 R_1}{\partial\varphi^2}$ meets the Hölder condition with respect to φ and*

$$\langle \frac{\partial^2 R_1}{\partial\varphi^2}\rangle^{(\alpha)}_{\varphi;s+2,G_T} + \sum_k \lambda_k^2 \max_{\bar{\mathbb{R}}_T} r^{-s-2}|R_{1,k}(r,t)| \le const.\langle f\rangle^{(\alpha)}_{\varphi;s,G_T}. \tag{5.1}$$

Proof. After a formal differentiation with respect to φ one can obtain

$$\frac{\partial^2 R_1}{\partial\varphi^2} = -\sum_k \lambda_k^2 \sin(\lambda_k\varphi)\int_0^t d\tau \int_0^\infty L_k(\rho,r,t-\tau)b_k(\rho,\tau)d\rho \tag{5.2}$$

where $b_k(r,t)$ are the Fourier coefficients of the function $f(r,\varphi,t)$. The function $f(r,\varphi,t)$ is continued odd onto the interval $(-\theta,0)$, and $f(r,\varphi,t)=0$ if $\varphi=0,\theta$ or $t<0$. In the case of a $2\theta-$ periodical function, the change of variables allows keeping the mentioned above argumentations regarding to use of the approximating trigonometric polynomial to a $2\pi-$ periodical function. Let us denote by

$$B_k = -\lambda_k^2 \int_0^t d\tau \int_0^\infty L_k(\rho,r,t-\tau)b_k(\rho,\tau)d\rho,$$

in view of Lemma 3.1,

$$|B_k| \le \text{const.} r^{2+s}\max_{r,t} r^{-s}|b_k| \tag{5.3}$$

with the constant is independent of k. After that, we put in (4.4): $x=\varphi$, $f(x,q):=f(r,\varphi,t)$, $b_k(q):=r^{-s}b_k(r,t)$, $\mu_k(b_k(q)):=r^{-2-s}B_k$, $Af(x,q):=r^{-2-s}\frac{\partial^2 R_1}{\partial\varphi^2}$. Then Lemma 4.3 together with the properties of the function $f(r,\varphi,t)$ (namely, $f\in \widehat{P}_s^{\alpha,\alpha/2}(\overline{G}_T)$, i.e. $r^{-s}f\in C^\alpha_\varphi([0,\theta])$ uniformly with respect to t and r, inequality like (4.3) holds) lead to estimate (5.1). □

Lemma 5.2. *The function $\frac{\partial^2 R_1}{\partial\varphi^2}(r,\varphi,t)$ satisfies the Hölder conditions with respect to t and r. Moreover,*

$$\sum_k \lambda_k^2 \langle R_{1,k}\rangle^{(\alpha/2)}_{t;s+2-\alpha,\mathbb{R}_T} \le const.\langle f\rangle^{(\alpha/2)}_{t;s-\alpha,G_T}, \tag{5.4}$$

$$\sum_k \lambda_k^2 \langle R_{1,k}\rangle^{(\alpha)}_{r;s+2-\alpha,\mathbb{R}_T} \le const.\langle f\rangle^{(\alpha)}_{r;s-\alpha,G_T}, \tag{5.5}$$

$$\begin{aligned}&[\frac{\partial^2 R_1}{\partial\varphi^2}]^{(\alpha,\alpha/2)}_{\varphi,t;s+2-\alpha,G_T} + \sum_k \lambda_k^2 [R_{1,k}]^{(\alpha,\alpha/2)}_{r,t;s+2-2\alpha,\mathbb{R}_T}\\ &\le const.([f]^{(\alpha,\alpha/2)}_{\varphi,t;s-\alpha,G_T} + [f]^{(\alpha,\alpha/2)}_{r,t;s-2\alpha,G_T}).\end{aligned} \tag{5.6}$$

Proof. The proof of estimates (5.4) and (5.5) follows from the properties of the function $f(r,\varphi,t)$, Lemma 3.1 and Lemma 4.4. Regarding inequality (5.6), it is obtained if one applies Lemmas 4.4 and 5.1 to the function $[\frac{\partial^2 R_1}{\partial\varphi^2}(r,\varphi,t_2) - \frac{\partial^2 R_1}{\partial\varphi^2}(r,\varphi,t_1)]$. □

5.2. **Estimates of the derivative of the function $R_1(r,\varphi,t)$ with respect to t.** First we obtain the representation of $\partial R_1/\partial t$. Let

$$v_k(r,t)=\frac{\partial}{\partial t}\int_0^t d\tau\int_0^\infty d\rho L_k(\rho,r,\tau)\int_0^\theta\frac{2}{\theta}f(\rho,\psi,t-\tau)\sin(\lambda_k\psi)d\psi. \tag{5.7}$$

Assume from the beginning that $f(r,\varphi,t)$ is differentiated with respect to t. Then differentiation under the integral sign acts on $f(r,\varphi,t)$, and integrating by parts gives

$$v_k(r,t)=\int_0^t d\tau\int_0^\infty d\rho\frac{\partial L_k}{\partial\tau}(\rho,r,\tau)b_k(\rho,t-\tau)+\lim_{\varepsilon\to0}\int_0^\infty d\rho L_k(\rho,r,\varepsilon)b_k(\rho,t). \tag{5.8}$$

Note that the derivative of the function $f(r,\varphi,t)$ is not required in (5.8). Using the relation above, we obtain the following representation

$$\begin{aligned}\frac{\partial R_1}{\partial t}(r,\varphi,t)&=\sum_k\sin(\lambda_k\varphi)\int_0^t d\tau\int_0^\infty d\rho\frac{\partial L_k}{\partial\tau}(\rho,r,\tau)\\&\quad\times\int_0^\theta\frac{2}{\theta}[f(\rho,\psi,t-\tau)-f(\rho,\psi,t)]\sin(\lambda_k\psi)d\psi\\&\quad+\sum_k\sin(\lambda_k\varphi)\int_0^\infty d\rho L_k(\rho,r,t)\int_0^\theta\frac{2}{\theta}f(\rho,\psi,t)\sin(\lambda_k\psi)d\psi\\&\equiv A_1+A_2.\end{aligned}$$

Straight away, we obtain another useful representation of $\frac{\partial R_1}{\partial t}(r,\varphi,t)$. Let

$$\begin{aligned}v_{1k}(\rho,t)&=\int_{-\infty}^t d\tau\int_0^\infty d\rho L_k(\rho,r,t-\tau)b_k(\rho,\tau),\\v_{1k}^h(\rho,t)&=\int_{-\infty}^{t-h} d\tau\int_0^\infty d\rho L_k(\rho,r,t-\tau)b_k(\rho,\tau).\end{aligned}$$

The derivative of $\partial v_{1k}/\partial t$ is $\lim_{h\to0}\frac{\partial v_{1k}^h}{\partial t}$. Non-complicated calculations and Corollary 3.4 give

$$\lim_{t\to+\infty}\lim L_k(\rho,r,t)=0,$$

and then

$$\frac{\partial v_{1k}}{\partial t}=\int_{-\infty}^t d\tau\int_0^\infty\frac{\partial L_k}{\partial t}(\rho,r,t-\tau)[b_k(\rho,\tau)-b_k(\rho,t)]d\rho, \tag{5.9}$$

$$\begin{aligned}\frac{\partial R_1}{\partial t}(r,\varphi,t)&=\sum_k\sin(\lambda_k\varphi)\int_{-\infty}^t d\tau\int_0^\infty d\rho\frac{\partial L_k}{\partial t}(\rho,r,t-\tau)\\&\quad\times\int_0^\theta\frac{2}{\theta}[f(\rho,\psi,\tau)-f(\rho,\psi,t)]\sin(\lambda_k\psi)d\psi.\end{aligned} \tag{5.10}$$

We will use the next representation

$$\begin{aligned}\frac{\partial L_k}{\partial t}(\rho,r,t)&=-\frac{\rho}{2t^2}e^{-\frac{\rho^2+r^2}{4t}}I_{\lambda_k}(\frac{\rho r}{2t})+\frac{\rho}{2t}\frac{\partial}{\partial t}\{e^{-\frac{\rho^2+r^2}{4t}}I_{\lambda_k}(\frac{\rho r}{2t})\}\\&=i_{1k}(\rho,r,t)+i_{2k}(\rho,r,t).\end{aligned} \tag{5.11}$$

Lemma 5.3. *The following estimate holds*

$$\sum_k \max_{\overline{\mathbb{R}}_T} r^{-s} |\frac{\partial R_{1,k}}{\partial t}| \le const. \|f\|_{P_s^{\alpha,\alpha/2}(G_T)}. \tag{5.12}$$

Proof. First we justify the estimate

$$|\frac{\partial R_{1,k}}{\partial t}| = \Big| \int_{-\infty}^t d\tau \int_0^\infty d\rho \frac{\partial L_k}{\partial t}(\rho,\varphi,t-\tau)[b_k(\rho,\tau) - b_k(\rho,t)]\Big| \le \text{const.} \langle b_k \rangle_{t,s-\alpha}^{(\alpha/2)} r^s \tag{5.13}$$

where

$$\begin{aligned}\frac{\partial}{\partial t} L_k(\rho,r,t) &= -\frac{1}{t} L_k(\rho,r,t) + \frac{\rho(\rho^2+r^2)}{8t^3} I_{\lambda_k}(\frac{r\rho}{2t}) e^{-\frac{\rho^2+r^2}{4t}} \\ &\quad - \frac{r\rho^2}{4t^3} e^{-\frac{\rho^2+r^2}{4t}} \frac{d}{dx} I_{\lambda_k}(x), \quad x = \frac{r\rho}{2t}.\end{aligned}$$

From the representation of the function $I_{\lambda_k}(x)$ (see [8, 8.431(1)])

$$I_{\lambda_k}(x) = \frac{(x/2)^{\lambda_k}}{\Gamma(\lambda_k+1/2)\Gamma(1/2)} \int_{-1}^1 e^{xy}(1-y^2)^{\lambda_k-1/2} dy,$$

it follows

$$\begin{aligned}\frac{dI_{\lambda_k}(x)}{dx} &= \frac{\lambda_k}{x} I_{\lambda_k}(x) + \frac{(x/2)^{\lambda_k}}{\Gamma(\lambda_k+1/2)\Gamma(1/2)} \int_{-1}^1 y e^{xy}(1-y^2)^{\lambda_k-1/2} dy \\ &\equiv \frac{\lambda_k}{x} I_{\lambda_k}(x) + Q_{\lambda_k}(x).\end{aligned}$$

On the other hand,(see [8, 8.486(4)])

$$x\frac{dI_{\lambda_k}(x)}{dx} = \lambda_k I_{\lambda_k}(x) + x I_{\lambda_k+1}(x),$$

and, hence, $xQ_{\lambda_k}(x) = xI_{\lambda_k+1}(x)$. From this equation and the definition of $Q_{\lambda_k}(x)$, we obtain

$$xQ_{\lambda_k}(x) \le \text{const.} \begin{cases} \frac{x^{\lambda_k+2}}{\Gamma(\lambda_k+1)}, & \text{for } x \le 1, \\ xI_{\lambda_k}(x), & \text{for } x > 1. \end{cases}$$

Returning to $\frac{\partial L_k}{\partial t}(\rho,r,t)$, we have

$$\begin{aligned}\frac{\partial L_k}{\partial t}(\rho,r,t) &= -\frac{1}{t} L_k + \frac{1}{t} L_k \frac{\rho^2+r^2}{4t} - \frac{\lambda_k}{t} L_k - \frac{\rho}{2t^2} x Q_{\lambda_k}(x) e^{-\frac{\rho^2+r^2}{4t}} \\ &\equiv -m_1(\rho,r,t) + m_2(\rho,r,t) - m_3(\rho,r,t) - m_4(\rho,r,t), \quad x = \frac{r\rho}{2t}.\end{aligned}$$

Let

$$M(r,t) = \int_{-\infty}^t d\tau \int_0^\infty d\rho \rho^{s-\alpha} (t-\tau)^{\alpha/2} L_k(\rho,r,t-\tau).$$

Since

$$M(r,t) = \int_0^\infty dz \int_0^\infty d\rho \rho^{s-\alpha} z^{\alpha/2} L_k(\rho,r,z),$$

then $\frac{\partial M}{\partial t}(r,t) = 0$. Due to Lemma 3.3,

$$\lim_{t\to 0} t^{\alpha/2} \int_0^\infty d\rho \rho^{s-\alpha} L_k(\rho,r,t) = 0,$$

and therefore

$$\begin{aligned}\frac{\partial M}{\partial t}(r,t) &= \int_{-\infty}^{t} d\tau \int_0^\infty d\rho [\frac{\partial}{\partial t}(t-\tau)^{\alpha/2}]\rho^{s-\alpha} L_k(\rho,r,t-\tau)\\ &\quad + \int_{-\infty}^{t} d\tau \int_0^\infty d\rho (t-\tau)^{\alpha/2}\rho^{s-\alpha}\frac{\partial}{\partial t} L_k(\rho,r,t-\tau).\end{aligned} \tag{5.14}$$

Let us consider the integral

$$M_1 = (1+\lambda_k)\int_0^\infty dt \int_0^\infty d\rho \rho^{s-\alpha} t^{-1+\alpha/2} L_k(\rho,r,t)$$

corresponding to $(m_1(\rho,r,t)+m_3(\rho,r,t))$ in the representation of $\frac{\partial L_k}{\partial t}(\rho,r,t)$. Then the following estimate holds

$$|M_1| \le r^s \frac{1}{\lambda_k^{\alpha/2-\mu}}, \quad \mu < \alpha/2. \tag{5.15}$$

We represent its proof in the Appendix (see Subsection 7.3). Now, using the integral representations of $I_{\lambda_k}(x)$ and $Q_{\lambda_k}(x)$, we have

$$\begin{aligned}&m_2(\rho,r,t) - m_4(\rho,r,t)\\ &= \frac{\rho}{8t^3} e^{-\frac{\rho^2+r^2}{4t}} \{(\rho^2+r^2) I_{\lambda_k}(r\rho/2t) - 2r\rho Q_{\lambda_k}(r\rho/2t)\}\\ &= \frac{\rho}{8t^3} e^{-\frac{\rho^2+r^2}{4t}} \{(\rho^2-2r\rho+r^2) I_{\lambda_k}(r\rho/2t)\\ &\quad + 2r\rho \frac{(r\rho/4t)^{\lambda_k}}{\Gamma(\lambda_k+1/2)\Gamma(1/2)} \int_{-1}^{1}(1-y)e^{xy}(1-y^2)^{\lambda_k-1/2}dy\} \ge 0.\end{aligned}$$

Estimate (5.15) implies

$$\int_0^\infty d\tau \int_0^\infty d\rho \rho^{s-\alpha}\tau^{\alpha/2}(m_1(\rho,r,\tau)+m_3(\rho,r,\tau)) \le \text{const.} r^s \frac{1}{\lambda_k^{\alpha/2-\mu}}. \tag{5.16}$$

Note that the estimate of the first term at the right part of (5.14) is contained in (5.15), thus, by the equation $\frac{\partial M}{\partial t}(r,t) = 0$, we have

$$\int_0^\infty d\tau \int_0^\infty d\rho \rho^{s-\alpha}\tau^{\alpha/2}(m_2(\rho,r,\tau)-m_4(\rho,r,\tau)) \le \text{const.} r^s \frac{1}{\lambda_k^{\alpha/2-\mu}}. \tag{5.17}$$

At last, we are ready with (5.16) and (5.17) to prove inequality (5.13):

$$\begin{aligned}|\frac{\partial R_{1k}}{\partial t}| &\le \langle b_k\rangle^{(\alpha/2)}_{t,s-\alpha,\mathbb{R}_T} r^s \int_0^\infty d\tau \int_0^\infty d\rho |\frac{\partial L_k}{\partial t}(\rho,\varphi,\tau)|\rho^{s-\alpha}\tau^{\frac{\alpha}{2}}\\ &= \langle b_k\rangle^{(\alpha/2)}_{t,s-\alpha,\mathbb{R}_T} r^s \int_{-\infty}^{t}\int_0^\infty \rho^{s-\alpha}\tau^{\frac{\alpha}{2}}(m_1(\rho,r,\tau)+m_3(\rho,r,\tau)\\ &\quad + m_2(\rho,r,\tau)-m_4(\rho,r,\tau))\\ &\le \text{const.} \langle b_k\rangle^{(\alpha/2)}_{t,s-\alpha,\mathbb{R}_T} \frac{r^s}{\lambda_k^{\alpha/2-\mu}}.\end{aligned}$$

As the series $\sum_k \langle b_k\rangle^{(\alpha/2)}_{t,s-\alpha,\mathbb{R}_T}$ converges, we arrive at (5.12) which completes the proof. □

5.3. **Hölder constant for $\frac{\partial R_1}{\partial t}(r,\varphi,t)$ with respect to φ.** We apply Theorem 4.2 to estimate the Hölder constant of the function $\frac{\partial R_1}{\partial t}(r,\varphi,t)$ with respect to φ. Let us consider the function

$$F(x,t,\tau)=g(x,t-\tau)-g(x,t),\quad g(x,t)\in C^{\alpha,\alpha/2}_{x,t}(\Omega_T)$$

where $\Omega_T:=[0,2\pi]\times[0,T]$, $\alpha\in(0,1)$, and the function $g(x,t)$ satisfies (4.7). The approximating polynomial to $F(x,t,\tau)$ is

$$u_n(x,t,\tau)=c(n)\int_0^{\pi/2}[F(x+2l,t,\tau)+F(x-2l,t,\tau)]K(2l)dl.$$

Let

$$T_n(x,t,\tau)=\overline{A}u_n(x,t,\tau)$$

where the operator $\overline{A}$, on the one hand, is the operator like A from (4.4) with $f(x,q):=F(x,t,\tau)$, and, on the other hand, models the operator from the right hand side in (5.10). In the same way as above,

$$\begin{aligned}&T_n(x,t,\tau)-\overline{A}F(x,t,\tau)\\&=c(n)\int_0^{\pi/2}\overline{A}\{F(x+2l,t,\tau)+F(x-2l,t,\tau)-2F(x,t,\tau)\}K(2l)dl.\end{aligned}\tag{5.18}$$

After applying the operator $\overline{A}$ and following the proof of Lemma 5.3, we have

$$\max_{x,t,\tau}|\overline{A}F(x,t,\tau)|\le\text{const.}\max_{x,t,\tau}\{\frac{|F(x,t,\tau)|}{\tau^{\alpha/2}}\}.$$

We apply this estimate to the integrand in (5.18) and obtain

$$\begin{aligned}&|T_n(x,t,\tau)-\overline{A}F(x,t,\tau)|\\&\le\text{const.}c(n)\int_0^{\pi/2}K(2l)\max_{x,t,\tau}\big\{\frac{|F(x+2l,t,\tau)+F(x-2l,t,\tau)-2F(x,t,\tau)|}{\tau^{\alpha/2}}\big\}dl.\end{aligned}$$

It is obvious that

$$\max_{x,t,\tau}\frac{|F(x+2l,t,\tau)+F(x-2l,t,\tau)-2F(x,t,\tau)|}{\tau^{\alpha/2}}\le\text{const.}[g]^{(\alpha,\alpha/2)}_{x,t;\Omega_T}l^\alpha.$$

That is why following the proof of Theorem 4.2, we obtain that the studied function $\overline{A}F(x,t,\tau)$ belongs to $C^\alpha_x[0,2\pi]$).

Thus, similar considerations as in the case of the function $\frac{\partial R_1}{\partial t}(r,\varphi,t)$ lead to

$$r^{-s}\frac{\partial R_1}{\partial t}(r,\varphi,t)\in C^\alpha_\varphi,\quad\langle\frac{\partial R_1}{\partial t}\rangle^{(\alpha)}_{\varphi;s,G_T}\le\text{const.}[f]^{(\alpha,\alpha/2)}_{\varphi,t;s-\alpha,G_T}.\tag{5.19}$$

This is the place where the additional smoothness of the function $f(r,\varphi,t)$; i.e., the boundedness of the seminorm $[f]^{(\alpha,\alpha/2)}_{\phi,t;s-\alpha,G_T}$, is used. That, of course, is stipulated by the approach to the investigation of the problem.

5.4. **Hölder constant for $\frac{\partial R_1}{\partial t}(r,\varphi,t)$ with respect to t.** In this section we make use representation (5.10) of the function $\frac{\partial R_1}{\partial t}(r,\varphi,t)$. Let $t_2>t_1$, and $\triangle t=t_2-t_1$. We have

$$\begin{aligned}
&\frac{\partial R_1}{\partial t}(r,\varphi,t_2)-\frac{\partial R_1}{\partial t}(r,\varphi,t_1)\\
&=\sum_k\sin(\lambda_k\varphi)\int_{2t_1-t_2}^{t_2}d\tau\int_0^\infty d\rho\frac{\partial L_k}{\partial\tau}(\rho,r,t_2-\tau)[b_k(\rho,\tau)-b_k(\rho,t_2)]\\
&\quad-\sum_k\sin(\lambda_k\varphi)\int_{2t_1-t_2}^{t_1}d\tau\int_0^\infty d\rho\frac{\partial L_k}{\partial\tau}(\rho,r,t_1-\tau)[b_k(\rho,\tau)-b_k(\rho,t_1)]\\
&\quad+\sum_k\sin(\lambda_k\varphi)\int_{-\infty}^{2t_1-t_2}d\tau\int_0^\infty d\rho[b_k(\rho,\tau)-b_k(\rho,t_1)]\\
&\quad\times[\frac{\partial L_k}{\partial\tau}(\rho,r,t_2-\tau)-\frac{\partial L_k}{\partial\tau}(\rho,r,t_1-\tau)]\\
&\quad+\sum_k\sin(\lambda_k\varphi)\int_{-\infty}^{2t_1-t_2}d\tau\int_0^\infty d\rho[b_k(\rho,t_1)-b_k(\rho,t_2)]\frac{\partial L_k}{\partial\tau}(\rho,r,t_2-\tau)\\
&=\sum_{i=1}^4\sum_k\sin(\lambda_k\varphi)\sum_{j=1}^2A_{j,k}^{(i)},
\end{aligned}\tag{5.20}$$

where $A_{1,k}^{(i)}$, $i=\overline{1,4}$, correspond to i_{1k} in representation (5.11) for the function $\partial L_k/\partial t$ and $A_{2,k}^{(i)}$, $i=\overline{1,4}$, do to i_{2k}. By the definition

$$A_{1,k}^{(1)}=-\int_{2t_1-t_2}^{t_2}d\tau\int_0^\infty d\rho\frac{\rho}{2(t_2-\tau)^2}e^{-\frac{\rho^2+r^2}{4(t_2-\tau)}}I_{\lambda_k}(\frac{\rho r}{2(t_2-\tau)})[b_k(\rho,\tau)-b_k(\rho,t_2)],$$

so that the inequality

$$|A_{1,k}^{(1)}|\le\text{const.}\int_{2t_1-t_2}^{t_2}d\tau\int_0^\infty d\rho\frac{\rho^{s+1-\alpha}}{(t_2-\tau)^{2-\alpha/2}}e^{-\frac{\rho^2+r^2}{4(t_2-\tau)}}I_{\lambda_k}(\frac{\rho r}{2(t_2-\tau)})\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)}$$

is valid. After applying Lemma 3.3, we obtain

$$\begin{aligned}
|A_{1,k}^{(1)}|&\le\text{const.}r^{s-\alpha}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)}\int_{2t_1-t_2}^{t_2}\frac{d\tau}{(t_2-\tau)^{1-\alpha/2}}\\
&\le\text{const.}r^{s-\alpha}(\Delta t)^{\alpha/2}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)}.
\end{aligned}\tag{5.21}$$

The estimate of $A_{1,k}^{(2)}$ has been done the same way. To estimate

$$A_{1,k}^{(3)}=\int_{-\infty}^{2t_1-t_2}d\tau\int_0^\infty d\rho[b_k(\rho,\tau)-b_k(\rho,t_1)][i_{1k}(\rho,r,t_2-\tau)-i_{1k}(\rho,r,t_1-\tau)],$$

we apply the mean value theorem. To this end we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\{\frac{\rho}{2t^2}e^{-\frac{\rho^2+r^2}{4t}}I_{\lambda_k}(\frac{\rho r}{2t})\}&=-\frac{\partial i_{1k}}{\partial t}\\
&=-\frac{\rho}{t^3}e^{-\frac{\rho^2+r^2}{4t}}I_{\lambda_k}(\frac{\rho r}{2t})+\frac{\rho}{2t^2}\frac{\partial}{\partial t}\{e^{-\frac{\rho^2+r^2}{4t}}I_{\lambda_k}(\frac{\rho r}{2t})\}\\
&=J_{1k}+J_{2k}.
\end{aligned}$$

Let $\bar t \in (t_1, t_2)$ and

$$A^{(3,1)}_{1,k} = \int_{-\infty}^{2t_1-t_2} d\tau \int_0^\infty d\rho [b_k(\rho,\tau) - b_k(\rho,t_1)] \frac{\rho(t_2-t_1)}{(\bar t-\tau)^3} e^{-\frac{\rho^2+r^2}{4(\bar t-\tau)}} I_{\lambda_k}\big(\frac{\rho r}{2(\bar t-\tau)}\big).$$

We restrict ourself only by the estimate of $A^{(3,1)}_{1,k}$, that is the part of $A^{(3)}_{1,k}$ corresponding to J_{1k}. The rest estimates are proved with the same way.

Note that $\bar t - \tau \geq t_1 - \tau$ and $\bar t - 2t_1 + t_2 \geq \Delta t$, thus, Lemma 3.3 gives

$$\begin{aligned} A^{(3,1)}_{1,k} &\leq (\Delta t) \langle b_k\rangle^{(\alpha/2)}_{t,s-\alpha,\mathbb{R}_T} \int_{-\infty}^{2t_1-t_2} d\tau \int_0^\infty d\rho \frac{\rho^{s+1-\alpha}}{(\bar t-\tau)^{3-\alpha/2}} e^{-\frac{\rho^2+r^2}{4(\bar t-\tau)}} I_{\lambda_k}\big(\frac{\rho r}{2(\bar t-\tau)}\big) \\ &\leq \text{const.} r^{s-\alpha}(\Delta t)^{\alpha/2} \langle b_k\rangle^{(\alpha/2)}_{t;s-\alpha,\mathbb{R}_T}. \end{aligned}$$

The estimate of $A^{(4)}_{j,k}$ in (5.20) is obtained simultaneously for $j = 1$ and $j = 2$. We have

$$\begin{aligned} \big|\sum_{j=1}^{2} A^{(4)}_{j,k}\big| &= \Big| \int_{-\infty}^{2t_1-t_2} d\tau \int_0^\infty d\rho [b_k(\rho,t_1) - b_k(\rho,t_2)] \frac{\partial L_k}{\partial \tau}(\rho,r,t_2-\tau)\Big| \\ &= \Big| \int_0^\infty d\rho [b_k(\rho,t_1) - b_k(\rho,t_2)] \int_{-\infty}^{2t_1-t_2} \frac{\partial L_k}{\partial \tau}(\rho,r,t_2-\tau) d\tau\Big| \\ &= \Big| \int_0^\infty d\rho [b_k(\rho,t_1) - b_k(\rho,t_2)] \frac{\rho}{2\Delta t} e^{-\frac{\rho^2+r^2}{4\Delta t}} I_{\lambda_k}\big(\frac{\rho r}{2\Delta t}\big)\Big| \\ &\leq \text{const.} r^{s-\alpha}(\Delta t)^{\alpha/2} \langle b_k\rangle^{(\alpha/2)}_{t;s-\alpha,\mathbb{R}_T}, \end{aligned}$$

where Lemma 3.3 has been applied.

The coefficients $A^{(i)}_{j,k}$, $i = \overline{1,3}$, $j = 2$, are evaluated similarly. Thus, the above gives an estimate for all $i = \overline{1,4}$, and $j = 1, 2$

$$|A^{(i)}_{j,k}| \leq \text{const.} r^{s-\alpha}(\Delta t)^{\alpha/2} \langle b_k\rangle^{(\alpha/2)}_{t;s-\alpha,\mathbb{R}_T}.$$

This inequality together with the convergence of $\sum_k \langle b_k\rangle^{(\alpha/2)}_{t;s-\alpha,\mathbb{R}_T}$ lead to

$$\sum_k \langle \frac{\partial R_{1,k}}{\partial t}\rangle^{(\alpha/2)}_{t;s-\alpha,\mathbb{R}_T} \leq \text{const.} \langle f\rangle^{(\alpha/2)}_{t;s-\alpha,G_T} \tag{5.22}$$

as it was to be proved.

5.5. **Hölder constant of the function $\frac{\partial R_1}{\partial t}(r,\varphi,t)$ with respect to r.** We change the variables in the representation $\frac{\partial R_1}{\partial t}(r,\varphi,t)$ from (5.10): $t-\tau \to \tau$, and consider as the example, the part of one which corresponds to $i_{1k}(\rho,r,t)$ in (5.11). Let

$$\begin{aligned} V_1(r,\varphi,t) &= \sum_k \sin(\lambda_k\varphi) \int_0^\infty d\tau \int_0^\infty d\rho\, i_{1k}(\rho,r,\tau)[b_k(\rho,t-\tau) - b_k(\rho,t)] \\ &\equiv \sum_k \sin(\lambda_k\varphi) V_{1,k}(r,t). \end{aligned}$$

Consider the difference as $h > 0$

$$
\begin{aligned}
&V_1(r+h,\varphi,t) - V_1(r,\varphi,t)\\
&= \sum_k \sin(\lambda_k\varphi)\int_0^{h^2} d\tau \int_0^\infty d\rho\, i_{1k}(\rho, r+h,\tau)[b_k(\rho,t-\tau) - b_k(\rho,t)]\\
&\quad - \sum_k \sin(\lambda_k\varphi)\int_0^{h^2} d\tau \int_0^\infty d\rho\, i_{1k}(\rho, r,\tau)[b_k(\rho,t-\tau) - b_k(\rho,t)]\\
&\quad + \sum_k \sin(\lambda_k\varphi)\int_{h^2}^{\infty} d\tau \int_0^\infty d\rho[i_{1k}(\rho, r+h,\tau) - i_{1k}(\rho,r,\tau)]\\
&\quad \times [b_k(\rho,t-\tau) - b_k(\rho,t)] \equiv \sum_{j=1}^3\sum_k V_{1,k}^j \sin(\lambda_k\varphi).
\end{aligned}
\tag{5.23}
$$

Let $r_h = r+h$. One can easy estimate the coefficients $V_{1,k}^j$, $j=1,2$ with Lemma 3.3. For instance,

$$
\begin{aligned}
|V_{1,k}^1| &\le \langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} \int_0^{h^2} d\tau \int_0^\infty d\rho \frac{\rho^{s+1-\alpha}}{2\tau^2}\tau^{\alpha/2} e^{-\frac{\rho^2+r_h^2}{4\tau}} I_{\lambda_k}(\frac{\rho r_h}{2\tau})\\
&\le \text{const.} r^{s-\alpha}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} \int_0^{h^2} d\tau \frac{1}{\tau^{1-\alpha/2}}\\
&\le \text{const.} r^{s-\alpha}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} h^\alpha.
\end{aligned}
$$

To estimate $V_{1,k}^3$ in (5.23), we apply the mean value theorem. We have

$$
\begin{aligned}
\frac{\partial i_{1k}(\rho,r,t)}{\partial r} &= \frac{r\rho}{4t^3} e^{-\frac{\rho^2+r^2}{4t}} I_{\lambda_k}(\frac{\rho r}{2t}) - \frac{\rho^2}{4t^3}e^{-\frac{\rho^2+r^2}{4t}}\frac{d}{dx}I_{\lambda_k}(x)\\
&= \frac{\rho}{2t^2}\Big\{\frac{r}{2t}I_{\lambda_k}(x) - \frac{\rho}{2t}\frac{d}{dx}I_{\lambda_k}(x)\Big\} e^{-\frac{\rho^2+r^2}{4t}}\\
&= \frac{\rho}{2t^2}\frac{r-\rho}{2t} I_{\lambda_k}(x) e^{-\frac{\rho^2+r^2}{4t}} - \frac{\rho}{2t^2}\frac{\rho}{2t}[\frac{d}{dx}I_{\lambda_k}(x) - I_{\lambda_k}(x)]e^{-\frac{\rho^2+r^2}{4t}}\\
&= j_{1k} + j_{2k}
\end{aligned}
\tag{5.24}
$$

where $\rho r/2t = x$. In compliance with (5.24) the Fourier coefficients $V_{1,k}^3$ can be represented as $V_{1,k}^3 = V_{1,k}^{3,1} + V_{1,k}^{3,2}$. First we estimate $V_{1,k}^{3,1}$

$$
V_{1,k}^{3,1} = h\int_{h^2}^\infty d\tau \int_0^\infty d\rho \frac{\rho(\overline{r}-\rho)}{4\tau^3} e^{-\frac{\rho^2+\overline{r}^2}{4\tau}} I_{\lambda_k}(\frac{\rho\overline{r}}{2\tau})[b_k(\rho,t-\tau) - b_k(\rho,t)]
$$

where $\overline{r} \in (r, r+h)$. We have by properties of the function $b_k(\rho,t)$

$$
|V_{1,k}^{3,1}| \le \langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} h\int_{h^2}^\infty d\tau \int_0^\infty d\rho \frac{\rho^{1+s-\alpha}|\overline{r}-\rho|}{4\tau^3}\tau^{\alpha/2} e^{-\frac{\rho^2+\overline{r}^2}{4\tau}} I_{\lambda_k}(\frac{\rho\overline{r}}{2\tau}),
$$

and as it follows from Subsection 7.4 in the appendix,

$$
\int_0^\infty d\rho \frac{\rho^{1+s-\alpha}|\overline{r}-\rho|}{4t^{3/2}} e^{-\frac{\rho^2+\overline{r}^2}{4t}} I_{\lambda_k}(\frac{\rho\overline{r}}{2t}) \le \text{const.}\overline{r}^{s-\alpha}.
$$

Therefore,

$$
\begin{aligned}
|V_{1,k}^{3,1}| &\le \text{const.}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} h\overline{r}^{s-\alpha}\int_{h^2}^{\infty}\tau^{\alpha/2-3/2}d\tau \le \text{const.}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} h^{\alpha}\overline{r}^{s-\alpha}\\
&\le \text{const.}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} h^{\alpha}(r+h)^{s-\alpha} \le \text{const.}\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)} h^{\alpha}r^{s-\alpha},
\end{aligned}
\tag{5.25}
$$

since the only $h \le r$ should be considered, to obtain the Hölder constant for $\partial R_1/\partial t$ with respect to r.

As for $V_{1,k}^{3,2}$ corresponding to j_{2k} in (5.24), it can be represented as ($x = r\rho/2t$)

$$
j_{2k} = \frac{-\rho^2}{4t^3}[\frac{d}{dx}I_{\lambda_k}(x) - I_{\lambda_k}(x)]e^{-\frac{\rho^2+r^2}{4t}} = \frac{\rho^2}{4t^3}e^{-\frac{\rho^2+r^2}{4t}}[I_{\lambda_k}(x) - \frac{\lambda_k}{x}I_{\lambda_k}(x) - Q_{\lambda_k}(x)]
$$

and

$$
j_{2k} \le \frac{\rho\lambda_k}{2rt^2}I_{\lambda_k}(r\rho/2t)e^{-\frac{\rho^2+r^2}{4t}} + \frac{\rho^2}{4t^3}e^{-\frac{\rho^2+r^2}{4t}}[I_{\lambda_k}(r\rho/2t) - I_{\lambda_k+1}(r\rho/2t)].
$$

Note that the first term in the right part of the last inequality is estimated in the proof of Lemma 5.3 (see (5.15)), as for the second term one is evaluated like $V_{1,k}^{3,1}$. If we take into account that from the equation $I_{\lambda_k+1}(x) = Q_{\lambda_k}(x)$, we have $I_{\lambda_k+1}(x) \le \text{const.}I_{\lambda_k}(x)$, and, hence, by Corollary 3.4, $x^{1/2}e^{-x}I_{\lambda_k+1}(x) \le const.$ uniformly in k. From here it follows that $I_{\lambda_k}(x) - I_{\lambda_k+1}(x) \sim \text{const.}x^{-3/2}$ for large value of x where the constant in independent of k. Using this fact, we can repeat the arguments from Subsection 7.4. Thus, the estimate like (5.25) holds for $V_{1,k}^{3,2}$. On account of convergence of $\sum_k\langle b_k\rangle_{t;s-\alpha,\mathbb{R}_T}^{(\alpha/2)}$ we have

$$
\sum_k\langle V_{1,k}\rangle_{r;s-\alpha,\mathbb{R}_T}^{(\alpha)} \le \text{const.}\langle f\rangle_{r,t;s-\alpha,G_T}^{(\alpha,\alpha/2)}.
$$

Finally, we note that the analogous methods are applied to treat the function (which corresponds to i_{2k} from (5.11))

$$
\begin{aligned}
V_2(r,\varphi,t) &= \sum_k \sin(\lambda_k\varphi)\int_0^{\infty}d\tau\int_0^{\infty}d\rho\, i_{2k}(\rho,r,\tau)[b_k(\rho,t-\tau) - b_k(\rho,t)]\\
&\equiv \sum_k V_{2,k}(r,t)\sin(\lambda_k\varphi),
\end{aligned}
$$

and the following is true

$$
\sum_k\langle V_{2,k}\rangle_{r;s-\alpha,\mathbb{R}_T}^{(\alpha)} \le \text{const.}\langle f\rangle_{r,t;s-\alpha,G_T}^{(\alpha,\alpha/2)}.
$$

The above estimates lead to

$$
\sum_k\langle\frac{\partial R_{1,k}}{\partial t}\rangle_{r;s-\alpha,\mathbb{R}_T}^{(\alpha)} \le \text{const.}\langle f\rangle_{r,t;s-\alpha,G_T}^{(\alpha,\alpha/2)}.
\tag{5.26}
$$

Remark 5.4. Note that the estimate of $\sum_k[\frac{\partial R_{1,k}}{\partial t}]_{r,t;s-2\alpha,\mathbb{R}_T}^{\alpha,\alpha/2}$ will be obtained in the same way if we apply the arguments above to the difference $[\frac{\partial R_1}{\partial t}(r,\varphi,t_2) - \frac{\partial R_1}{\partial t}(r,\varphi,t_1)]$.

5.6. **Estimate for the seminorm** $[\frac{\partial R_1}{\partial t}]^{(\alpha,\alpha/2)}_{\varphi,t;s-\alpha,G_T}$**.** We will use representation (5.20) to the difference of $\frac{\partial R_1}{\partial t}(r,\varphi,t_2)-\frac{\partial R_1}{\partial t}(r,\varphi,t_1)$ to obtain the desired estimate. Let us consider the item $A^{(1)}_{j,k}=A^{(1)}_{j,k}(r,\varphi,t_1,t_2)$. Let A_1 be the operator corresponding to $A^{(1)}_{j,k}$; i.e.,

$$A_1(f(r,\varphi,\tau)-f(r,\varphi,t_2))=A^{(1)}_{j,k}(r,\varphi,t_1,t_2)=v(r,\varphi),$$

and

$$\begin{aligned} u_n(f(r,\varphi,\tau)-f(r,\varphi,t_2))=c(n)\int_0^{\pi/2}&[f(r,\varphi+2l\theta/\pi,\tau)-f(r,\varphi-2l\theta/\pi,\tau)\\ &-f(r,\varphi+2l\theta/\pi,t_2)+f(r,\varphi-2l\theta/\pi,t_2)]K(2l)dl \end{aligned}$$

be the approximating trigonometric polynomial of $f(r,\varphi,\tau)-f(r,\varphi,t_2)$. After that, we introduce the approximating trigonometric polynomial of $A_1(f(r,\varphi,\tau)-f(r,\varphi,t_2))$ as $T_n(r,\varphi,\tau,t_2)=A_1u_n(f(r,\varphi,\tau)-f(r,\varphi,t_2))$. Then, as before,

$$\begin{aligned} v-T_n=c(n)\int_0^{\pi/2}&A_1\big\{f(r,\varphi+2l\theta/\pi,\tau)-f(r,\varphi-2l\theta/\pi,\tau)\\ &-f(r,\varphi+2l\theta/\pi,t_2)+f(r,\varphi-2l\theta/\pi,t_2)\\ &-2[f(r,\varphi,\tau)-f(r,\varphi,t_2)]\big\}K(2l)dl. \end{aligned}$$

Estimate (5.21) ensured that the value $A_1\{\dots\}$ where $\{\dots\}$ is the expression in the braces in the integrand can be evaluated as

$$|A_1\{\dots\}|\le \text{const.}l^\alpha r^{s-\alpha}|\Delta t|^{\alpha/2}[f]^{(\alpha,\alpha/2)}_{\varphi,t;s-\alpha,G_T}.$$

After that, ending the estimate as well as the proof of Theorem 4.2 and applying Theorem 4.1, we obtain $\frac{A^{(1)}_{j,k}(r,\varphi,t_1,t_2)}{r^{s-\alpha}|\Delta t|^{\alpha/2}}\in C^\alpha_\varphi$ uniformly with respect to the rest variables. The same arguments are true in the case of other terms in (5.20). This implies

$$[\frac{\partial R_1}{\partial t}]^{(\alpha,\alpha/2)}_{\varphi,t;s-\alpha,G_T}\le \text{const.}[f]^{(\alpha,\alpha/2)}_{\varphi,t;s-\alpha,G_T}. \tag{5.27}$$

6. Proof of Theorem 2.1 and applications

To complete the proof of Theorem 2.1, we note the following. The exact representation of the solution in (2.5) has been got. We have shown the proof of the estimates to the higher derivatives of the solution with respect to φ and t. After that the derivatives of the solution with respect to r are evaluated with these estimates and the equation. We have given the estimates of the solution corresponding to the bulk potential, and the estimates of the potential corresponding to the initial data are done with the same way. This proves estimate (2.10). A uniqueness of the solution in the wider class has been proved in [13]. Thus, Theorem 2.1 has been proved.

Remark 6.1. Problem (2.2) with not uniform boundary conditions can be studied with reduction one to the problem with uniformly boundary value problem if the boundary functions are extended into the domain G_T (see [13]).

Remark 6.2. The described method makes possible to consider the homogeneous Dirichlet initial problem in an arbitrary domain in $\mathbb{R}^2$ with an corner point on the boundary.

In this section we formulate only results relating to the problem for the parabolic equation with singular coefficients of the form

$$\frac{\partial u}{\partial t} - \Big(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial u}{\partial r} + \frac{b}{r}\frac{\partial u}{\partial r}\Big) - \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \varphi^2} + \frac{b}{\varphi}\frac{\partial u}{\partial \varphi}\Big) = f(r,\varphi,t), \quad (r,\varphi,t) \in G_T, \tag{6.1}$$

$$\frac{\partial u}{\partial \varphi}|_{\varphi=0} = 0, \quad u|_{\varphi=\theta} = 0, \quad u|_{t=0} = u_0(r,\varphi), \tag{6.2}$$

where $b = \text{const.} > 0$.

Equation (6.1) is the main part of the parabolic equation with the Bessel operator

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \Big(\frac{\partial^2 u}{\partial y^2} + \frac{b}{y}\frac{\partial u}{\partial y}\Big) = f(x,y,t), \quad (x,y,t) \in G_T,$$

which can be also rewritten in the form

$$\frac{\partial u}{\partial t} - \Big(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial u}{\partial r} + \frac{b}{r}\frac{\partial u}{\partial r}\Big) - \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \varphi^2} + b\frac{\cos\varphi}{\sin\varphi}\frac{\partial u}{\partial \varphi}\Big) = f(r,\varphi,t), \quad (r,\varphi,t) \in G_T. \tag{6.3}$$

If $b = 0$, we get the problem for the heat equation.

We shall use the representation of a solution to problem (6.1), (6.2) in the form of the Fourier series by using eigenfunctions of the problem

$$\frac{\partial^2 v}{\partial \varphi^2} + \frac{b}{\varphi}\frac{\partial v}{\partial \varphi} = -\lambda^2 v \quad \varphi \in (0,\theta), \tag{6.4}$$

$$\frac{\partial v}{\partial \varphi}|_{\varphi=0} = 0, \quad v|_{\varphi=\theta} = 0. \tag{6.5}$$

Equation (6.4) has the two linearly independent solutions:

$$v_1(\varphi) = \varphi^{q/2} J_{q/2}(\lambda_k \varphi), \quad v_2(\varphi) = \varphi^{q/2} J_{-q/2}(\lambda_k \varphi), \quad q = 1-b, b \neq 1,$$

and if $b = 1$

$$v_1(\varphi) = J_0(\lambda_k \varphi), \quad v_2(\varphi) = N_0(\lambda_k \varphi),$$

where $J_\nu(x)$ and $N_\nu(x)$ are the Bessel functions of the first and second kind. The Bessel functions $J_\nu(x)$ has the power series representation

$$J_\nu(x) = \frac{x^\nu}{2^\nu}\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{2^{2k} k!\Gamma(\nu+k+1)}.$$

In view of this expansion the eigenfunctions $v_2(\phi)$ for $b \neq 1$ and $v_1(\phi)$ for $b = 1$ are appropriate for our purpose. They have the bounded second derivative and satisfy the first boundary condition in (6.5). To satisfy the second one, we define $\lambda = \lambda_k$ as the solutions of the equation $J_{-q/2}(\lambda_k \theta) = 0$, $k = 1, 2, \dots$. We will say about the case $b \neq 1$, the case $b = 1$ can be studied similarly.

The formal solution of problem (6.1), (6.2) is represented as

$$u(r,\varphi,t) = R_{1b}(r,\varphi,t) + R_{2b}(r,\varphi,t), \tag{6.6}$$

where $R_{1b}(r,\varphi,t)$ is the volume potential

$$\begin{aligned} &R_{1b}(r,\varphi,t) \\ &= \sum_k \varphi^{q/2} J_{-q/2}(\lambda_k\varphi) \int_0^t d\tau \int_0^\infty \Big(\frac{\rho}{r}\Big)^{b/2} \frac{\rho}{2(t-\tau)} e^{-\frac{\rho^2+r^2}{4(t-\tau)}} I_{\nu_k}\Big(\frac{\rho r}{2(t-\tau)}\Big) a_k d\rho \end{aligned}$$

with

$$a_k = \Big(\frac{\theta^2}{2} J^2_{1-q/2}(\lambda_k\theta)\Big)^{-1} \int_0^\theta \psi^{1-q/2} J_{-q/2}(\lambda_k\psi) f(\rho,\psi,\tau) d\psi,$$

and $R_{2b}(r,\varphi,t)$ is the initial data potential

$$R_{2b}(r,\varphi,t)=\sum_k \varphi^{q/2}J_{-q/2}(\lambda_k\varphi)\int_0^\infty \Big(\frac{\rho}{r}\Big)^{b/2}\frac{\rho}{2t}e^{-\frac{\rho^2+r^2}{4t}}I_{\nu_k}\Big(\frac{\rho r}{2t}\Big)a_{0k}d\rho$$

with

$$a_{0k}=\Big(\frac{\theta^2}{2}J^2_{1-q/2}(\lambda_k\theta)\Big)^{-1}\int_0^\theta \psi^{1-q/2}J_{-q/2}(\lambda_k\psi)u_0(\rho,\psi)d\psi,$$

and $\nu_k^2=\lambda_k^2+b^2/4$.

It turns out that the natural space for solutions of problem (6.1), (6.2) is the space $P^{l+\alpha,(l+\alpha)/2}_{s,b}(\overline{G}_T)$ of the functions with the finite norm (l is an integer, $\alpha\in(0,1)$)

$$\begin{aligned}
\|u\|_{P^{l+\alpha,(l+\alpha)/2}_{s,b}(\overline{G}_T)}&=\sum_{0\le\beta_1+\beta_2+2a\le l}\ \sup_{(r,\phi,t)\in\overline{G}_T} r^{-s+\beta_1+2a}\varphi^{b/2}|D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u|\\
&+\sum_{0<l+\alpha-(\beta_1+\beta_2+2a)<2}\Big\{\langle\varphi^{b/2}D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u\rangle^{(\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}_{t;s-\beta_1-2a-\alpha,G_T}\\
&+[\varphi^{b/2}D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u]^{(\alpha,\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}_{r,t;s-\beta_1-2a-2\alpha,G_T}\\
&+[\varphi^{b/2}D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u]^{(\alpha,\frac{l+\alpha-\beta_1-\beta_2-2a}{2})}_{\varphi,t;s-\beta_1-2a-\alpha,G_T}\Big\}\\
&+\sum_{\beta_1+\beta_2+2a=l}\Big\{\langle\varphi^{b/2}D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u\rangle^{(\alpha)}_{r;s-\beta_1-2a-\alpha,G_T}\\
&+\langle\varphi^{b/2}D_r^{\beta_1}D_\varphi^{\beta_2}D_t^a u\rangle^{(\alpha)}_{\varphi;s-\beta_1-2a,G_T}\Big\}.
\end{aligned}$$

We introduce the subspace $\widehat{P}^{l+\alpha,\frac{l+\alpha}{2}}_{s,b}(\overline{G}_T)$ ($\widehat{P}^{l+\alpha}_{s,b}(\overline{G})$) of the space $P^{l+\alpha,\frac{l+\alpha}{2}}_{s}(\overline{G}_T)$ ($P^{l+\alpha}_{s}(\overline{G})$) like the definition of $\widehat{P}^{l+\alpha,\frac{l+\alpha}{2}}_{s}(\overline{G}_T)$ ($\widehat{P}^{l+\alpha}_{s}(\overline{G})$). We are looking for the solution to the problem in the form of the series and waiting that these series converge in $\overline{G}_T$. All their terms are equal to zero at $\varphi=0$, thus, the condition

$$f(r,\theta,t)=0 \tag{6.7}$$

is necessary for the solvability of the problem in $P^{2+\alpha,(2+\alpha)/2}_{s,b}(\overline{G}_T)$.

Theorem 6.3. *Assume the consistency conditions of the first order and condition (6.7) are fulfilled. The functions $f\in\widehat{P}^{\alpha,\alpha/2}_{s,b}(\overline{G}_T)$ and $u_0\in\widehat{P}^{2+\alpha}_{s+2,b}(\overline{G})$ Then there exists a unique solution $u(r,\varphi,t)\in\widehat{P}^{2+\alpha,\frac{2+\alpha}{2}}_{s,b}(\overline{G}_T)$ and*

$$\begin{aligned}
&\|u\|_{P^{2+\alpha,(2+\alpha)/2}_{s+2,b}(\overline{G}_T)}+S(u)_{\widehat{P}^{2+\alpha,(2+\alpha)/2}_{s+2,b}(\overline{G}_T)}\\
&\le const.(\|f\|_{P^{\alpha,\alpha/2}_{s,b}(\overline{G}_T)}+\|u_0\|_{P^{2+\alpha}_{s+2,b}(\overline{G})}+S(f)_{\widehat{P}^{\alpha,\alpha/2}_{s,b}(\overline{G}_T)}+S(u_0)_{\widehat{P}^{2+\alpha}_{s+2,b}(\overline{G})}),
\end{aligned}\tag{6.8}$$

where the constant in (6.8) is independent of $u(r,\varphi,t)$, $\alpha\in(0,1)$ and $s+2<(\lambda_1^2+b^2/4)^{1/2}$, $\lambda_1\theta$ is the smallest root of the equation $J_{-q/2}(\lambda_k\theta)=0$.

In general, the proof of Theorem 6.3 repeats our arguments from the proof of Theorem 2.1. We note only that if $k>>1$,

$$J_{-q/2}(\lambda_k\varphi)\sim\sqrt{\frac{1}{\lambda_k\varphi}}\cos(\lambda_k\varphi+\pi(q-1)/4),\quad \lambda_k\theta\sim(k-(q+1)/4)\pi+O(1/k),$$

that gives the possibility to apply here the theorems from the trigonometric series theory.

7. Appendix

7.1. **Formal representation of the solution to** (2.2)**.** To obtain the formal solution of (2.2), we applied the method of the separation of the variables. In detail, one consists in the following. Let us consider case of $u_0(r,\varphi) \equiv 0$ (this case corresponds to $R_2(r,\varphi,t) = 0$ in (2.3)). We look for the solution $u(r,\varphi,t)$ of the problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = f(r,\varphi,t), \quad (r,\varphi,t) \in G_T, \\ u|_{g_{iT}} = 0, \quad u|_{t=0} = 0, \quad (r,\varphi) \in G,\end{gathered} \tag{7.1}$$

as

$$u(r,\varphi,t) = \sum_k V_k(r,t)\Phi_k(\varphi). \tag{7.2}$$

After the substitution of the function $V_k(r,t)\Phi_k(\varphi)$ into the homogenous equation and boundary condition from (7.1), we obtain

$$r^2 \frac{\frac{\partial V_k}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial V_k}{\partial r}}{V_k} = \frac{\frac{\partial^2 \Phi_k}{\partial \varphi^2}}{\Phi_k} \equiv -\lambda_k^2, \tag{7.3}$$

$$\Phi_k(0) = \Phi_k(\theta) = 0. \tag{7.4}$$

Conditions (7.3) and (7.4) lead to the function Φ_k being the solution of the problem

$$\begin{gathered}\Phi_k''(\varphi) + \lambda_k^2 \Phi_k = 0, \\ \Phi_k(0) = \Phi_k(\theta) = 0.\end{gathered} \tag{7.5}$$

The solution of (7.5) is the function

$$\Phi_k = \sin \lambda_k \varphi, \quad \lambda_k = \pi k/\theta, \ k = 1, 2 \dots. \tag{7.6}$$

Now we return to problem (7.1) and represent $f(r,\varphi,t)$ as

$$f(r,\varphi,t) = \sum_k b_k(r,t) \sin \lambda_k \varphi, \tag{7.7}$$

with

$$b_k(r,t) = \frac{2}{\theta} \int_0^\theta f(r,\psi,t) \sin \lambda_k \psi d\psi. \tag{7.8}$$

After that we substitute (7.2), (7.7) and (7.8) to the equation and the initial condition of (7.1) and have

$$\begin{gathered}\frac{\partial V_k}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial V_k}{\partial r} + \lambda_k^2 \frac{V_k}{r^2} = b_k(r,t), \\ V_k(r,0) = 0.\end{gathered} \tag{7.9}$$

Here we use that the function $\Phi_k(\varphi)$ satisfies the equation in (7.5).

Let us denote the Hankel transformation (see, for example, [8, 12, 15] for discussion) with respect to r of the functions $V_k(r,t)$ and $b_k(r,t)$ by $\widehat{V}_k(\mu,t)$ and $\widehat{b}_k(\mu,t)$, respectively, μ is the parameter under the transformation:

$$\begin{gathered}\widehat{V}_k(\mu,t) = \int_0^\infty V_k(r,t) r J_{\lambda_k}(\mu r) dr; \\ \widehat{b}_k(\mu,t) = \int_0^\infty b_k(r,t) r J_{\lambda_k}(\mu r) dr,\end{gathered} \tag{7.10}$$

where $J_{\lambda_k}(\mu r)$ is the Bessel function [8].

Expressions (7.9) and (7.10) lead to the following problem for the function $\widehat{V}_k(\mu,t)$,

$$\begin{gathered}\frac{d\widehat{V}_k}{dt}+\mu^2\widehat{V}_k=\widehat{b}_k(\mu,t),\\ \widehat{V}_k(\mu,0)=0.\end{gathered}\tag{7.11}$$

It is easy to check that the function

$$\widehat{V}_k(\mu,t)=\int_0^t e^{-\mu^2(t-\tau)}\widehat{b}_k(\mu,\tau)d\tau\tag{7.12}$$

gives the solution of problem (7.11).

After applying the inverse Hankel transformation in (7.12), we obtain

$$V_k(r,t)=\int_0^\infty \mu J_{\lambda_k}(\mu r)\int_0^t e^{-\mu^2(t-\tau)}\widehat{b}_k(\mu,\tau)d\tau d\mu.\tag{7.13}$$

Then the formal solution (7.1) follows from (7.2), (7.6) and (7.13), so,

$$u(r,\varphi,t)=\sum_{k=1}\sin\lambda_k\varphi\int_0^\infty \mu J_{\lambda_k}(\mu r)\int_0^t e^{-\mu^2(t-\tau)}\widehat{b}_k(\mu,\tau)d\tau d\mu.\tag{7.14}$$

To obtain formula (2.6), we transform (7.14) applying formula [8, 6.633(2)]:

$$\int_0^\infty \mu e^{-\mu^2(t-\tau)}J_{\lambda_k}(\mu r)J_{\lambda_k}(\mu\rho)d\mu=\frac{1}{2(t-\tau)}I_{\lambda_k}\Big(\frac{r\rho}{2(t-\tau)}\Big)\exp\Big(-\frac{r^2+\rho^2}{4(t-\tau)}\Big)$$

where $I_{\lambda_k}(x)$ is the modified Bessel function. Thus,

$$\begin{aligned}&u(r,\varphi,t)\\ &=\sum_{k=1}\sin\lambda_k\varphi\int_0^t d\tau\int_0^\infty d\rho b_k(\rho,\tau)\rho\int_0^\infty \mu J_{\lambda_k}(\mu r)e^{-\mu^2(t-\tau)}J_{\lambda_k}(\mu\rho)d\mu\\ &=\sum_{k=1}\sin\lambda_k\varphi\int_0^t d\tau\int_0^\infty d\rho b_k(\rho,\tau)\frac{\rho}{2(t-\tau)}I_{\lambda_k}\Big(\frac{r\rho}{2(t-\tau)}\Big)\exp\Big(-\frac{r^2+\rho^2}{4(t-\tau)}\Big).\end{aligned}$$

That gives (2.5), (2.6) with $R_2\equiv 0$ (due to $u_0\equiv 0$). To obtain the complete formula (2.5); i.e., with $R_2\neq 0$, it is enough to consider the problem

$$\begin{gathered}\frac{\partial u}{\partial t}-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}-\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}=0,\quad (r,\varphi,t)\in G_T,\\ u|_{g_{iT}}=0,\quad u|_{t=0}=u_0(r,\varphi),(r,\varphi)\in G,\end{gathered}\tag{7.15}$$

and apply all reasoning mentioned above to this problem.

After that, the solution of (2.2) is represented as

$$u(r,\varphi,t)=R_1(r,\varphi,t)+R_2(r,\varphi,t)\tag{7.16}$$

where $R_1(r,\varphi,t)$ and $R_2(r,\varphi,t)$ are the solutions of (7.1) and (7.15), correspondingly.

$$\begin{gathered}R_1(r,\varphi,t)=\sum_k\sin(\lambda_k\varphi)\int_0^t d\tau\int_0^\infty d\rho\frac{\rho}{2(t-\tau)}e^{-\frac{\rho^2+r^2}{4(t-\tau)}}I_{\lambda_k}\Big(\frac{\rho r}{2(t-\tau)}\Big)b_k(\rho,\tau),\\ R_2(r,\varphi,t)=\sum_k\sin(\lambda_k\varphi)\int_0^\infty d\rho\frac{\rho}{2t}e^{-\frac{\rho^2+r^2}{4t}}I_{\lambda_k}\Big(\frac{\rho r}{2t}\Big)u_{0k}(\rho),\end{gathered}\tag{7.17}$$

$$u_{0k}(r)=\frac{2}{\theta}\int_0^\theta u_0(r,\psi)\sin(\lambda_k\psi)d\psi,\quad b_k(r,t)=\frac{2}{\theta}\int_0^\theta f(r,\psi,t)\sin(\lambda_k\psi)d\psi. \tag{7.18}$$

Equation (2.5) implies that the desired solution is the sum of the volume potential $R_1(r,\phi,t)$ and the potential of the initial data $R_2(r,\phi,t)$.

The representation for $R_1(r,\varphi,t)$ can be rewritten also as

$$R_1(r,\varphi,t)=\sum_k\sin(\lambda_k\varphi)\int_{-\infty}^t d\tau\int_0^\infty d\rho\frac{\rho}{2(t-\tau)}e^{-\frac{\rho^2+r^2}{4(t-\tau)}}I_{\lambda_k}\Big(\frac{\rho r}{2(t-\tau)}\Big)b_k(\rho,\tau), \tag{7.19}$$

if $f(r,\varphi,t)=0$ for $t<0$, that was assumed. Thus representations (7.16)-(7.19) give (2.5)-(2.8).

7.2. **Proof of Corollary 3.4.** In the integral from (3.7), we change the variable $\frac{\rho r}{2t}=x$ and then $x^2=y$,

$$\begin{aligned} D_s&=\int_0^\infty\Big(\frac{2xt}{r}\Big)^{1+s}\frac{2t}{r}\frac{1}{2t}e^{-\frac{r^2}{4t}-t\frac{x^2}{r^2}}I_{\lambda_k}(x)dx\\ &=\Big(\frac{2t}{r}\Big)^{1+s}\frac{1}{2r}e^{-\frac{r^2}{4t}}\int_0^\infty y^{s/2}e^{-t\frac{y}{r^2}}I_{\lambda_k}(y^{1/2})dy. \end{aligned}$$

Using tabular integral [8, 6.643(2)], we obtain

$$D_s=2^{1+s}r^s(t/r^2)^{\frac{1+s}{2}}e^{-\frac{r^2}{8t}}\frac{\Gamma(\frac{\lambda_k+s+2}{2})}{\Gamma(\lambda_k+1)}M_{-\frac{1+s}{2},\frac{\lambda_k}{2}}(r^2/4t).$$

In our case (see [8, 9.221])

$$M_{-\frac{1+s}{2},\frac{\lambda_k}{2}}(r^2/4t)=\frac{(r^2/4t)^{\frac{\lambda_k+1}{2}}}{2^{\lambda_k}B(\frac{\lambda_k-s}{2},\frac{\lambda_k+s+2}{2})}N\Big(\lambda_k,\frac{r^2}{8t}\Big),$$

where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $\Gamma(x)$ is the Gamma function, and

$$N\Big(\lambda_k,\frac{r^2}{8t}\Big)=\int_{-1}^1(1+z)^{\frac{\lambda_k+s}{2}}(1-z)^{\frac{\lambda_k-2-s}{2}}e^{z\frac{r^2}{8t}}dz.$$

By the substitution $1+z=x$ we go to

$$N\Big(\lambda_k,\frac{r^2}{8t}\Big)=\int_0^2 e^{-\frac{r^2}{8t}}x^{\frac{\lambda_k+s}{2}}(2-x)^{\frac{\lambda_k-2-s}{2}}e^{x\frac{r^2}{8t}}dx.$$

In this equality, we put $s=-1$ and use tabular integral [8, 3.383(2)], then

$$N\Big(\lambda_k,\frac{r^2}{8t}\Big)=\sqrt{\pi}\Big(\frac{16t}{r^2}\Big)^{\lambda_k/2}\Gamma\Big(\frac{\lambda_k+1}{2}\Big)I_{\lambda_k/2}\Big(\frac{r^2}{8t}\Big).$$

Finally, we gather our calculations and obtain

$$D_{-1}=\mathrm{const.}r^{-1}e^{-z}z^{\frac12}I_{\frac{\lambda_k}{2}}(z),\quad z=r^2/8t.$$

Lemma 3.3 leads to

$$e^{-z}z^{\frac12}I_{\frac{\lambda_k}{2}}(z)\le\mathrm{const.},$$

where the constant does not depend on k. Recall that $\lambda_k=\frac{\pi}{\theta}k$, so, if we take $k=2n$, $n=1,2,\dots$, we will obtain our assertion. This ends the proof of Corollary 3.4.

7.3. **Estimate for the integral** $M_1=(1+\lambda_k)\int_0^\infty dt\int_0^\infty d\rho\frac{\rho^{s-\alpha}}{t^{1-\alpha/2}}L_k(\rho,r,t)$**.** Using the representation of the function $I_{\lambda_k}(x)$ from [3, 7.7.3(25)], we obtain

$$\begin{aligned}
M_1&=(1+\lambda_k)\int_0^\infty dt\int_0^\infty d\rho\frac{\rho^{1+s-\alpha}}{t^{1-\alpha/2}}\int_0^\infty J_{\lambda_k}(\rho\mu)J_{\lambda_k}(r\mu)e^{-t\mu^2}\mu d\mu\\
&=(1+\lambda_k)\int_0^\infty dt\int_0^\infty d\rho\frac{\rho^{-1+s-\alpha}}{t^{1-\alpha/2}}\int_0^\infty J_{\lambda_k}(zr/\rho)J_{\lambda_k}(z)e^{-z^2t\rho^{-2}}zdz\\
&=(1+\lambda_k)\int_0^\infty dyy^{-1-s+\alpha}\int_0^\infty dzzJ_{\lambda_k}(zry)J_{\lambda_k}(z)\int_0^\infty dtt^{-1+\alpha/2}e^{-z^2ty^2},
\end{aligned}$$

In the first equality above we used $\mu=z/\rho$, and in the second $\rho=y^{-1}$. The last integral can be calculated (see [8, 3.381(4)])

$$\int_0^\infty dtt^{-1+\alpha/2}e^{-z^2ty^2}=\frac{\Gamma(\alpha/2)}{z^\alpha y^\alpha},$$

so that after the changing of the variable $y=q/r$,

$$\begin{aligned}
&M_1\\
&=(1+\lambda_k)r^s\Gamma(\alpha/2)\int_0^\infty dqq^{-1-s}\int_0^\infty dzz^{1-\alpha}J_{\lambda_k}(zq)J_{\lambda_k}(z)\\
&=(1+\lambda_k)r^s\Gamma(\alpha/2)\Big\{\int_0^{1-\varepsilon}+\int_{1-\varepsilon}^{1+\varepsilon}+\int_{1+\varepsilon}^\infty\Big\}dqq^{-1-s}\int_0^\infty dzz^{1-\alpha}J_{\lambda_k}(zq)J_{\lambda_k}(z)\\
&\equiv(1+\lambda_k)r^s\Gamma(\alpha/2)(M_1^{(1)}+M_1^{(2)}+M_1^{(3)}).
\end{aligned}\tag{7.20}$$

For $q\in(0,1-\varepsilon)$ the integral (see [3, 7.7.4(29)])

$$\begin{aligned}
d_1&=\int_0^\infty dzz^{1-\alpha}J_{\lambda_k}(zq)J_{\lambda_k}(z)\\
&=\frac{q^{\lambda_k}\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)}F(\lambda_k+1-\alpha/2,1-\alpha/2;\lambda_k+1;q^2).
\end{aligned}$$

The function $F(\lambda_k+1-\alpha/2,1-\alpha/2;\lambda_k+1;q^2)$ is bounded (see [8, 9.102]) so that

$$d_1\le\text{const.}\frac{q^{\lambda_k}}{2^{\alpha-1}\Gamma(\alpha/2)}\lambda_k^{-\alpha/2}\le\text{const.}q^{\lambda_k}\lambda_k^{-\alpha/2}.\tag{7.21}$$

For $q\in(1+\varepsilon,\infty)$,

$$\begin{aligned}
d_3&=\int_0^\infty dzz^{1-\alpha}J_{\lambda_k}(zq)J_{\lambda_k}(z)\\
&=\frac{q^{-\lambda_k+\alpha-2}\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)}F(\lambda_k+1-\alpha/2,1-\alpha/2;\lambda_k+1;q^{-2})\\
&\le\text{const.}q^{-\lambda_k+\alpha-2}\lambda_k^{-\alpha/2}.
\end{aligned}\tag{7.22}$$

Estimates (7.21) and (7.22) lead to

$$M_1^{(1)}\le\text{const.}\lambda_k^{-1-\alpha/2},\quad M_1^{(3)}\le\text{const.}\lambda_k^{-1-\alpha/2}.\tag{7.23}$$

Now we estimate the integral

$$M_1^{(2)}=\Big\{\int_{1-\varepsilon}^1+\int_1^{1+\varepsilon}\Big\}q^{-1-s}dq\int_0^\infty z^{1-\alpha}J_{\lambda_k}(z)J_{\lambda_k}(qz)dz=M_1^{(2,1)}+M_1^{(2,2)},$$

$$
\begin{aligned}
M_1^{(2,1)} &= \int_{1-\varepsilon}^{1} q^{-1-s}dq \int_0^\infty z^{1-\alpha} J_{\lambda_k}(z) J_{\lambda_k}(qz)dz \\
&= \int_0^\varepsilon (1-x)^{-1-s}dx \int_0^\infty z^{1-\alpha} J_{\lambda_k}(z) J_{\lambda_k}((1-x)z)dz \\
&= \int_0^\varepsilon (1-x)^{-1-s} d_{21}(x)dx
\end{aligned}
$$

(in the second inequality above, we used $q = 1-x$),

$$
d_{21} = \frac{(1-x)^{\lambda_k}\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)} F(\lambda_k+1-\alpha/2, 1-\alpha/2; \lambda_k+1; (1-x)^2),
$$

$$
\begin{aligned}
M_1^{(2,2)} &= \int_1^{1+\varepsilon} q^{-1-s}dq \int_0^\infty z^{1-\alpha} J_{\lambda_k}(z) J_{\lambda_k}(qz)dz \\
&= \int_0^\varepsilon (1+x)^{-1-s} d_{22}(x)dx
\end{aligned}
$$

(in the inequality above, we used $q = 1+x$),

$$
d_{22} = \frac{(1+x)^{\alpha-2-\lambda_k}\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)} F(\lambda_k+1-\alpha/2, 1-\alpha/2; \lambda_k+1; (1+x)^{-2}),
$$

where

$$
F(\alpha,\beta;\gamma;x) = 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \dots = 1 + \sum_{p=1}^\infty a_p x^p.
$$

In our case

$$
a_1 = \frac{(\lambda_k+1-\alpha/2)(1-\alpha/2)}{(\lambda_k+1)}, \quad a_2 = a_1 \cdot \frac{(\lambda_k+2-\alpha/2)(2-\alpha/2)}{2\cdot(\lambda_k+2)}, \dots;
$$

i.e., $a_p \le const.$ with respect to p and λ_k. After that,

$$
\begin{aligned}
& M_1^{(2,1)} + M_1^{(2,2)} \\
&= \frac{\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)} \sum_{p=0} a_p \int_0^\varepsilon [(1-x)^{\lambda_k-1-s+2p} + (1+x)^{-\lambda_k-1-s-2p-2+\alpha}]dx \\
&= \frac{\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)} \sum_{p=0} a_p [(1-x)^{\lambda_k-s+2p}(\lambda_k-s+2p)^{-1} \\
&\quad + (1+x)^{-\lambda_k-s-2p-2+\alpha}(-\lambda_k-s-2p-2+\alpha)^{-1}]|_{x=0}^{x=\varepsilon} \\
&= \frac{\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)} \sum_{p=0} a_p \Big\{[(1-\varepsilon)^{\lambda_k-s+2p}(\lambda_k-s+2p)^{-1} \\
&\quad + (1+\varepsilon)^{-\lambda_k-s-2p-2+\alpha}(-\lambda_k-s-2p-2+\alpha)^{-1}] \\
&\quad - (\alpha-2-2s)(\lambda_k-s+2p)^{-1}(-\lambda_k-s-2p-2+\alpha)^{-1}\Big\} \\
&= \frac{\Gamma(\lambda_k+1-\alpha/2)}{2^{\alpha-1}\Gamma(\lambda_k+1)\Gamma(\alpha/2)} \Big\{\sum_{p=0} a_p [(1-\varepsilon)^{\lambda_k-s+2p}(\lambda_k-s+2p)^{-1} \\
&\quad + (1+\varepsilon)^{-\lambda_k-s-2p-2+\alpha}(-\lambda_k-s-2p-2+\alpha)^{-1}] \\
&\quad + \sum_{p=0} a_p(\alpha-2-2s)(\lambda_k-s+2p)^{-1}(\lambda_k+s+2p+2-\alpha)^{-1}\Big\}.
\end{aligned}
$$

The first series converges because, for example, for every fixed $\varepsilon \geq \varepsilon_0 > 0$

$$a_p(1-\varepsilon)^{\lambda_k - s+2p}(\lambda_k - s + 2p)^{-1} \leq \frac{\text{const.}}{\lambda_k} q^{\lambda_k}, \quad q < 1,$$

so

$$\begin{aligned}
&\big|\sum_{p=0} a_p[(1-\varepsilon)^{\lambda_k - s+2p}(\lambda_k - s+2p)^{-1} \\
&+ (1+\varepsilon)^{-\lambda_k - s-2p-2+\alpha}(-\lambda_k - s - 2p - 2 + \alpha)^{-1}]\big| \leq \frac{\text{const.}}{\lambda_k}.
\end{aligned}$$

As for the second series,

$$\big|\sum_{p=0} a_p(\alpha - 2 - 2s)(\lambda_k - s + 2p)^{-1}(\lambda_k + s + 2p + 2 - \alpha)^{-1}\big| \leq \frac{\text{const.}}{\lambda_k^{1-\mu}}, \quad \mu > 0.$$

Taking into account that

$$\frac{\Gamma(\lambda_k + 1 - \alpha/2)}{\Gamma(\lambda_k + 1)} \approx \lambda_k^{-\alpha/2}$$

for large λ_k, we have

$$|M_1^{(2)}| = |M_1^{(2,1)} + M_1^{(2,2)}| \leq \text{const.}\lambda_k^{-1+\mu-\alpha/2}. \tag{7.24}$$

Finally, the following inequality follows from (7.20), (7.23) and (7.24):

$$|M_1| \leq \text{const.}\frac{r^s}{\lambda_k^{\alpha/2-\mu}} \tag{7.25}$$

with $0 < \mu < \alpha/2$.

7.4. **Estimate for the integral $\int_0^\infty \frac{\rho^{1+s-\alpha}}{t^{3/2}}|r-\rho| I_{\lambda_k}(r\rho/2t) e^{-\frac{r^2+\rho^2}{4t}} d\rho$ from Subsection 5.5.** In this subsection we show the estimate

$$I = \int_0^\infty \frac{\rho^{1+s-\alpha}}{t^{3/2}}|r-\rho| I_{\lambda_k}(r\rho/2t) e^{-\frac{r^2+\rho^2}{4t}} d\rho \leq \text{const.} r^{s-\alpha}.$$

Denote

$$u = \frac{\rho}{2t^{1/2}}, \quad v = \frac{r}{2t^{1/2}}, \quad 2uv = \frac{r\rho}{2t},$$

and change the integration variable ρ by u. We obtain

$$\begin{aligned}
I &= \int_0^\infty 4t\frac{(2t^{1/2}u)^{1+s-\alpha}}{t^{3/2}}|v-u| e^{-(u^2+v^2)} I_{\lambda_k}(2uv)du \\
&\leq \text{const.}\int_0^\infty t^{\frac{s-\alpha}{2}} u^{1+s-\alpha} e^{-\gamma(u-v)^2} I_{\lambda_k}(2uv) e^{-2uv} du
\end{aligned}$$

where $\gamma \in (0,1)$. Next we consider the integral

$$A(v) = \int_0^\infty u^\beta e^{-\gamma(u-v)^2} I_{\lambda_k}(2uv) e^{-2uv} du.$$

Introduce the new integration variable $z = uv$ so that

$$\begin{aligned}
A(v) &= v^{-\beta-1}\int_0^\infty z^\beta e^{-\gamma(\frac{z}{v}-v)^2} I_{\lambda_k}(2z) e^{-2z} dz \\
&= v^{-\beta-1}\int_0^1 z^\beta e^{-\gamma(\frac{z}{v}-v)^2} I_{\lambda_k}(2z) e^{-2z} dz
\end{aligned}$$

$$
\begin{aligned}
&+ v^{-\beta-1}\int_1^\infty z^\beta e^{-\gamma(\frac{z}{v}-v)^2} I_{\lambda_k}(2z)e^{-2z}dz\\
&\equiv A_1(v)+A_2(v).
\end{aligned}
$$

To estimate $A_2(v)$, we use Corollary 3.4, and obtain

$$
A_2(v)\le \text{const.}v^{-\beta-1}\int_1^\infty z^{\beta-1/2}e^{-\gamma(\frac{z}{v}-v)^2}dz.
$$

Now let $\xi=\frac{z}{v}-v$. Then

$$
\begin{aligned}
A_2(v)&\le \text{const.}v^{-\beta-1}\int_{-v+1/v}^\infty v^{\beta+1/2}(\xi+v)^{\beta-1/2}e^{-\gamma\xi^2}d\xi\\
&\le \text{const.}v^{-1/2}\int_{-v+1/v}^\infty \max_\xi[(\xi+v)^{\beta-1/2}e^{-\gamma\xi^2/2}]e^{-\gamma\xi^2/2}d\xi.
\end{aligned}
$$

One can verify that

$$
\varphi(\xi,v)=(\xi+v)^{\beta-1/2}e^{-\gamma\xi^2/2}\le \text{const.}v^{\beta-1/2}
$$

under $v\ge v_0(\beta,\gamma)>0$. It implies

$$
A_2(v)\le \text{const.}v^{\beta-1}\quad \text{for } v\ge 1.
$$

To estimate $A_2(v)$ for $v<1$, notice that $e^{-\gamma\frac{z^2}{2v^2}}\le e^{\frac{-\gamma}{2v^2}}$ if $z\ge 1$ and $e^{-\gamma\frac{z^2}{2v^2}}\le e^{\frac{-\gamma z^2}{2}}$ if $v<1$. Therefore,

$$
\begin{aligned}
A_2(v)&\le \text{const.}v^{-1-\beta}\int_1^\infty z^{\beta-1/2}e^{-\gamma\frac{z^2}{2v^2}}e^{-\gamma\frac{z^2}{2v^2}}e^{2z}e^{-\gamma v^2}dz\\
&\le \text{const.}v^{-1-\beta}e^{-\frac{\gamma}{2v^2}}\int_1^\infty z^{\beta-1/2}e^{-\gamma\frac{z^2}{2}+2z}dz\\
&\le \text{const.}v^{-1-\beta}e^{-\frac{\gamma}{2v^2}}\le \text{const.}v^{-1+\beta}
\end{aligned}
$$

for $v<1$.

After that we evaluate the integral $A_1(v)$ for $v\ge 1$. For $z\le 1$, we use the estimate

$$
I_{\lambda_k}(2z)\le \text{const.}z^{\lambda_k}\le \text{const.}z^{\lambda_1},\quad \lambda_1=\pi/\theta.
$$

Then

$$
\begin{aligned}
A_1(v)&\le \text{const.}v^{-\beta-1}\int_0^1 z^{\beta+\lambda_1}e^{-\gamma(v^2-2z+z^2/v^2)}dz\\
&\le \text{const.}v^{-\beta-1}e^{-\gamma v^2}\le \text{const.}v^{-1+\beta}
\end{aligned}
$$

for $v\ge 1$. At last, for $v<1$,

$$
\begin{aligned}
A_1(v)&\le \text{const.}v^{-\beta-1}\int_0^1 z^{\beta+\lambda_1}e^{-\gamma(v^2-2z+z^2/v^2)}dz\\
&\le \text{const.}v^{-\beta}\int_0^{1/v} dy(yv)^{\beta+\lambda_1}e^{-\gamma y^2}\\
&\le \text{const.}v^{\lambda_1}=\text{const.}v^{-1+\beta}v^{\lambda_1+1-\beta}\le \text{const.}v^{\beta-1}
\end{aligned}
$$

if $\lambda_1+1-\beta\ge 0$.

If we take $\beta = 1 + s - \alpha$, then the condition $\lambda_1 + 1 - \beta \geq 0$ means $1 - \alpha \leq \pi/\theta$ that is fulfilled under conditions of Theorem 2.1. Thus, our calculations lead to

$$I \leq \text{const.} t^{(s-\alpha)/2} \Big(\frac{r}{2t^{1/2}} \Big)^{-1+1+s-\alpha} \leq \text{const.} r^{s-\alpha},$$

that was to be proved.

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References

[1] H. Amann; *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr., 186 (1997), 5-56.

[2] N. K. Bari; *Trigonometric series* (Russian), GITTL, Moscow, 1961. [N.K. Bari, A treatise on trigonometric series, v. I,II, The Macmillan Co, New York, 1964].

[3] H. Bateman, A. Erdélyi; *Higher transcendental functions*, v. II, New York Toronto London, MC Graw-Hill Company, 1953.

[4] V. A. Chernyatin; *Existence of a classical solution of a mixed problem for a second-order one-dimensional parabolic equation*, Moscow Univ. Math. Bull., **46**, 6 (1990), 7-11.

[5] F. Colombo, D. Guidetti, and A. Lorenzi; *Elliptic equations in nonsmooth plane domanis with an applictaion to a parabolic problem*, Adv. Diff. Eq., **7**, 6 (2002), 695-716.

[6] E. V. Frolova; *On a certain nonstationary problem in a dihedral angle I*, J. Soviet. Math., **70** (1994), 1828-1840.

[7] M. G. Garroni, V. A. Solonnikov, and M. A. Vivaldi; *Existence and regularity results for oblique derivative problem for heat equations in an angle*, Proceedings of the Royal Society of Edinburgh, **128 A**, (1998), 47-79.

[8] I. S. Gradshteyn and I. M. Ryzhik; *Table of integrals, sums, series and products*, 5-th ed., Academic Press, London, UK, 1994.

[9] P. Grisvard; *Elliptic problems in nonsmooth domains*, Pitman, London, 1985.

[10] D. Guidetti; *The mixed Cauchy-Dirichlet problem for the heat equation in a plane angle in spaces of Hölder continuous functions*, Adv. Diff. Eq., **6**, 8 (2001), 897-930.

[11] I. P. Natanson; *Constructive function theory* (Russian), GITTL, Moscow, 1949. [I. P. Natanson, Constructive function theory, v. I, II, III, Frederic Ungar Publishing Co, New York, 1964].

[12] A. D. Polyanin, and A. V. Manzhirov; *Handbook of Integral Equations*, CRC Press, Boca Raton, 1998.

[13] V. A. Solonnikov; *Solvability of classical initial boundary-value problems for a heat equation in dihedral angle*, J. Soviet. Math., **32**, (1986), 526-546.

[14] V. A. Solonnikov; *Solvability of a problem on the motion of a viscous incompressible fluid bounded by a free surface*, Math. USSR-Izv., **11**, 6 (1978), 1323-1358.

[15] W. R. Smythe; *Static and Dynamic Electricity* (3rd ed.), New York: McGraw-Hill, 1968.

[16] N. Vasylyeva; *On solvability of the Hele-Shaw problem in weighted Hölder spaces in a plane domain with a corner point*, UMB, **2**, 3 (2005), 317-343.

[17] A. Zygmund; *Trigonometric series*, 2nd. ed. v. I,II, Cambridge University Press, New York, 1959.

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