Electronic Journal of Differential Equations, Vol. 2010(2010), No. 91, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

IMPULSIVE DISCONTINUOUS HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER ON BANACH ALGEBRAS

SAÏD ABBAS, RAVI P. AGARWAL, MOUFFAK BENCHOHRA

Abstract. This article studies the existence of solutions and extremal solutions to partial hyperbolic differential equations of fractional order with impulses in Banach algebras under Lipschitz and Carathéodory conditions and certain monotonicity conditions.

1. Introduction

This article studies the existence of solutions to fractional order initial-value problems (IVP for short), for the system

$$^cD_0^r\Big(\frac{u(x,y)}{f(x,y,u(x,y))}\Big) = g(x,y,u(x,y)), \quad (x,y)\in J,\ x\neq x_k,\ k=1,\dots,m, \tag{1.1}$$

$$u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)), \quad y\in[0,b];\ k=1,\dots,m, \tag{1.2}$$

$$u(x,0)=\varphi(x), \quad u(0,y)=\psi(y), \quad x\in[0,a],\ y\in[0,b], \tag{1.3}$$

where $J=[0,a]\times[0,b]$, $a,b>0$, $^cD_0^r$ is the Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $0=x_0<x_1<\dots<x_m<x_{m+1}=a$, $f : J\times\mathbb{R}^n\to\mathbb{R}^n\backslash\{0_{\mathbb{R}^n}\}$, $g : J\times\mathbb{R}^n\to\mathbb{R}^n$ and $I_k:\mathbb{R}^n\to\mathbb{R}^n$, $k=1,\dots,m$ are given functions satisfying suitable conditions and $\varphi:[0,a]\to\mathbb{R}^n$, $\psi:[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions with $\varphi(0)=\psi(0)$.

There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas [20], Lakshmikantham *et al.* [22], Podlubny [24], Samko [25], the papers by Abbas and Benchohra [1, 2, 3], Agarwal *et al.* [5], Belarbi *et al.* [7], Benchohra *et al.* [8, 9, 10], Diethelm [16], Vityuk and Golushkov [27] and the references therein. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [18, 23]).

The theory of impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a

2000 *Mathematics Subject Classification.* 26A33.
Key words and phrases. Impulsive differential equations; fractional order; upper solution; lower solution; extremal solutions; left-sided mixed Riemann-Liouville integral; Caputo fractional-order derivative; fixed point; Banach algebra.

Submitted May 6, 2010. Published July 7, 2010.

significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [11], Lakshmikantham *et al.* [21], and Samoilenko and Perestyuk [26], and the references therein.

In this article, we prove the existence of extremal solutions under discontinuous nonlinearity under certain Lipschitz and monotonicity conditions. These results extend to the Banach algebra setting those considered with integer order derivative [12, 19] and those with fractional derivative [4]. Also, we extend some results considered on Banach algebras with integer order derivative and without impulses [6, 15]. Finally, an example illustrating the abstract results is presented in the last Section.

This paper initiates the study of fractional hyperbolic differential equations with impulses on Banach algebras.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper. Let $L^\infty(J,\mathbb{R}^n)$ be the Banach space of measurable functions $u : J \to \mathbb{R}^n$ which are bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : \|u(x,y)\| \le c, \text{ a.e. } (x,y) \in J\},$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. By $L^1(J,\mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b \|u(x,y)\|\, dy\, dx.$$

$AC(J,\mathbb{R}^n)$ is the space of absolutely continuous valued functions on J. Denote by $D^2_{xy} := \frac{\partial^2}{\partial x \partial y}$ the mixed second order partial derivative. In all what follows set

$$J_k := (x_k, x_{k+1}] \times [0,b], \quad k = 0,1,\dots,m.$$

Consider the space

$$\begin{aligned} PC(J,\mathbb{R}^n) = \big\{ & u : J \to \mathbb{R}^n : u \in C(J_k,\mathbb{R}^n); k = 1,\dots,m, \text{ and there exist} \\ & u(x_k^-,y), u(x_k^+,y); k = 1,\dots,m, \text{ with } u(x_k^-,y) = u(x_k,y)\big\}. \end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J} \|u(x,y)\|.$$

Define a multiplication " $\cdot$ " by

$$(u \cdot v)(x,y) = u(x,y)v(x,y) \quad \text{for } (x,y) \in J.$$

Then $PC(J,\mathbb{R}^n)$ is a Banach algebra with the above norm and multiplication.

Let $a_1 \in [0,a]$, $z^+ = (a_1^+,0) \in J$, $J_z = [a_1,a] \times [0,b]$, $r_1, r_2 > 0$ and $r = (r_1,r_2)$. For $u \in L^1(J_z,\mathbb{R}^n)$, the expression

$$(I^r_{z^+}u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)\, dt\, ds,$$

where $\Gamma(.)$ is the Euler gamma function, is called the left-sided mixed Riemann-Liouville integral of order r.

Definition 2.1 ([27])**.** For $u \in L^1(J_z, \mathbb{R}^n)$, the Caputo fractional-order derivative of order r is defined by the expression $({}^cD^r_{z^+}u)(x,y) = (I^{1-r}_{z^+}D^2_{xy}u)(x,y)$.

Let X be a Banach algebra with norm $\|\cdot\|$. An operator $T : X \to X$ is called *compact* if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. Similarly $T : X \to X$ is called *totally bounded* if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \to X$ is called *completely continuous* operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true.

Definition 2.2. A function $\gamma : J \times \mathbb{R}^n \to \mathbb{R}^n$ is called *Carathéodory*'s if

- (i) the function $(x,y) \to \gamma(x,y,u)$ is measurable for each $u \in \mathbb{R}^n$,
- (ii) the function $u \to \gamma(x,y,u)$ is continuous for almost each $(x,y) \in J$.

A non-empty closed set K in a Banach algebra X is called a *cone* if

- (i) $K + K \subseteq K$,
- (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \geq 0$ and
- (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X.

The cone K is called to be *positive* if

- (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X.

We introduce an order relation, $\leq$, in X as follows. Let $u, v \in X$. Then $u \leq v$ if and only if $v - u \in K$. A cone K is called to be *normal* if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded.

Lemma 2.3 ([14])**.** *Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1u_2 \leq v_1v_2$.*

For any $v, w \in X, v \leq w$, the order interval $[v,w]$ is a set in X given by

$$[v,w] = \{u \in X : v \leq u \leq w\}.$$

The nonlinear alternative of Schaefer type proved by Dhage [13] is embodied in the following theorem.

Theorem 2.4. *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

- (a) *A is Lipschitz with a Lipschitz constant α,*
- (b) *B is compact and continuous, and*
- (c) *$\alpha M < 1$, where $M = \|B(X)\| := \sup\{\|Bu\| : u \in X\}$.*

Then either

- (i) *the equation $\lambda[Au\,Bu] = u$ has a solution for $0 < \lambda < 1$, or*
- (ii) *the set $\mathcal{E} = \{u \in X : \lambda[Au\,Bu] = u,\ 0 < \lambda < 1\}$ is unbounded.*

We use the following fixed point theorems by Dhage [14] for proving the existence of extremal solutions for our problem under certain monotonicity conditions.

Theorem 2.5. *Let K be a cone in a Banach algebra X and let $v, w \in X$. Suppose that $A, B : [v,w] \to K$ are two operators such that*

- (a) *A is completely continuous,*
- (b) *B is totally bounded,*
- (c) *$Au_1Bu_2 \in [v,w]$ for all $u_1, u_2 \in [v,w]$, and*

(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Au\,Bu = u$ has a least and a greatest positive solution in $[v, w]$.

Theorem 2.6. *Let K be a cone in a Banach algebra X and let $v, w \in X$. Suppose that $A, B : [v, w] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*
(b) *B is totally bounded,*
(c) *$Au_1\,Bu_2 \in [v, w]$ for all $u_1, u_2 \in [v, w]$, and*
(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Au\,Bu = u$ has least and a greatest positive solution in $[v, w]$, whenever $\alpha M < 1$, where $M = \|B([v, w])\| := \sup\{\|Bu\| : u \in [v, w]\}$.

Remark 2.7. Note that hypothesis (c) of Theorems 2.5 and 2.6 holds if the operators A and B are positive monotone increasing and there exist elements v and w in X such that $v \le Av\,Bv$ and $Aw\,Bw \le w$.

3. Auxiliary Results

Let us start by defining what we mean by a solution of problem (1.1)-(1.3). Set $J' := J \backslash \{(x_1, y), \dots, (x_m, y),\ y \in [0, b]\}$.

Definition 3.1. A function $u \in PC(J, \mathbb{R}^n)$ whose r-derivative exists on J' is said to be a solution of (1.1)-(1.3) if

(i) the function $(x, y) \mapsto \frac{u(x,y)}{f(x,y,u(x,y))}$ is absolutely continuous, and
(ii) u satisfies ${}^cD_0^r\big(\frac{u(x,y)}{f(x,y,u(x,y))}\big) = g(x, y, u(x, y))$ on J' and conditions (1.2), (1.3) are satisfied.

Let $f \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n \backslash \{0_{\mathbb{R}^n}\})$, $g \in L^1([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$, $z_k = (x_k, 0)$, and

$$\mu_{0,k}(x, y) = \frac{u(x, 0)}{f(x, 0)} + \frac{u(x_k^+, y)}{f(x_k^+, y)} - \frac{u(x_k^+, 0)}{f(x_k^+, 0)}, \quad k = 0, \dots, m.$$

For the existence of solutions for the problem (1.1)-(1.3), we need the following lemma.

Lemma 3.2. *A function $u \in AC([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$, $k = 0, \dots, m$ is a solution of the differential equation*

$$ {}^cD_{z_k}^r\Big(\frac{u}{f}\Big)(x, y) = g(x, y), \quad (x, y) \in [x_k, x_{k+1}] \times [0, b], \tag{3.1}$$

if and only if $u(x, y)$ satisfies

$$u(x, y) = f(x, y)\Big(\mu_{0,k}(x, y) + (I_{z_k}^r g)(x, y)\Big), \quad (x, y) \in [x_k, x_{k+1}] \times [0, b]. \tag{3.2}$$

Proof. Let $u(x, y)$ be a solution of (3.1). Then, taking into account the definition of the derivative ${}^cD_{z_k}^r$, we have

$$I_{z_k^+}^{1-r}(D_{xy}^2 \frac{u}{f})(x, y) = g(x, y).$$

Hence, we obtain

$$I^r_{z_k^+}(I^{1-r}_{z_k}D^2_{xy}\frac{u}{f})(x,y)=(I^r_{z_k^+}g)(x,y),$$

then

$$I^1_{z_k^+}(D^2_{xy}\frac{u}{f})(x,y)=(I^r_{z_k^+}g)(x,y).$$

Since

$$I^1_{z_k^+}(D^2_{xy}\frac{u}{f})(x,y)=\frac{u(x,y)}{f(x,y)}-\frac{u(x,0)}{f(x,0)}-\frac{u(x_k^+,y)}{f(x_k^+,y)}+\frac{u(x_k^+,0)}{f(x_k^+,0)},$$

we have

$$u(x,y)=f(x,y)\Big(\mu_{0,k}(x,y)+(I^r_{z_k}g)(x,y)\Big).$$

Now let $u(x,y)$ satisfies (3.2). It is clear that $u(x,y)$ satisfies (3.1). □

Corollary 3.3. *The function $u\in AC([x_k,x_{k+1}]\times[0,b],\mathbb{R}^n)$, $k=0,\dots,m$ is a solution of the differential equation* (1.1) *if and only if u satisfies the equation*

$$\begin{aligned}u(x,y)=\big[f(x,y,u(x,y))\big]\Big(&\mu_k(x,y)\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\Big),\end{aligned}\tag{3.3}$$

for $(x,y)\in[x_k,x_{k+1}]\times[0,b]$, where

$$\mu_k(x,y)=\frac{u(x,0)}{f(x,0,u(x,0))}+\frac{u(x_k^+,y)}{f(x_k^+,y,u(x_k^+,y))}-\frac{u(x_k^+,0)}{f(x_k^+,0,u(x_k^+,0))},\quad k=0,\dots,m.$$

Let $\mu':=\mu_{0,0}$.

Lemma 3.4. *Let $0<r_1,r_2\le 1$ and let $f:J\to\mathbb{R}^n\backslash\{0_{\mathbb{R}^n}\}$, $g:J\to\mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x,y)=\begin{cases}f(x,y)\Big[\mu'(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big],\\ \quad \text{if } (x,y)\in[0,x_1]\times[0,b],\\ f(x,y)\Big[\mu'(x,y)+\sum_{i=1}^k\Big(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y)}-\frac{I_i(u(x_i^-,0))}{f(x_i^+,0)}\Big)\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big],\\ \quad \text{if } (x,y)\in(x_k,x_{k+1}]\times[0,b],\ k=1,\dots,m,\end{cases}\tag{3.4}$$

if and only if u is a solution of the fractional initial-value problem

$${}^cD^r(\frac{u}{f})(x,y)=g(x,y),\quad (x,y)\in J',\tag{3.5}$$

$$u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y)),\ y\in[0,b],\ k=1,\dots,m.\tag{3.6}$$

Proof. Assume u satisfies (3.5)-(3.6). If $(x,y)\in[0,x_1]\times[0,b]$, then

$${}^cD^r(\frac{u}{f})(x,y)=g(x,y).$$

Lemma 3.2 implies

$$u(x,y)=f(x,y)\Big(\mu'(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)\,dt\,ds\Big).$$

If $(x,y) \in (x_1,x_2]\times[0,b]$, then Lemma 3.2 implies

$$\begin{aligned}
&u(x,y)\\
&= f(x,y)\Big(\mu_{0,1}(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)\\
&= f(x,y)\Big(\frac{\varphi(x)}{f(x,0)} + \frac{u(x_1^+,y)}{f(x_1^+,y)} - \frac{u(x_1^+,0)}{f(x_1^+,0)}\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)\\
&= f(x,y)\Big(\frac{\varphi(x)}{f(x,0)} + \frac{u(x_1^-,y)}{f(x_1^+,y)} - \frac{u(x_1^-,0)}{f(x_1^+,0)} + \frac{I_1(u(x_1^-,y))}{f(x_1^+,y)} - \frac{I_1(u(x_1^-,0))}{f(x_1^+,0)}\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)\\
&= f(x,y)\Big(\frac{\varphi(x)}{f(x,0)} + \frac{u(x_1,y)}{f(x_1^+,y)} - \frac{u(x_1,0)}{f(x_1^+,0)} + \frac{I_1(u(x_1^-,y))}{f(x_1^+,y)} - \frac{I_1(u(x_1^-,0))}{f(x_1^+,0)}\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)\\
&= f(x,y)\Big(\frac{\varphi(x)}{f(x,0)} + \frac{\psi(y)}{f(0,y)} - \frac{u(0,0)}{f(0,0)} + \frac{I_1(u(x_1^-,y))}{f(x_1^+,y)} - \frac{I_1(u(x_1^-,0))}{f(x_1^+,0)}\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_0^y (x_1-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)\\
&= f(x,y)\Big(\mu'(x,y) + \frac{I_1(u(x_1^-,y))}{f(x_1^+,y)} - \frac{I_1(u(x_1^-,0))}{f(x_1^+,0)}\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_0^y (x_1-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big).
\end{aligned}$$

If $(x,y) \in (x_2,x_3]\times[0,b]$, then from Lemma 3.2 we obtain

$$\begin{aligned}
&u(x,y)\\
&= f(x,y)\Big(\mu_{0,2}(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)\\
&= f(x,y)\Big(\frac{\varphi(x)}{f(x,0)} + \frac{u(x_2^+,y)}{f(x_2^+,y)} - \frac{u(x_2^+,0)}{f(x_2^+,0)}\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)\\
&= f(x,y)\Big(\frac{\varphi(x)}{f(x,0)} + \frac{u(x_2^-,y)}{f(x_2^+,y)} - \frac{u(x_2^-,0)}{f(x_2^+,0)} + \frac{I_2(u(x_2^-,y))}{f(x_2^+,y)} - \frac{I_2(u(x_2^-,0))}{f(x_2^+,0)}\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big)
\end{aligned}$$

$$
\begin{aligned}
&= f(x,y)\Big(\frac{\varphi(x)}{f(x,0)} + \frac{u(x_2,y)}{f(x_2^+,y)} - \frac{u(x_2,0)}{f(x_2^+,0)} + \frac{I_2(u(x_2^-,y))}{f(x_2^+,y)} - \frac{I_2(u(x_2^-,0))}{f(x_2^+,0)} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big) \\
&= f(x,y)\Big(\mu'(x,y) + \frac{I_1(u(x_1^-,y))}{f(x_1^+,y)} - \frac{I_1(u(x_1^-,0))}{f(x_1^+,0)} + \frac{I_2(u(x_2^-,y))}{f(x_2^+,y)} - \frac{I_2(u(x_2^-,0))}{f(x_2^+,0)} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{x_1}\int_0^y (x_1-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_0^y (x_2-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dt\,ds\Big).
\end{aligned}
$$

If $(x,y) \in (x_k, x_{k+1}] \times [0,b]$ then again from Lemma 3.2 we get (3.4).

Conversely, assume that u satisfies the impulsive fractional integral equation (3.4). If $(x,y) \in [0,x_1] \times [0,b]$ and using the fact that ${}^cD^r$ is the left inverse of I^r we get

$$
{}^cD^r(\frac{u}{f})(x,y) = g(x,y), \quad \text{for each } (x,y) \in [0,x_1] \times [0,b].
$$

If $(x,y) \in [x_k, x_{k+1}) \times [0,b]$, $k = 1,\dots,m$ and using the fact that ${}^cD^rC = 0$, where C is a constant, we get

$$
{}^cD^r(\frac{u}{f})(x,y) = g(x,y), \text{for each } (x,y) \in [x_k, x_{k+1}) \times [0,b].
$$

Also, we can easily show that

$$
u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)), \quad y \in [0,b], k = 1,\dots,m.
$$

□

Let $\mu := \mu_0$.

Corollary 3.5. *Let $0 < r_1, r_2 \le 1$ and let $f : J \times \mathbb{R}^n \to \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$, $g : J \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$
u(x,y) = \begin{cases}
f(x,y,u(x,y))\Big[\mu(x,y) \\
+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\Big], \\
\quad \text{if } (x,y) \in [0,x_1] \times [0,b], \\
f(x,y,u(x,y))\Big[\mu(x,y) + \sum_{i=1}^k \Big(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\Big) \\
+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds \\
+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\Big], \\
\quad \text{if } (x,y) \in (x_k, x_{k+1}] \times [0,b],\ k = 1,\dots,m,
\end{cases} \tag{3.7}
$$

if and only if u is a solution of the fractional initial-value problem

$$
{}^cD^r\Big(\frac{u(x,y)}{f(x,y,u(x,y))}\Big) = g(x,y,u(x,y)), \quad (x,y) \in J', \tag{3.8}
$$

$$
u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)),\ y \in [0,b],\ k = 1,\dots,m. \tag{3.9}
$$

4. Existence of Solutions

In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.3). The following hypotheses will be used in the sequel.

(A1) The function f is continuous on $J \times \mathbb{R}^n$.

(A2) There exists a function $\alpha \in C(J, \mathbb{R}_+)$ such that

$$\|f(x,y,u) - f(x,y,\overline{u})\| \le \alpha(x,y)\|u - \overline{u}\|; \quad \text{for all } (x,y) \in J \text{ and } u, \overline{u} \in \mathbb{R}^n.$$

(A3) The function g is Carathéodory, and there exists $h \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|g(x,y,u)\| \le h(x,y); \quad \text{a.e. } (x,y) \in J, \text{ for all } u \in \mathbb{R}^n.$$

(A4) There exists a function $\beta \in C(J, \mathbb{R}_+)$ such that

$$\Big\|\frac{I_k(u)}{f(x,y,u)}\Big\| \le \beta(x,y); \quad \text{for all } (x,y) \in J \text{ and } u \in \mathbb{R}^n.$$

Theorem 4.1. *Assume that hypotheses* (A1)–(A4) *hold. If*

$$\|\alpha\|_\infty \Big[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big] < 1, \tag{4.1}$$

Then the initial-value problem (1.1)-(1.3) *has at least one solution on J.*

Proof. Let $X := PC(J, \mathbb{R}^n)$. Define two operators A and B on X by

$$Au(x,y) = f(x,y,u(x,y)); \quad (x,y) \in J, \tag{4.2}$$

and

$$\begin{aligned} Bu(x,y) = \mu(x,y) &+ \sum_{i=1}^m \Big(\frac{I_i(u(x_i^-,y))}{f(x_i^+,y,u(x_i^+,y))} - \frac{I_i(u(x_i^-,0))}{f(x_i^+,0,u(x_i^+,0))}\Big) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^m \int_{x_{i-1}}^{x_i} \int_0^y (x_i - s)^{r_1-1}(y-t)^{r_2-1} g(s,t,u(s,t))\,dt\,ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_m}^{x} \int_0^y (x - s)^{r_1-1}(y-t)^{r_2-1} g(s,t,u(s,t))\,dt\,ds; \end{aligned} \tag{4.3}$$

with $(x,y) \in J$. Solving (1.1)-(1.3) is equivalent to solving (3.3), which is further equivalent to solving the operator equation

$$Au(x,y)\,Bu(x,y) = u(x,y), \quad (x,y) \in J. \tag{4.4}$$

We show that operators A and B satisfy all the assumptions of Theorem 2.4. First we shall show that A is a Lipschitz. Let $u_1, u_2 \in X$. Then by (A2),

$$\begin{aligned} \|Au_1(x,y) - Au_2(x,y)\| &= \|f(x,y,u_1(x,y)) - f(x,y,u_2(x,y))\| \\ &\le \alpha(x,y)\|u_1(x,y) - u_2(x,y)\| \\ &\le \|\alpha\|_\infty \|u_1 - u_2\|_{PC}. \end{aligned}$$

Taking the maximum over (x,y), in the above inequality yields

$$\|Au_1 - Au_2\|_{PC} \le \|\alpha\|_\infty \|u_1 - u_2\|_{PC},$$

and so A is a Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$.

Next, we show that B is compact operator on X. Let $\{u_n\}$ be a sequence in X. From (A3) and (A4) it follows that

$$\|Bu_n\|_{PC} \leq \|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

As a result $\{Bu_n : n \in \mathbb{N}\}$ is a uniformly bounded set in X.

Let $(\tau_1, y_1), (\tau_2, y_2) \in J$, $\tau_1 < \tau_2$ and $y_1 < y_2$, then for each $(x, y) \in J$,

$$\begin{aligned}
&\|B(u_n)(\tau_2, y_2) - B(u_n)(\tau_1, y_1)\| \\
&\leq \|\mu(\tau_1, y_1) - \mu(\tau_2, y_2)\| + \sum_{k=1}^m \Big\| \frac{I_k(u(x_k^-, y_1))}{f(x_k^+, y_1, u(x_i^+, y_1))} - \frac{I_k(u(x_k^-, y_2))}{f(x_k^+, y_2, u(x_i^+, y_2))} \Big\| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1-1}[(y_2 - t)^{r_2-1} - (y_1 - t)^{r_2-1}] \\
&\quad \times g(s, t, u(s, t))\, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1-1}(y_2 - t)^{r_2-1} \|g(s, t, u(s, t))\|\, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_0^{y_1} [(\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} - (\tau_1 - s)^{r_1-1}(y_1 - t)^{r_2-1}] \\
&\quad \times g(s, t, u(s, t))\, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} \|g(s, t, u(s, t))\|\, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} \|g(s, t, u(s, t))\|\, dt\, ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_0^{y_1} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} \|g(s, t, u(s, t))\|\, dt\, ds \\
&\leq \|\mu(\tau_1, y_1) - \mu(\tau_2, y_2)\| + \sum_{k=1}^m \Big\| \frac{I_k(u(x_k^-, y_1))}{f(x_k^+, y_1, u(x_i^+, y_1))} - \frac{I_k(u(x_k^-, y_2))}{f(x_k^+, y_2, u(x_i^+, y_2))} \Big\| \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_0^{y_1} (x_k - s)^{r_1-1}[(y_2 - t)^{r_2-1} - (y_1 - t)^{r_2-1}]\, dt\, ds \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^m \int_{x_{k-1}}^{x_k} \int_{y_1}^{y_2} (x_k - s)^{r_1-1}(y_2 - t)^{r_2-1}\, dt\, ds \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_0^{y_1} [(\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} - (\tau_1 - s)^{r_1-1}(y_1 - t)^{r_2-1}]\, dt\, ds \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1}\, dt\, ds \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1}\, dt\, ds \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_0^{y_1} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1}\, dt\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\| + \sum_{k=1}^{m}\Big\|\frac{I_k(u(x_k^-,y_1))}{f(x_k^+,y_1,u(x_i^+,y_1))} - \frac{I_k(u(x_k^-,y_2))}{f(x_k^+,y_2,u(x_i^+,y_2))}\Big\| \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y_1}(x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\,dt\,ds \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_{y_1}^{y_2}(x_k-s)^{r_1-1}(y_2-t)^{r_2-1}\,dt\,ds \\
&\quad + \frac{\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}[2y_2^{r_2}(\tau_2-\tau_1)^{r_1}+2\tau_2^{r_1}(y_2-y_1)^{r_2} \\
&\quad + \tau_1^{r_1}y_1^{r_2}-\tau_2^{r_1}y_2^{r_2}-2(\tau_2-\tau_1)^{r_1}(y_2-y_1)^{r_2}].
\end{aligned}
$$

As $\tau_1 \to \tau_2$ and $y_1 \to y_2$, the right-hand side of the above inequality tends to zero. From this we conclude that $\{Bu_n : n \in \mathbb{N}\}$ is an equicontinuous set in X. Hence $B : X \to X$ is compact by Arzelà-Ascoli theorem. Moreover,

$$
M = \|B(X)\| \leq \|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)},
$$

and so,

$$
\alpha M \leq \|\alpha\|_\infty\Big(\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) < 1,
$$

by assumption (4.1). To finish, it remain to show that either the conclusion (i) or the conclusion (ii) of Theorem 2.4 holds. We now will show that the conclusion (ii) is not possible. Let $u \in X$ be any solution to (1.1)-(1.3), then for any $\lambda \in (0,1)$ we have

$$
\begin{aligned}
&u(x,y) \\
&= \lambda f(x,y,u(x,y))\Big[\mu(x,y) + \sum_{0<x_k<x}\Big(\frac{I_k(u(x_k^-,y))}{f(x_k^+,y,u(x_i^+,y))} - \frac{I_k(u(x_k^-,0))}{f(x_k^+,0,u(x_i^+,0))}\Big) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^{y}(x_k-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^{y}(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\Big].
\end{aligned}
$$

for $(x,y) \in J$. Therefore,

$$
\begin{aligned}
\|u(x,y)\| &\leq \|f(x,y,u(x,y))\|\Big(\|\mu(x,y)\| + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \\
&\leq \big[\|f(x,y,u(x,y)) - f(x,y,0)\| + \|f(x,y,0)\|\big] \\
&\quad \times \Big(\|\mu(x,y)\| + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \\
&\leq \big[\|\alpha\|_\infty\|u(x,y)\| + f^*\big]\Big(\|\mu(x,y)\| + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big) \\
&\leq [\|\alpha\|_\infty\|u\|_{PC} + f^*]\Big(\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big),
\end{aligned}
$$

where $f^* = \sup\{\|f(x,y,0)\| : (x,y) \in J\}$, and consequently

$$\|u\|_{PC} \leq \frac{f^*\big[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\big]}{1-\|\alpha\|_\infty\big[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\big]} := M.$$

Thus the conclusion (ii) of Theorem 2.4 does not hold. Therefore (1.1)-(1.3) has a solution on J. □

5. Existence of extremal solutions

We equip the space $PC(J,\mathbb{R}^n)$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{u \in PC(J,\mathbb{R}^n) : u(x,y) \geq 0,\ \forall (x,y) \in J\}.$$

Thus $u \leq \bar{u}$ if and only if $u(x,y) \leq \bar{u}(x,y)$ for each $(x,y) \in J$.

It is well-known that the cone K is positive and normal in $PC(J,\mathbb{R}^n)$ ([17]). If $\underline{u}, \bar{u} \in C(J,\mathbb{R}^n)$ and $\underline{u} \leq \bar{u}$, we put

$$[\underline{u}, \overline{u}] = \{u \in PC(J,\mathbb{R}^n) : \underline{u} \leq u \leq \bar{u}\}.$$

Definition 5.1. A function $\gamma : J \times \mathbb{R}^n \to \mathbb{R}^n$ is called *Chandrabhan* if

- (i) the function $(x,y) \to \gamma(x,y,u)$ is measurable for each $u \in \mathbb{R}^n$,
- (ii) the function $u \to \gamma(x,y,u)$ is nondecreasing for almost each $(x,y) \in J$.

Definition 5.2. A function $\underline{u}(\cdot,\cdot) \in PC(J,\mathbb{R}^n)$ is said to be a lower solution of (1.1)-(1.3) if

$$\begin{gathered}{}^cD_0^r\Big[\frac{\underline{u}(x,y)}{f(x,y,\underline{u}(x,y))}\Big] \leq g(x,y,\underline{u}(x,y)), \quad (x,y) \in J,\ x \neq x_k,\ k = 1,\dots,m,\\ \underline{u}(x_k^+,y) \leq \underline{u}(x_k^-,y) + I_k(\underline{u}(x_k^-,y)),\ y \in [0,b];\ k = 1,\dots,m,\\ \underline{u}(x,0) \leq \varphi(x), \quad \underline{u}(0,y) \leq \psi(y), \quad (x,y) \in J.\end{gathered}$$

Similarly a function $\bar{u}(\cdot,\cdot) \in PC(J,\mathbb{R}^n)$ is said to be an upper solution of (1.1)-(1.3) if

$$\begin{gathered}{}^cD_0^r\Big[\frac{\bar{u}(x,y)}{f(x,y,\bar{u}(x,y))}\Big] \geq g(x,y,\bar{u}(x,y)), \quad (x,y) \in J,\ x \neq x_k,\ k = 1,\dots,m,\\ \bar{u}(x_k^+,y) \geq \bar{u}(x_k^-,y) + I_k(\bar{u}(x_k^-,y)), \quad y \in [0,b];\ k = 1,\dots,m,\\ \bar{u}(x,0) \geq \varphi(x), \quad \bar{u}(0,y) \geq \psi(y), \quad (x,y) \in J.\end{gathered}$$

Definition 5.3. A solution u_M of the problem (1.1)-(1.3) is said to be maximal if for any other solution u to the problem (1.1)-(1.3) one has $u(x,y) \leq u_M(x,y)$, for all $(x,y) \in J$. Again a solution u_m of the problem (1.1)-(1.3) is said to be minimal if $u_m(x,y) \leq u(x,y)$, for all $(x,y) \in J$ where u is any solution of the problem (1.1)-(1.3) on J.

The following hypotheses will be used in the sequel.

- (H1) $f : J \times \mathbb{R}_+^n \to \mathbb{R}_+^n \setminus \{0\}$, $g : J \times \mathbb{R}_+^n \to \mathbb{R}_+^n$, $\psi(y) \geq 0$ on $[0,b]$ and
$$\frac{\varphi(x)}{f(x,0,\varphi(x))} \geq \frac{\varphi(0)}{f(0,0,\varphi(0))} \quad \text{for all } x \in [0,a].$$
- (H2) The functions f and g are Chandrabhan.

(H3) There exists a function $\tilde{h} \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|g(x,y,u)\| \le \tilde{h}(x,y), \quad \text{a.e. } (x,y) \in J, \text{ for all } u \in \mathbb{R}^n.$$

(H4) There exists a function $\tilde{\beta} \in C(J, \mathbb{R}_+)$ such that

$$\Big\| \frac{I_k(u)}{f(x,y,u)} \Big\| \le \tilde{\beta}(x,y), \quad \text{for all } (x,y) \in J, \text{ for all } u \in \mathbb{R}^n.$$

(H5) The problem (1.1)-(1.3) has a lower solution $\underline{u}$ and an upper solution $\overline{u}$ with $\underline{u} \le \overline{u}$.

Theorem 5.4. *Assume that hypotheses* (A2), (H1)–(H5) *hold. If*

$$\|\alpha\|_\infty \big[\|\mu\|_\infty + 2m\|\tilde{\beta}\| + \frac{2a^{r_1} b^{r_2} \|\tilde{h}\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)} \big] < 1,$$

then (1.1)-(1.3) *has a minimal and a maximal positive solution on J.*

Proof. Let $X = PC(J, \mathbb{R}^n)$ and consider a closed interval $[\underline{u}, \overline{u}]$ in X which is well defined in view of hypothesis (H5). Define two operators $A, B : [\underline{u}, \overline{u}] \to X$ by (4.2) and (4.3), respectively. Clearly A and B define the operators $A, B : [\underline{u}, \overline{u}] \to K$.

Now solving (1.1)-(1.3) is equivalent to solving (3.3), which is further equivalent to solving the operator equation

$$Au(x,y)\, Bu(x,y) = u(x,y), \quad (x,y) \in J. \tag{5.1}$$

We show that operators A and B satisfy all the assumptions of Theorem 2.6. As in Theorem 4.1 we can prove that A is Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$ and B is completely continuous operator on $[\underline{u}, \overline{u}]$. Now hypothesis (H2) implies that A and B are nondecreasing on $[\underline{u}, \overline{u}]$. To see this, let $u_1, u_2 \in [\underline{u}, \overline{u}]$ be such that $u_1 \le u_2$. Then by (H2), we obtain

$$Au_1(x,y) = f(x,y,u_1(x,y)) \le f(x,y,u_2(x,y)) = Au_2(x,y), \quad \forall (x,y) \in J,$$

and

$$\begin{aligned}
&Bu_1(x,y)\\
&= \mu(x,y) + \sum_{0<x_k<x} \Big(\frac{I_k(u_1(x_k^-, y))}{f(x_k^+, y, u(x_i^+, y))} - \frac{I_k(u_1(x_k^-, 0))}{f(x_k^+, 0, u(x_i^+, 0))} \Big)\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y-t)^{r_2-1} g(s,t,u_1(s,t))\, dt\, ds\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} g(s,t,u_1(s,t))\, dt\, ds\\
&\le \mu(x,y) + \sum_{0<x_k<x} \Big(\frac{I_k(u_2(x_k^-, y))}{f(x_k^+, y, u(x_i^+, y))} - \frac{I_k(u_2(x_k^-, 0))}{f(x_k^+, 0, u(x_i^+, 0))} \Big)\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1} (y-t)^{r_2-1} g(s,t,u_2(s,t))\, dt\, ds\\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} g(s,t,u_2(s,t))\, dt\, ds\\
&= Bu_2(x,y), \quad \forall (x,y) \in J.
\end{aligned}$$

So A and B are nondecreasing operators on $[\underline{u},\overline{u}]$. Again hypothesis (H5) implies

$$\begin{aligned}
&\underline{u}(x,y)\\
&=[f(x,y,\underline{u}(x,y))]\Big(\mu(x,y)+\sum_{0<x_k<x}\Big(\frac{I_k(\underline{u}(x_k^-,y))}{f(x_k^+,y,u(x_i^+,y))}-\frac{I_k(\underline{u}(x_k^-,0))}{f(x_k^+,0,u(x_i^+,0))}\Big)\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,\underline{u}(s,t))\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,\underline{u}(s,t))\,dt\,ds\Big)\\
&\le[f(x,y,u(x,y))]\Big(\mu(x,y)+\sum_{0<x_k<x}\Big(\frac{I_k(u(x_k^-,y))}{f(x_k^+,y,u(x_i^+,y))}-\frac{I_k(u(x_k^-,0))}{f(x_k^+,0,u(x_i^+,0))}\Big)\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\Big)\\
&\le[f(x,y,\overline{u}(x,y))]\Big(\mu(x,y)+\sum_{0<x_k<x}\Big(\frac{I_k(\overline{u}(x_k^-,y))}{f(x_k^+,y,u(x_i^+,y))}-\frac{I_k(\overline{u}(x_k^-,0))}{f(x_k^+,0,u(x_i^+,0))}\Big)\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,\overline{u}(s,t))\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,\overline{u}(s,t))\,dt\,ds\Big)\\
&\le\overline{u}(x,y),
\end{aligned}$$

for all $(x,y)\in J$ and $u\in[\underline{u},\overline{u}]$. As a result

$$\underline{u}(x,y)\le Au(x,y)Bu(x,y)\le\overline{u}(x,y),\quad\forall(x,y)\in J\text{ and }u\in[\underline{u},\overline{u}].$$

Hence $Au\,Bu\in[\underline{u},\overline{u}]$, for all $u\in[\underline{u},\overline{u}]$.

Notice for any $u\in[\underline{u},\overline{u}]$,

$$\begin{aligned}
M&=\|B([\underline{u},\overline{u}])\|\\
&\le\|\mu(x,y)\|+\Big\|\sum_{0<x_k<x}\Big(\frac{I_k(u(x_k^-,y))}{f(x_k^+,y,u(x_i^+,y))}-\frac{I_k(u(x_k^-,0))}{f(x_k^+,0,u(x_i^+,0))}\Big)\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{0<x_k<x}\int_{x_{k-1}}^{x_k}\int_0^y(x_k-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t,u(s,t))\,dt\,ds\Big\|\\
&\le\|\mu\|_\infty+2m\|\tilde{\beta}\|+\frac{2a^{r_1}b^{r_2}\|\tilde{h}\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}.
\end{aligned}$$

and so,

$$\alpha M\le\|\alpha\|_\infty\Big(\|\mu\|_\infty+2m\|\tilde{\beta}\|+\frac{2a^{r_1}b^{r_2}\|\tilde{h}\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big)<1.$$

Thus the operators A and B satisfy all the conditions of Theorem 2.6 and so the operator equation (4.3) has a least and a greatest solution in $[\underline{u}, \overline{u}]$. This further implies that the problem (1.1)-(1.3) has a minimal and a maximal positive solution on J. □

Theorem 5.5. *Assume that hypotheses* (A1), (H1)–(H5) *hold. Then* (1.1)-(1.3) *has a minimal and a maximal positive solution on J.*

Proof. Let $X = PC(J, \mathbb{R}^n)$. Consider the order interval $[\underline{u}, \overline{u}]$ in X and define two operators A and B on $[\underline{u}, \overline{u}]$ by (4.2) and (4.3) respectively. Then the problem (1.1)-(1.3) is transformed into an operator equation $Au(x, y)\, Bu(x, y) = u(x, y)$, $(x, y) \in J$ in a Banach algebra X. Notice that(H1) implies $A, B : [\underline{u}, \overline{u}] \to K$. Since the cone K in X is normal, $[\underline{u}, \overline{u}]$ is a norm bounded set in X.

Next we show that A is completely continuous on $[\underline{u}, \overline{u}]$. Now the cone K in X is normal, so the order interval $[\underline{u}, \overline{u}]$ is norm-bounded. Hence there exists a constant $r > 0$ such that $\|u\| \le r$ for all $u \in [\underline{u}, \overline{u}]$. As f is continuous on compact set $J \times [-r, r]$, it attains its maximum, say M. Therefore, for any subset S of $[\underline{u}, \overline{u}]$ we have

$$\begin{aligned}
\|A(S)\| &= \sup\{\|Au\| : u \in S\} \\
&= \sup\Big\{ \sup_{(x,y)\in J} \|f(x, y, u(x, y))\| : u \in S\Big\} \\
&\le \sup\Big\{ \sup_{(x,y)\in J} \|f(x, y, u)\| : u \in [-r, r]\Big\} \le M.
\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

We note that the function $f(x, y, u)$ is uniformly continuous on $J \times [-r, r]$. Therefore, for any $(\tau_1, y_1), (\tau_2, y_2) \in J$ we have

$$\|f(\tau_1, y_1, u) - f(\tau_2, y_2, u)\| \to 0 \quad \text{as } (\tau_1, y_1) \to (\tau_2, y_2),$$

for all $u \in [-r, r]$. Similarly for any $u_1, u_2 \in [-r, r]$

$$\|f(x, y, u_1) - f(x, y, u_2)\| \to 0 \quad \text{as } u_1 \to u_2,$$

for all $(x, y) \in J$. Hence for any $(\tau_1, y_1), (\tau_2, y_2) \in J$ and for any $u \in S$ one has

$$\begin{aligned}
\|Au(\tau_1, y_1) - Au(\tau_2, y_2)\| &= \|f(\tau_1, y_1, u(\tau_1, y_1)) - f(\tau_2, y_2, u(\tau_2, y_2))\| \\
&\le \|f(\tau_1, y_1, u(\tau_1, y_1)) - f(\tau_2, y_2, u(\tau_1, y_1)\| \\
&\quad + \|f(\tau_2, y_2, u(\tau_1, y_1)) - f(\tau_2, y_2, u(\tau_2, y_2))\| \\
&\to 0 \quad \text{as } (\tau_1, y_1) \to (\tau_2, y_2).
\end{aligned}$$

This shows that $A(S)$ is an equicontinuous set in K. Now an application of Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[\underline{u}, \overline{u}]$. □

Next it can be shown as in the proof of Theorem 5.4 that B is a compact operator on $[\underline{u}, \overline{u}]$. Now an application of Theorem 2.5 yields that the problem (1.1)-(1.3) has a minimal and maximal positive solution on J.

6. An Example

As an application of our results we consider the following partial hyperbolic functional differential equations of the form

$$ {}^cD_0^r\Big(\frac{u(x,y)}{f(x,y,u(x,y))}\Big) = g(x,y,u(x,y)), \quad (x,y)\in[0,1]\times[0,1], \tag{6.1}$$

$$ u\Big(\frac{1}{2}^+,y\Big) = u\Big(\frac{1}{2}^-,y\Big) + I_1\Big(u\Big(\frac{1}{2}^-,y\Big)\Big), \quad y\in[0,1], \tag{6.2}$$

$$ u(x,0)=\varphi(x), \quad x\in[0,1], \quad u(0,y)=\psi(y), \quad y\in[0,1], \tag{6.3}$$

where $f,g:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ and $I_1:\mathbb{R}\to\mathbb{R}$ are defined by

$$\begin{gathered} f(x,y,u) = \frac{1}{e^{x+y+10}(1+|u|)}, \\ g(x,y,u) = \frac{1}{e^{x+y+8}(1+u^2)}, \\ I_1(u) = \frac{(8+e^{-10})^2}{512e^{10}(1+|u|)^2}. \end{gathered}$$

The functions $\varphi,\psi:[0,1]\to\mathbb{R}$ are defined by

$$\varphi(x) = \begin{cases} \frac{x^2}{2}e^{-10}; & \text{if } x\in[0,\frac{1}{2}], \\ x^2e^{-10}; & \text{if } x\in(\frac{1}{2},1], \end{cases}$$

and

$$\psi(y) = ye^{-10}, \quad \text{for all } y\in[0,1].$$

We show that the functions φ,ψ,f,g and I_1 satisfy all the hypotheses of Theorem 4.1. Clearly, the function f satisfies (A1) and (A2) with $\alpha(x,y)=\frac{1}{e^{x+y+10}}$ and

$$\|\alpha\|_\infty = 1/e^{10}.$$

Also, the function g satisfies (A3) with $h(x,y)=\frac{1}{e^{x+y+8}}$ and

$$\|h\|_{L^\infty} = 1/e^8.$$

Finally, condition (A4) holds with $\beta(x,y)=\frac{81e^{x+y}}{512}$ and $\|\beta\|_\infty=\frac{81e^2}{512}$. A simple computation gives $\mu(x,y)<4e$. Condition (4.1) holds. Indeed

$$\begin{aligned} &\|\alpha\|_\infty\Big[\|\mu\|_\infty + 2m\|\beta\|_\infty + \frac{2a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big] \\ &< \frac{1}{e^{10}}\Big[4e + \frac{81e^2}{256} + \frac{2}{e^8\Gamma(r_1+1)\Gamma(r_2+1)}\Big] < 1, \end{aligned}$$

for each $(r_1,r_2)\in(0,1]\times(0,1]$. Hence by Theorem 4.1, problem (6.1)-(6.3) has a solution defined on $[0,1]\times[0,1]$.

References

[1] S. Abbas and M. Benchohra; Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative, *Commun. Math. Anal.* **7** (2009), 62-72.

[2] S. Abbas and M. Benchohra; Darboux problem for perturbed partial differential equations of fractional order with finite delay, *Nonlinear Anal. Hybrid Syst.* **3** (2009), 597-604.

[3] S. Abbas and M. Benchohra; Upper and lower solutions method for impulsive partial hyperbolic differential equations with fractional order, *Nonlinear Anal. Hybrid Syst.* **4** (2010), 604-613.

[4] S. Abbas and M. Benchohra; Darboux problem for fractional order discontinuous hyperbolic partial differential equations in Banach algebras, *Math. Bohemica* (to appear).

[5] R.P Agarwal, M. Benchohra and S. Hamani; A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta. Appl. Math.* **109** (3) (2010), 973-1033.

[6] A. Arara, B. D. Karande, M. Benchohra and B. C. Dhage; Existence theorems for a class of hyperbolic differential equations in Banach algebras, *J. Nat. Physic. Sci.* **20** (1-2) (2006), 51-63.

[7] A. Belarbi, M. Benchohra and A. Ouahab; Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces, *Appl. Anal.* **85** (2006), 1459-1470.

[8] M. Benchohra, J. R. Graef and S. Hamani; Existence results for boundary value problems of nonlinear fractional differential equations with integral conditions, *Appl. Anal.* **87** (7) (2008), 851-863.

[9] M. Benchohra, S. Hamani and S. K. Ntouyas; Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[10] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab; Existence results for functional differential equations of fractional order, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[11] M. Benchohra, J. Henderson and S. K. Ntouyas; *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, Vol 2, New York, 2006.

[12] M. Dawidowski and I. Kubiaczyk; An existence theorem for the generalized hyperbolic equation $z''_{xy} \in F(x, y, z)$ in Banach space, *Ann. Soc. Math. Pol. Ser. I, Comment. Math.*, **30** (1) (1990), 41-49.

[13] B. C. Dhage; A nonlinear alternative in Banach algebras with applications to functional differential equations, *Nonlinear Funct. Anal. & Appl.* **8** (2004), 563–575.

[14] B. C. Dhage; Some algebraic fixed point theorems for multi-valued mappings with applications, *Diss. Math. Differential Inclusions, Control & Optim.* **26** (2006), 5-55.

[15] B. C. Dhage, S. K. Ntouyas; Existence of solutions for discontinuous hyperbolic partial differential equations in Banach algebras, *Electron. J. Differential Equations* Vol. 2006 (2006), No. 28, pp. 1-10.

[16] K. Diethelm and N. J. Ford; Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.

[17] S. Heikkila and V. Lakshmikantham; *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Pure and Applied Mathematics, Marcel Dekker, New York, 1994.

[18] R. Hilfer; *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[19] Z. Kamont; *Hyperbolic Functional Differential Inequalities and Applications*. Kluwer Academic Publishers, Dordrecht, 1999.

[20] A. A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo; *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam, 2006.

[21] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov; *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[22] V. Lakshmikantham, S. Leela and J. Vasundhara; *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[23] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher; Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.

[24] I. Podlubny; *Fractional Differential Equation*, Academic Press, San Diego, 1999.

[25] S. G. Samko, A. A. Kilbas and O. I. Marichev; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[26] A. M. Samoilenko and N. A. Perestyuk; *Impulsive Differential Equations, World Scientific*, Singapore, 1995.

[27] A. N. Vityuk and A. V. Golushkov; Existence of solutions of systems of partial differential equations of fractional order, *Nonlinear Oscil.* **7** (3) (2004), 318-325.

Saïd Abbas
Laboratoire de Mathématiques, Université de Saïda, B.P. 138, 20000, Saïda, Algérie
E-mail address: abbas_said_dz@yahoo.fr

Ravi P. Agarwal
Department of Mathematical Sciences, Florida Institute of Technology, Melboune, Florida, 32901-6975, USA
KFUPM Chair Professor, Mathematics and Statistics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail address: agarwal@fit.edu

Mouffak Benchohra
Laboratoire de Mathématiques, Université de Sidi Bel-Abbès, B.P. 89, 22000, Sidi Bel-Abbès, Algérie
E-mail address: benchohra@univ-sba.dz