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SOLUTIONS TO THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS FOR INCOMPRESSIBLE FLUIDS

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Abstract. This article gives explicit solutions to the space-periodic Navier-Stokes problem with non-periodic pressure. These type of solutions are not unique and by using such solutions one can construct a periodic, smooth, divergence-free initial vector field allowing a space-periodic and time-bounded external force such that there exists a smooth solution to the 3-dimensional Navier-Stokes equations for incompressible fluid with those initial conditions, but the solution cannot be continued to the whole space.

1. Introduction

Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ denote the position, $t \geq 0$ be the time, $p(x,t) \in \mathbb{R}$ be the pressure and $u(x,t) = (u_i(x,t))_{1\leq i\leq 3} \in \mathbb{R}^3$ be the velocity vector. Let $f_i(x,t)$ be the external force. The Navier-Stokes equations for incompressible fluids filling all of $\mathbb{R}^3$ for $t \geq 0$ are [1]

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = v\Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), \quad x \in \mathbb{R}^3,\ t \geq 0,\ 1 \leq i \leq 3 \tag{1.1}$$

$$\operatorname{div} u = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0, \quad x \in \mathbb{R}^3,\ t \geq 0 \tag{1.2}$$

with initial conditions

$$u(x,0) = u^0(x), \quad x \in \mathbb{R}^3. \tag{1.3}$$

Here $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables, v is a positive coefficient and $u^0(x)$ is $C^\infty(\mathbb{R}^3)$ vector field on $\mathbb{R}^3$ required to be divergence-free; i.e., satisfying $\operatorname{div} u^0 = 0$. The time derivative $\frac{\partial u_i}{\partial t}$ at $t = 0$ in (1.1) is taken to mean the limit when $t \to 0^+$.

This article shows that there exists $C^\infty(\mathbb{R}^3)$, periodic, divergence-free initial vector fields u^0 defined at $\mathbb{R}^3$ such that there exists a family of smooth (here, in the class $C^\infty(\mathbb{R}^3 \times [0,\infty))$) functions $u(x,t)$ and $p(x,t)$ satisfying (1.1), (1.2) and (1.3). We also show that there exist a periodic and bounded external force $f_i(x,t)$ such that the solution cannot be continued to the whole $\mathbb{R}^3 \times [0,\infty)$.

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2. Theorems and Lemmas

The simple explicit case of $u(x,t)$ in Lemma 2.1 satisfies the conditions given in (1.3) and allows a free function $g(t)$ that only satisfies $g(0) = g'(0) = 0$. The solution is then not unique. If the time derivatives of $u(x,t)$ are specified at $t = 0$ then the solution in the lemma is unique.

Lemma 2.1. *Let*

$$\begin{aligned} u_1^0 &= 2\pi\sin(2\pi x_2) + 2\pi\cos(2\pi x_3),\\ u_2^0 &= 2\pi\sin(2\pi x_3) + 2\pi\cos(2\pi x_1),\\ u_3^0 &= 2\pi\sin(2\pi x_1) + 2\pi\cos(2\pi x_2) \end{aligned}$$

be the initial vector field, and let $f_i(x,t)$ be chosen identically zero for $1 \le i \le 3$. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with $g(0) = g'(0) = 0$ and $\beta = (2\pi)^2 v$. The following family of functions u and p satisfy (1.1)-(1.3)*:*

$$\begin{aligned} u_1 &= e^{-\beta t}2\pi\left(\sin(2\pi(x_2+g(t))) + \cos(2\pi(x_3+g(t)))\right) - g'(t),\\ u_2 &= e^{-\beta t}2\pi\left(\sin(2\pi(x_3+g(t))) + \cos(2\pi(x_1+g(t)))\right) - g'(t),\\ u_3 &= e^{-\beta t}2\pi\left(\sin(2\pi(x_1+g(t))) + \cos(2\pi(x_2+g(t)))\right) - g'(t),\\ p &= -e^{-2\beta t}(2\pi)^2\sin(2\pi(x_1+g(t)))\cos(2\pi(x_2+g(t)))\\ &\quad - e^{-2\beta t}(2\pi)^2\sin(2\pi(x_2+g(t)))\cos(2\pi(x_3+g(t)))\\ &\quad - e^{-2\beta t}(2\pi)^2\sin(2\pi(x_3+g(t)))\cos(2\pi(x_1+g(t))) + g''(t)\sum_{j=1}^{3} x_j. \end{aligned} \tag{2.1}$$

Proof. The initial vector field is smooth, periodic, bounded and divergence-free. Let (i,k,m) be any of the permutations $(1,2,3)$, $(2,3,1)$ or $(3,1,2)$. We can write all definitions in (2.1) shorter as (here $g'(t) = dg/dt$):

$$\begin{aligned} u_i &= e^{-\beta t}2\pi\left(\sin(2\pi(x_k+g(t))) + \cos(2\pi(x_m+g(t)))\right) - g'(t),\\ p &= -e^{-2\beta t}(2\pi)^2\sin(2\pi(x_i+g(t)))\cos(2\pi(x_k+g(t)))\\ &\quad - e^{-2\beta t}(2\pi)^2\sin(2\pi(x_k+g(t)))\cos(2\pi(x_m+g(t)))\\ &\quad - e^{-2\beta t}(2\pi)^2\sin(2\pi(x_m+g(t)))\cos(2\pi(x_i+g(t))) + g''(t)\sum_{j=1}^{3} x_j. \end{aligned}$$

It is sufficient to proof the claim for these permutations. The permutations $(1,3,2)$, $(2,1,3)$ and $(3,2,1)$ only interchange the indices k and m. The functions (2.1) are smooth and $u(x,t)$ in (2.1) satisfies (1.2) and (1.3) for the initial vector field in Lemma 2.1. We will verify (1.1) by directly computing:

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= -\beta e^{-\beta t}2\pi\left(\sin(2\pi(x_k+g(t))) + \cos(2\pi(x_m+g(t)))\right) - g''(t)\\ &\quad + g'(t)e^{-\beta t}(2\pi)^2\left(\cos(2\pi(x_k+g(t))) - \sin(2\pi(x_m+g(t)))\right),\\ -v\Delta u_i &= ve^{-\beta t}(2\pi)^3\left(\sin(2\pi(x_k+g(t))) + \cos(2\pi(x_m+g(t)))\right),\\ \frac{\partial p}{\partial x_i} &= -e^{-2\beta t}(2\pi)^3\cos(2\pi(x_i+g(t)))\cos(2\pi(x_k+g(t)))\\ &\quad + e^{-2\beta t}(2\pi)^3\sin(2\pi(x_m+g(t)))\sin(2\pi(x_i+g(t))) + g''(t). \end{aligned}$$

The functions u_k and u_m are

$$u_k = e^{-\beta t} 2\pi \left(\sin(2\pi(x_m + g(t))) + \cos(2\pi(x_i + g(t)))\right) - g'(t),$$
$$u_m = e^{-\beta t} 2\pi \left(\sin(2\pi(x_i + g(t))) + \cos(2\pi(x_k + g(t)))\right) - g'(t).$$

The remaining term to be computed in (1.1) is

$$\begin{aligned}\sum_{j\in\{i,k,m\}} u_j \frac{\partial u_i}{\partial x_j} &= u_i \frac{\partial u_i}{\partial x_i} + u_k \frac{\partial u_i}{\partial x_k} + u_m \frac{\partial u_i}{\partial x_m} \\ &= e^{-2\beta t}(2\pi)^3 \cos(2\pi(x_i + g(t))) \cos(2\pi(x_k + g(t))) \\ &\quad - e^{-2\beta t}(2\pi)^3 \sin(2\pi(x_m + g(t))) \sin(2\pi(x_i + g(t))) \\ &\quad - g'(t) e^{-\beta t}(2\pi)^2 \left(\cos(2\pi(x_k + g(t))) - \sin(2\pi(x_m + g(t)))\right).\end{aligned}$$

Inserting the parts to (1.1) shows that

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - v\Delta u_i + \frac{\partial p}{\partial x_i} = 0.$$

This completes the proof . □

Theorem 2.2. *There exists a periodic, $C^\infty(\mathbb{R}^3)$, and divergence-free vector field $u^0(x) = (u_i^0(x))_{1\le i\le 3}$ on $\mathbb{R}^3$ such that the following two claims hold:*

- C1: *The solution to (1.1)-(1.3) is not necessarily unique. In fact, there are infinitely many $C^\infty\left(\mathbb{R}^3 \times [0,\infty)\right)$ functions $u(x,t) = (u_i(x,t))_{1\le i\le 3}$ and $p(x,t)$ satisfying (1.1), (1.2) and (1.3).*
- C2: *Periodic initial values do not guarantee that the solution is bounded. Indeed, there exist unbounded $u(x,t)$ and $p(x,t)$ satisfying (1.1), (1.2) and the initial values (1.3). There also exist bounded solutions that are periodic as functions of x.*

Proof. Let $f_i(x,t)$ be chosen identically zero for $1 \le i \le 3$, and let us select $g(t) = \frac{1}{2}ct^2$ in Lemma 2.1. The value $c \in \mathbb{R}$ can be freely chosen. This shows C1. If $c = 0$ then the solution is bounded and it is periodic as a function of x. If $c \ne 0$ then $u_i(x,t)$ for every i and $p(x,t)$ are all unbounded. In $u_i(x,t)$ the $ct = g'(t)$ term and in p the term $c(x_1 + x_2 + x_3) = g''(t)\sum_{j=1}^{3} x_j$ are not bounded. This shows C2. The failure of uniqueness is caused by the fact that (1.1)-(1.3) do not determine the limits of the higher time derivatives of $u(x,t)$ when $t \to 0+$. These derivatives can be computed by differentiating (2.1) but the function $g(t)$ is needed and it determines the higher time derivatives. As $g(t)$ can be freely chosen, the solutions are not unique. □

Theorem 2.3. *There exists a smooth, divergence-free vector field $u^0(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ and a number $C_{\alpha,m,K} > 0$ satisfying*

$$u^0(x + e_j) = u^0(x), \quad f(x + e_j, t) = f(x,t), \quad 1 \le j \le 3 \tag{2.2}$$

(here e_j is the unit vector), and

$$|\partial_x^\alpha \partial_t^m f(x,t)| \le C_{\alpha m K}(1 + |t|)^{-K} \tag{2.3}$$

for any α, m and K, such that there exists $a > 0$ and a solution (p,u) of (1.1), (1.2), (1.3) satisfying

$$u(x,t) = u(x + e_j, t) \tag{2.4}$$

on $\mathbb{R}^3 \times [0, a)$ *for* $1 \le j \le 3$*, and*

$$p, u \in C^\infty(\mathbb{R}^3 \times [0, a)) \tag{2.5}$$

that cannot be smoothly continued to $\mathbb{R}^3 \times [0, \infty)$*.*

Proof. Let us make a small modification to the solution in Lemma 2.1. In Lemma 2.1, $g : \mathbb{R} \to \mathbb{R}$ is a smooth function with $g(0) = g'(0) = 0$, but we select

$$g(t) = \frac{1}{2}ct^2 \frac{1}{a - t}$$

where $c \neq 0$ and $a > 0$.

The initial vector field $u^0(x)$ in Lemma 2.1 is smooth, periodic and divergence-free. The period is scaled to one in (2.1). The $f(x, t)$ is zero and therefore is periodic in space variables with the period as one. Thus, (2.2) holds. The constant $C_{\alpha,m,K}$ is selected after the numbers α, m, K are selected. The force $f(x, t)$ is identically zero, thus (2.3) holds. The solution (2.1) in Lemma 2.1 is periodic in space variables with the period as one. Thus (2.4) holds. The solution $u(x, t)$ in (2.1) is smooth if $t < a$, thus (2.5) holds. The function $g'(t)$ has a singularity at a finite value $t = a$ and $g'(t)$ becomes infinite at $t = a$. From (2.1) it follows that if t approaches a from either side, there is no limit to the the oscillating sine and cosine term in u_1, and the $g'(t)$ additive term approaches infinity. Thus, the solution $u(x, t)$ cannot be continued to the whole $\mathbb{R}^3 \times [0, \infty)$. This completes the proof. $\square$

Theorem 2.4. *There exists a smooth, divergence-free vector field* $u^0(x)$ *on* $\mathbb{R}^3$ *and a smooth* $f(x, t)$ *on* $\mathbb{R}^3 \times [0, \infty)$ *defined as a feedback control function using the values of* $u(x, t)$ *and a number* $C_{\alpha,m,K} > 0$ *satisfying*

$$u^0(x + e_j) = u^0(x), \quad f(x + e_j, t) = f(x, t), \quad 1 \le j \le 3. \tag{2.6}$$

(here e_j *is the unit vector), and*

$$|\partial_x^\alpha \partial_t^m f(x, t)| \le C_{\alpha m K}(1 + |t|)^{-K} \tag{2.7}$$

for any α*,* m *and* K*, such that there exist no solutions* (p, u) *of* (1.1)*,* (1.2)*,* (1.3) *on* $\mathbb{R}^3 \times [0, \infty)$ *satisfying*

$$u(x, t) = u(x + e_j, t) \tag{2.8}$$

on $\mathbb{R}^3 \times [0, \infty)$ *for* $1 \le j \le 3$*, and*

$$p, u \in C^\infty(\mathbb{R}^3 \times [0, \infty)). \tag{2.9}$$

Proof. Let the solution in Theorem 2.3 with the particular g be denoted by U and a be larger than 1. A feedback control force $f(x, t)$ is defined by using the values of the function $u(x, t')$ for $t' \le t$. In practise there is a control delay and $t' < t$ but we allow zero control delay and select $f(x, t)$ as

$$f_i(x, t) = \frac{\partial}{\partial t} u_i(x, t) - \frac{\partial}{\partial t} U_i(x, t).$$

Inserting this force to (1.1) yields a differential equation in space variables

$$\frac{\partial U_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = v\Delta u_i - \frac{\partial p}{\partial x_i} \quad x \in \mathbb{R}^3,\ t \ge 0,\ 1 \le i \le 3.$$

This force is defined in the open interval $t \in [0, a)$ and can be smoothly continued as zero to $[a, \infty)$. There is a solution $u(x, t) = U(x, t)$ to this equation. We notice that if $u(x, t) = U(x, t)$ then the force $f_i(x, t)$ takes zero value at every point. This is correct: if we apply external control force without any control delay, it is possible to keep the solution u exactly at the selected solution U. This is not the same as to say that there is no force. If the solution $u(x,t)$ would be different from $U(x,t)$, then the force would not be zero. Since $u_1(x,t) = U_1(x,t)$ becomes infinite when t approaches a, the solution cannot be continued to the whole space $\mathbb{R}^3 \times [0,\infty)$. As in Theorem 2.3 the conditions (2.6)-(2.8) hold, but (2.9) does not hold.

We must still discuss if the feedback control force can control the equation (1.1) or if there can be several solutions. When the higher order time derivatives of $u(x,t)$ are fixed at $t = 0$ the solution to (1.1) is unique because of the local-in-time existence and uniqueness theorem. This means that if a solution $u(x,t)$ starts as $U(x,t)$ in some small interval $t < \epsilon$ for some small $\epsilon < 1$, then it will continue as $U(x,t)$ for all times $t < a$ if the external force is zero. The question is whether the feedback control force $f(x,t)$ can steer the solution $u(x,t)$ to a possible solution $U(x,t)$. The external force can freely change the higher order time derivatives of $u(x,t)$ in the equation (1.1). Thus, the external control force can set the higher order time derivatives of $u(x,t)$ to those of $U(x,t)$, therefore the answer is that the external control force can control the equation and the solution can be set to $U(x,t)$.

The difference between Theorems 2.3 and 2.4 is that if the force $f(x,t)$ is zero in Theorem 2.3, there is a family of solutions corresponding to different selections of $g(t)$, but if the force $f(x,t)$ is zero in Theorem 2.4, then necessarily the solution $u(x,t)$ equals $U(x,t)$ because otherwise the force is not zero.

Let us mention that we may select a force that does not take the value zero at all points e.g. by adding a control delay that has a zero value at $t = 0$ and when $t > t_1$ for some fixed t_1 satisfying $0 < t_1 < a$ and smoothing the force to C^∞. At some points $t < t_1$ the control delay is selected as nonzero and consequently the force is not zero at all points. This completes the proof. □

Let us continue by partially solving (1.1)-(1.3). Firstly, it is good to eliminate p by integrability conditions as in Lemma 2.5. We introduce new unknowns $h_{i,k}$. The relation of $h_{i,k}$ and u_i is given by Lemma 2.6.

Lemma 2.5. *Let $u(x,t)$ and $p(x,t)$ be $C^\infty\left(\mathbb{R}^3 \times [0,\infty)\right)$ functions satisfying (1.1) and (1.2) with $f_i(x,t)$ being identically zero for $1 \le i \le 3$, and let (i,k,m) be a permutation of $(1,2,3)$. The functions*

$$h_{i,k} = \frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \tag{2.10}$$

satisfy

$$\frac{\partial h_{i,k}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial h_{i,k}}{\partial x_j} - v\Delta h_{i,k} = \frac{\partial u_m}{\partial x_m} h_{i,k} - \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m} \tag{2.11}$$

for all $x \in \mathbb{R}^3$ and $t \ge 0$.

Proof. As $p \in C^\infty\left(\mathbb{R}^3 \times [0,\infty)\right)$,

$$\frac{\partial}{\partial x_i}\frac{\partial p}{\partial x_k} = \frac{\partial}{\partial x_k}\frac{\partial p}{\partial x_i}.$$

Thus, from (1.1) when $f_i(x,t)$ is identically zero for $1 \le i \le 3$, we obtain

$$\begin{aligned} &\frac{\partial p}{\partial\partial t}\frac{\partial u_i}{\partial x_k} + \sum_{j=1}^{3}\Big(u_j\frac{\partial^2 u_i}{\partial x_j\partial x_k} + \frac{\partial u_j}{\partial x_k}\frac{\partial u_i}{\partial x_j}\Big) - v\Delta\frac{\partial u_i}{\partial x_k} \\ &= \frac{\partial p}{\partial\partial t}\frac{\partial u_k}{\partial x_i} + \sum_{j=1}^{3}\Big(u_j\frac{\partial^2 u_k}{\partial x_j\partial x_i} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_j}\Big) - v\Delta\frac{\partial u_k}{\partial x_i}. \end{aligned}$$

This yields

$$\frac{\partial p}{\partial\partial t}h_{i,k} + \sum_{j=1}^{3} u_j\frac{\partial h_{i,k}}{\partial x_j} - v\Delta h_{i,k} = \sum_{j=1}^{3}\frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_j} - \sum_{j=1}^{3}\frac{\partial u_j}{\partial x_k}\frac{\partial u_i}{\partial x_j}. \tag{2.12}$$

The right-hand side of (2.12) can be written in the form

$$\begin{aligned} &\frac{\partial u_i}{\partial x_i}\frac{\partial u_k}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_k} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m} - \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_i} - \frac{\partial u_k}{\partial x_k}\frac{\partial u_i}{\partial x_k} - \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} \\ &= \frac{\partial u_i}{\partial x_i}\Big(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\Big) + \frac{\partial u_k}{\partial x_k}\Big(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\Big) - \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m}. \end{aligned} \tag{2.13}$$

In (2.13) we have replaced the sum $\sum_{j=1}^{3}$ by $\sum_{j\in\{i,k,m\}}$ which is possible since (i,k,m) is a permutation of $(1,2,3)$. Due to (1.2) the right-hand side of (2.13) can be further transformed into

$$\begin{aligned} &-\frac{\partial u_m}{\partial x_m}\Big(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\Big) - \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m} \\ &= \frac{\partial u_m}{\partial x_m}h_{i,k} - \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m}. \end{aligned}$$

The proof is complete. □

Lemma 2.6. *Let $u(x,t)$ and $p(x,t)$ be $C^\infty\big(\mathbb{R}^3\times[0,\infty)\big)$ functions satisfying* (1.1) *and* (1.2) *with $f_i(x,t)$ being identically zero for $1 \le i \le 3$, and let (i,k,m) be a permutation of $(1,2,3)$. The following relations hold for all $x \in \mathbb{R}^3$ and $t \ge 0$:*

$$\frac{\partial h_{i,k}}{\partial x_k} + \frac{\partial h_{i,m}}{\partial x_m} = \Delta u_i$$

where $h_{i,k}$ is defined by (2.10) *and Δu_i is the Laplacian of u_i in the space variables.*

Proof. From (2.10) we have

$$\frac{\partial h_{i,k}}{\partial x_k} + \frac{\partial h_{i,m}}{\partial x_m} = \frac{\partial^2 u_i}{\partial {x_k}^2} - \frac{\partial^2 u_k}{\partial x_i\partial x_k} + \frac{\partial^2 u_i}{\partial {x_m}^2} - \frac{\partial^2 u_m}{\partial x_i\partial x_m}. \tag{2.14}$$

The right-hand side of (2.14) can be rewritten, by (1.2), as

$$\begin{aligned} &\frac{\partial^2 u_i}{\partial {x_k}^2} + \frac{\partial^2 u_i}{\partial {x_m}^2} - \frac{\partial p}{\partial\partial x_i}\Big(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_m}{\partial x_m}\Big) \\ &= \frac{\partial^2 u_i}{\partial {x_k}^2} + \frac{\partial^2 u_i}{\partial {x_m}^2} + \frac{\partial p}{\partial\partial x_i}\frac{\partial u_i}{\partial x_i} = \Delta u_i. \end{aligned}$$

This completes the proof. □

As an example of (2.11) let us find another solution to (1.1)-(1.3).

Lemma 2.7. *Let $b_j, \alpha_j \in \mathbb{R}$ satisfy $\sum_{j=1}^3 \frac{b_j}{\alpha_j} = 0$. Let $f_i(x,t)$ be chosen identically zero for $1 \le i \le 3$. Then*

$$u_i^0 = b_i \sin\Big(\sum_{s=1}^3 \frac{x_s}{\alpha_s}\Big), \quad 1 \le i \le 3,$$

is a smooth, periodic, bounded and divergence-free initial vector field. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function with $g(0) = g'(0) = 0$. The following family of functions u and p satisfy (1.1)-(1.3)*:*

$$\begin{gathered} u_i(x,t) = b_i e^{\beta t} \sin\Big(\sum_{s=1}^3 \frac{x_s}{\alpha_s} + g(t)\Big) - g_0(t), \\ p(x,t) = g_0'(t) \sum_{j=1}^3 x_j, \end{gathered} \tag{2.15}$$

where $g'(t) = g_0(t) \sum_{j=1}^3 \frac{1}{\alpha_j}$ and $\beta = -v \sum_{j=1}^3 \frac{1}{\alpha_j^2}$.

Proof. Let us write $z = \sum_{s=1}^3 \frac{x_s}{\alpha_s} + g(t)$ for brevity. Then

$$\begin{gathered} \frac{\partial u_i}{\partial t} = \beta b_i e^{\beta t} \sin(z) - g_0'(t) + g'(t) b_i e^{\beta t} \cos(z), \\ -v\Delta u_i = v b_i e^{\beta t} \sum_{j=1}^3 \frac{1}{\alpha_j^2} \sin(z), \\ \frac{\partial p}{\partial x_i} = g_0'(t), \end{gathered}$$

$$\begin{aligned} \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} &= b_i e^{2\beta t} \sin(z)\cos(z) \sum_{j=1}^3 \frac{b_j}{\alpha_j} - b_i g_0(t) e^{\beta t} \cos(z) \sum_{j=1}^3 \frac{1}{\alpha_j} \\ &= -b_i g_0(t) e^{\beta t} \cos(z) \sum_{j=1}^3 \frac{1}{\alpha_j} \end{aligned}$$

since $\sum_{j=1}^3 \frac{b_j}{\alpha_j} = 0$. Inserting the parts to (1.1) shows that

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} - v\Delta u_i + \frac{\partial p}{\partial x_i} \\ &= \Big(\beta + v \sum_{j=1}^3 \frac{1}{\alpha_j^2}\Big) b_i e^{\beta t} \sin(z) + \Big(g'(t) - g_0(t) \sum_{j=1}^3 \frac{1}{\alpha_j}\Big) b_i e^{\beta t} \cos(z) = 0 \end{aligned}$$

by the conditions on $g'(t)$ and β in Lemma 2.7. □

The simple reasoning leading to the solutions in Lemmas 2.1 and 2.7 is as follows. Looking at (2.11) it seems that the leading terms of

$$\sum_{j=1}^3 u_j \frac{\partial h_{i,k}}{\partial x_j} - \frac{\partial u_m}{\partial x_m} h_{i,k} + \frac{\partial u_m}{\partial x_k} \frac{\partial u_i}{\partial x_m} - \frac{\partial u_m}{\partial x_i} \frac{\partial u_k}{\partial x_m} = g(x,t) \tag{2.16}$$

should cancel and leave a reminder $g(x,t)$ that can be obtained from the time derivative of $h_{i,k}$. Then there is a first order differential equation

$$\frac{\partial h_{i,k}}{\partial t} - v\Delta h_{i,k} + g(x,t) = 0$$

which suggests that the solution is exponential and in order to get periodic initial values, trigonometric functions were selected.

In Lemma 2.7 we first select $u_i = b_i f\big(\sum_{s=1}^{3} \frac{x_s}{\alpha_s}\big)$ where f is a smooth function to be determined. This choice automatically gives

$$\sum_{j=1}^{3} u_j \frac{\partial h_{i,k}}{\partial x_j} = 0$$

because expanding it shows that it has the multiplicative term $\sum_{s=1}^{3} \frac{b_s}{\alpha_s}$ which is zero by divergence-free condition (1.2). The terms

$$\begin{aligned} &-\frac{\partial u_m}{\partial x_m} h_{i,k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m} \\ &= \Big(-\frac{\partial u_m}{\partial x_m}\frac{\partial u_i}{\partial x_k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} \Big) + \Big(\frac{\partial u_m}{\partial x_m}\frac{\partial u_k}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m} \Big) \end{aligned}$$

also cancel automatically for the chosen function u_i. Another way to cancel the terms is used in Lemma 2.1. In Lemma 2.1 we originally set $u_m = h_{i,k}$ by which

$$u_m \frac{\partial h_{i,k}}{\partial x_m} - \frac{\partial u_m}{\partial x_m} h_{i,k} = 0.$$

The remaining terms in the left side of (2.16) are

$$\begin{aligned} &\frac{\partial u_m}{\partial x_i}\Big(u_i - \frac{\partial u_k}{\partial x_m} \Big), \\ &\frac{\partial u_m}{\partial x_k}\Big(u_k + \frac{\partial u_i}{\partial x_m} \Big). \end{aligned}$$

The divergence-free condition (1.2) assuming $u_m = h_{i,k}$ yields

$$0 = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = \frac{\partial p}{\partial \partial x_i}\Big(u_i - \frac{\partial u_k}{\partial x_m} \Big) + \frac{\partial}{\partial x_k}\Big(u_k + \frac{\partial u_i}{\partial x_m} \Big).$$

The form (2.1) is constructed such that it is divergence-free and the term $g(x,t)$ in (2.16) can be obtained from (2.16). The way to obtain it is adding a function of t to x_i, $1 \le i \le 3$. The basic solution can be further modified by a function $g(t)$ as in (2.1) and (2.15).

Lemmas 2.8 and 2.9, below, generalize Lemma 2.7. Lemma 2.8 shows that any periodic smooth function $u_i^0 = b_j g(\sum_j \frac{x_j}{\alpha_j})$ with $\sum_j \frac{b_j}{\alpha_j} = 0$ can be continued to smooth u for zero external force since it can be expressed by its Fourier series. Lemma 2.9 generalizes the solution to a case where there are two different functions.

Lemma 2.8. *Let* $f_i(x,t) = 0$ *and*

$$u_i^0(x) = b_i \sum_{n=1}^{\infty} \Big(c_n \sin\Big(\sum_{j=1}^{3} \frac{n x_j}{\alpha_j} \Big) + d_n \cos\Big(\sum_{j=1}^{3} \frac{n x_j}{\alpha_j} \Big) \Big)$$

where $\sum_{j=1}^{3} \frac{b_j}{\alpha_j} = 0$. *The following functions solve* (1.1)-(1.3)*:*

$$u_i(x,t) = b_i \sum_{n=1}^{\infty} e^{\beta n^2 t}\Big(c_n \sin\Big(\sum_{j=1}^{3} \frac{nx_j}{\alpha_j}\Big) + d_n \cos\Big(\sum_{j=1}^{3} \frac{nx_j}{\alpha_j}\Big)\Big),$$
$$p(x,t) = 0,$$

where $\beta = -v\sum_{j=1}^{3} \alpha_j^{-2}$.

Proof. Direct calculations show that

$$\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = 0, \quad \frac{\partial u_i}{\partial t} - v\Delta u_i = 0.$$

which completes the proof. □

Lemma 2.9. *Let* $f_i(x,t) = 0$ *and*

$$u_i^0(x) = b_{i,1} \sin\Big(\sum_{s=1}^{3} \frac{x_s}{\alpha_{s,1}}\Big) + b_{i,2} \sin\Big(\sum_{s=1}^{3} \frac{x_s}{\alpha_{s,2}}\Big)$$

where $\sum_j \frac{b_{j,n}}{\alpha_{j,n}} = 0$ *for* $n = 1,2$ *and*

$$b_{i,1} \sum_{j=1}^{3} \frac{b_{j,2}}{\alpha_{j,1}} = \frac{1}{\alpha_{i,2}}, \quad b_{i,2} \sum_{j=1}^{3} \frac{b_{j,1}}{\alpha_{j,2}} = \frac{1}{\alpha_{i,1}}.$$

Let $\beta_n = -v\sum_j \alpha_{j,n}^{-2}$ *for* $n = 1,2$. *The following two functions solve* (1.1)-(1.3)*:*

$$u_i(x,t) = b_{i,1} e^{\beta_1 t} \sin\Big(\sum_{s=1}^{3} \frac{x_s}{\alpha_{s,1}}\Big) + b_{i,2} e^{\beta_1 t} \sin\Big(\sum_{s=1}^{3} \frac{x_s}{\alpha_{s,2}}\Big),$$
$$p(x,t) = \cos\Big(\sum_{j=1}^{3} \frac{x_j}{\alpha_{j,1}}\Big) \cos\Big(\sum_{j=1}^{3} \frac{x_j}{\alpha_{j,2}}\Big) e^{(\beta_1+\beta_2)t}.$$

Proof. Computing $\sum_j u_j \frac{\partial u_i}{\partial x_j}$ shows that the term equals $-\frac{\partial p(x,t)}{\partial x_i}$. We mention that the conditions in the lemma imply $\sum_j \frac{1}{\alpha_{j,1}\alpha_{j,2}} = 0$, $\sum_j \frac{1}{\alpha_{j,1}^2} = \sum_j \frac{1}{\alpha_{j,2}^2}$. □

3. Approaches to general initial values

Let us first notice that the transform by the function $g(t)$ in Lemma 2.1 and Lemma 2.7 is not a coordinate transform of (x,t) to (x',t') where $x'_j = x_j + g(t)$, $j = 1,2,3$, and $t' = t$. The equation (1.1) is not invariant in this transform and if

$$\frac{\partial u_i(x,t)}{\partial t} + \sum_{j=1}^{3} u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j} - v\Delta u_i(x,t) + \frac{\partial p(x,t)}{\partial x_i} - f_i(x,t) = 0$$

then

$$\begin{aligned} &\frac{\partial u_i(x',t')}{\partial t'} + \sum_{j=1}^{3} u_j(x',t') \frac{\partial u_i(x',t')}{\partial x'_j} - v\Delta' u_i(x',t') + \frac{\partial p(x',t')}{\partial x'_i} \\ &- f_i(x',t') + g'(t) \sum_{j=1}^{3} \frac{\partial u_i(x',t')}{\partial x'_i} = 0. \end{aligned}$$

The transform that is used in Lemma 2.1 and Lemma 2.7,

$$u(x,t) \to u(x',t') - g'(t),$$
$$p(x,t) \to p(x',t') + g''(t)\sum_{j=1}^{3} x_j$$

keeps the initial values $u_i^0(x)$ and $f(x,t)$ fixed if $g(0) = g'(0) = 0$. It is a transform that can be done to any solution of (1.1)-(1.3) but it works only for equation (1.1). It is certainly not a generally valid coordinate transform. Such should work with any equation. It is not valid to think of the transform as

$$u(x,t) \to u(x',t') - g'(t),$$
$$p(x,t) \to p(x',t'),$$
$$f_i(x,t) \to f_i(x',t') - g''(t)$$

because this changes the previously selected force $f_i(x,t)$. In fact, what is done in Lemma 2.1 is not a change of the coordinate system. The force is kept at the selected value at zero. The coordinates are kept at (x,t) as they originally are. The pressure is eliminated from (1.1) and (1.2) as in (2.11). The equation (2.11) has several solutions for $u(x,t)$ and we find a family of solutions for the initial values of Lemma 2.1 and some solutions cannot be smoothly continued to the whole space-time. In Theorem 2.4 we notice that it is possible to select a force that picks up any of these solutions.

The equations (1.1)-(1.3) can be solved in a Taylor series form, though summing the Taylor series can be difficult. We write

$$u_j = \sum_{n=0}^{\infty} \psi_{n,j}(x)t^n, \quad p = \sum_{n=0}^{\infty} p_n(x)t^n, \quad f_j = \sum_{n=0}^{\infty} f_{n,j}(x)t^n.$$

Equation (1.1) yields

$$(n+1)\psi_{n+1,i} + \sum_{m=0}^{n} \psi_{m,j}\frac{\partial \psi_{n-m,i}}{\partial x_j} - v\Delta\psi_{n,i} + p_n - f_{n,i} = 0.$$

These equations can be solved recursively by dividing

$$v_{n,i} = \sum_{m=0}^{n} \psi_{m,j}\frac{\partial \psi_{n-m,i}}{\partial x_j} - f_{n,i}$$

into two parts

$$v_{n,i} = v_{n,i,1} + v_{n,i,2},$$

where $v_{n,i,1}$ is divergency-free and $v_{n,i,2}$ has no turbulence; i.e., it can be obtained from some function $g(x,t)$ as

$$v_{n,i,2} = \frac{\partial g}{\partial x_i}.$$

Thus, what needs to be solved is a system

$$(n+1)\psi_{n+1,i} - v\Delta\psi_{n,i} + v_{n,i,1} = 0,$$
$$v_{n,i,2} = \frac{\partial p_n}{\partial x_i}.$$

We can see the non-uniqueness of the solution. The division of $v_{n,i}$ into the two parts is not unique: if $\Delta g = 0$, then $\frac{\partial g}{\partial x_i}$ can be inserted to either $v_{n,i,1}$ or to $v_{n,i,2}$. We selected a linear symmetric g in Lemma 2.1 in order to have a nice periodic u_i.

The following approach is another way of finding $u(x,t)$ in (1.1) for a general initial vector field $u^0(x,t)$. In some cases the method may yield closed form results easier than the Taylor series method. If (1.1), (1.2) and (1.3) are satisfied, p can be derived by integration. Let us assume $u(x,t)$ exists. We separate a multiplicative part $Y(t)$ such that $u(x,t) = Y(t)X(x,t)$ where the scaling is $Y(0) = 1$. If there is no nontrivial multiplicative factor $Y(t)$ then let us set $Y(t) \equiv 1$. If $(u^0)^{-1}$ exists (e.g. locally for a local solution), and $g(x,t) = (u^0)^{-1}(X(x,t))$ is smooth then $g(x,t)$ can be expanded as a power series $\sum_{s=0}^{\infty} t^s g_s(x)$ of t and the series converges in some small neighborhood of the origin. Since $u(x,0) = u^0(x)$ we have $g_0(x) = x$. Let us write $x'(x,t) = (x'_i(x,t))_{1\le i\le 3}$ where

$$x'_i(x,t) = x_i + \sum_{s=1}^{\infty} t^s g_{s,i}(x).$$

Thus $u_i(x,t) = Y(t)u_i^0(x')$. Let

$$h_{i,k}^0 = \frac{\partial u_i^0}{\partial x_k} - \frac{\partial u_k^0}{\partial x_i}$$

and

$$\begin{aligned} f_{0,i,k}(x,t) &= \frac{\partial h_{i,k}^0}{\partial t} + Y(t)\sum_{j=1}^{3} u_j^0 \frac{\partial h_{i,k}^0}{\partial x_j} - v\Delta h_{i,k}^0 \\ &\quad - Y(t)\Big(\frac{\partial u_m^0}{\partial x_m} h_{i,k}^0 - \frac{\partial u_m^0}{\partial x_k}\frac{\partial u_i^0}{\partial x_m} + \frac{\partial u_m^0}{\partial x_i}\frac{\partial u_k^0}{\partial x_m}\Big). \end{aligned}$$

Clearly

$$\begin{aligned} \frac{\partial u_i(x,t)}{\partial x_j} &= Y(t)\sum_{r=1}^{3} \frac{\partial x'_r(x,t)}{\partial x_j}\frac{\partial u_i^0(x')}{\partial x'_r} \\ &= Y(t)\frac{\partial u_i^0(x')}{\partial x'_j} + Y(t)\sum_{s=1}^{\infty} t^s \sum_{r=1}^{3} \frac{\partial g_{s,r}(x)}{\partial x_j}\frac{\partial u_i^0(x')}{\partial x'_r} \end{aligned}$$

and thus

$$h_{i,k}(x) = Y(t)h_{i,k}^0(x') + Y(t)\sum_{s=1}^{\infty} t^s \sum_{r=1}^{3}\Big(\frac{\partial g_{s,r}(x)}{\partial x_k}\frac{\partial u_i^0(x')}{\partial x'_r} - \frac{\partial g_{s,r}(x)}{\partial x_i}\frac{\partial u_k^0(x')}{\partial x'_r}\Big).$$

The interesting term is (here $Y' = dY/dt$)

$$\begin{aligned} \frac{\partial h_{i,k}(x,t)}{\partial t} &= Y(t)\frac{\partial h_{i,k}^0(x)}{\partial t} + Y'(t)h_{i,k}^0(x') \\ &\quad + Y'(t)\sum_{s=1}^{\infty} t^s \sum_{r=1}^{3}\Big(\frac{\partial g_{s,r}(x)}{\partial x_k}\frac{\partial u_i^0(x')}{\partial x'_r} - \frac{\partial g_{s,r}(x)}{\partial x_i}\frac{\partial u_k^0(x')}{\partial x'_r}\Big) \\ &\quad + Y(t)\sum_{s=1}^{\infty} st^{s-1} \sum_{r=1}^{3}\Big(\frac{\partial g_{s,r}(x)}{\partial x_k}\frac{\partial u_i^0(x')}{\partial x'_r} - \frac{\partial g_{s,r}(x)}{\partial x_i}\frac{\partial u_k^0(x')}{\partial x'_r}\Big) \end{aligned}$$

because it lowers powers of t and allows recursion. Computing the terms in (2.11) shows that for some functions $Q_{s,i,k}(x,t)$, it holds

$$\begin{aligned} 0 &= \frac{\partial h_{i,k}}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial h_{i,k}}{\partial x_j} - v\Delta h_{i,k} - \frac{\partial u_m}{\partial x_m} h_{i,k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_i}{\partial x_m} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_k}{\partial x_m} \\ &= Y(t)\sum_{s=1}^{\infty} s t^{s-1} \sum_{r=1}^{3} \Big(\frac{\partial g_{s,r}(x)}{\partial x_k}\frac{\partial u_i^0(x')}{\partial x'_r} - \frac{\partial g_{s,r}(x)}{\partial x_i}\frac{\partial u_k^0(x')}{\partial x'_r} \Big) + \sum_{s=0}^{\infty} t^s Q_{s,i,k}(x,t) \end{aligned}$$

where $Q_{0,i,k}(x,t) = Y'(t) h^0_{i,k}(x') + Y(t) f^0_{0,i,k}(x')$. Comparing the coefficients of the powers of t individually, and then inserting $t = 0$ to the equation of each coefficient yields, for $s \ge 1$,

$$\sum_{r=1}^{3} \Big(\frac{\partial g_{s,r}(x)}{\partial x_k}\frac{\partial u_i^0(x)}{\partial x_r} - \frac{\partial g_{s,r}(x)}{\partial x_i}\frac{\partial u_k^0(x)}{\partial x_r} \Big) = -\frac{1}{s} Q_{s-1,i,k}(x,0). \tag{3.1}$$

From (1.2) we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \\ &= Y(t)\sum_{i=1}^{3}\sum_{r=1}^{3} \frac{\partial x'_r(x,t)}{\partial x_i}\frac{\partial u_i^0(x')}{\partial x'_r} \\ &= Y(t)\sum_{i=1}^{3} \frac{\partial u_i^0(x')}{\partial x'_i} + Y(t)\sum_{s=1}^{\infty} t^s \sum_{i=1}^{3}\sum_{r=1}^{3} \frac{\partial g_{s,r}(x)}{\partial x_i}\frac{\partial u_i^0(x')}{\partial x'_r} \end{aligned}$$

The first term in the right-hand side vanishes because u^0 is divergence-free. Again, comparing the coefficients of the powers of t individually, and then inserting $t = 0$ to the equation of each coefficient yields, for $s \ge 1$,

$$\sum_{i=1}^{3}\sum_{r=1}^{3} \frac{\partial g_{s,r}(x)}{\partial x_i}\frac{\partial u_i^0(x)}{\partial x_r} = 0. \tag{3.2}$$

It seems that solving (3.1) and (3.2) for $s = 1$ gives $g_1(x)$, then we can derive for $s = 2$ and obtain $g_2(x)$ and so on, and that function $Q_{s,i,k}$ contains only terms $g_{s',j}$, $s' \le s$. However, (3.1) and (3.2) do not necessarily determine even g_1 and this approach must be modified. This may be a direction for research how to obtain linear systems, like (3.1) and (3.2), from which to continue, but we will not study this method more in this short paper. Notice that this approach cannot show that a solution $u(x,t)$ exists.

4. Conclusions

Theorem 2.2 proves that the solutions to the 3-dimensional Navier-Stokes equations for incompressible fluid are not always unique for the initial values and for the periodic solutions discussed in Statement D of [1]. This is not surprising, as the proof of uniqueness requires periodicity of $p(x,t)$. Periodicity is not required in Theorem 2.2 or in [1]. Another (different) counterexample to uniqueness of (1.1) is given in [3].

Some changes are needed to the official problem setting [1] of the Clay Mathematics Institute's Navier-Stokes problem.

In [1] it is stated that we know for a long time that the initial data $u^0(x)$ can be continued uniquely to some finite time. This statement needs a clarification. In [2] there is one proof of uniqueness. Temam implicitly assumes that $p(x,t)$ is periodic, which [1] does not do. All other easy ways to solve the problem have been explicitly excluded. It does not seem to be intended that the pressure $p(x,t)$ should be implicitly assumed as periodic.

If periodicity is not demanded for pressure there are many solutions for zero external force, and we can construct a solution that cannot be continued to the whole space, as is done in Theorem 2.3. If feedback forces are not excluded, we can select the solution from Theorem 2.3 by using a feedback force, as in Theorem 2.4. Unless Theorem 2.4 is accepted as a proof of Statement D in [1], the official problem statement for the millennium problem must be corrected.

If unique solutions are required then conditions must be imposed on the pressure. In the space-periodic case the condition is periodicity of $p(x,t)$, while in the non-periodic case growth conditions must be set for $p(x,t)$. It is not enough to make these changes only, because if feedback forces are allowed and we require the solution to satisfy conditions on $p(x,t)$ then a feedback force can select a solution, which does not satisfy these conditions. Such a solution is then a counterexample since it is a solution and violates the required conditions.

The problem statement [1] has an expression *a given, externally applied force (e.g. gravitation).* This is not clear enough to rule out feedback forces since a feedback control force is an external force to the controlled system. If the intention has been to say that the external force must be similar to gravitation then a clarification is needed. Gravitation is time-independent while $f(x,t)$ can be time-dependent, thus the similarity to gravitation is not full and the extent of similarity needs to be stated. The word *given* does not clarify the issue since this word has many meanings. It could be understood as a weak indication that the external force should be selected as a point function and any solution (u,p) is looked for. However, the most common meaning of *given* in this mathematical context is *chosen*. We have to take this meaning to *given* since otherwise there is a contradiction in [1]: the official problem setting claims that the solutions can be uniquely continued to some finite time. The only way to understand the uniqueness claim to be correct without a requirement of periodicity of $p(x,t)$ is to use a feedback force in the case of Lemma 2.1. Thus, some expressions in the official problem setting are unclear and understanding *given* as *chosen* i.e., not excluding feedback forces, seems to be the only one where [1] does not make incorrect statements. The easiest approach for improving clarity is to exclude feedback forces. However, while correcting [1] one should carefully investigate if there are other unnoticed transforms or other problems. It is not a small straightforward correction.

Let us mention that in a weak formulation of (1.1)-(1.3) it is essential that the test function has compact support in $\mathbb{R}^3 \times \mathbb{R}$ because the presented solutions (u,p) with non-periodic pressure cannot be integrated over the whole $\mathbb{R}^3 \times [0,\infty)$.

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