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EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR A p-LAPLACE EQUATION WITH POTENTIALS IN $\mathbb{R}^N$

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Abstract. We study the p-Laplace equation with Potentials
$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda V(x)|u|^{p-2}u = |u|^{q-2}u,$$
$u \in W^{1,p}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ where $2 \le p$, $p < q < p^*$. Using a concentration-compactness principle from critical point theory, we obtain existence, multiplicity solutions, and concentration of solutions.

1. Introduction

This article concerns the existence and the multiplicity of decaying solutions for the equation
$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N \tag{1.1}$$
and for the related equations
$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda V(x)|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.2}$$
and
$$-\varepsilon^p \operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N \tag{1.3}$$
respectively as $\lambda \to \infty$, and $\varepsilon \to 0$. We also consider concentration of solutions as $\lambda \to \infty$ or $\varepsilon \to 0$.

We assume throughout that V and p, q satisfy the following conditions:

- (V1) $V \in C(\mathbb{R}^N)$ and V is bounded.
- (V2) There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure.
- (P1) $2 \le p$, $p < q < p^*$ where $p^* = \frac{pN}{N-p}$ if $N > p$ and $p^* = \infty$ if $1 \le N \le p$.

Note that if $\varepsilon^p = \lambda^{-1}$, then u is a solution of (1.2) if and only if $v = \lambda^{\frac{-1}{q-p}}u$ is a solution of (1.3), hence as far as the existence and the number of solutions are concerned, these two problems are equivalent.

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$\|u\|_p$ will denote the usual $L^p(\mathbb{R}^N)$ norm and $V^{\pm}(x) = \max\{\pm V(x), 0\}$. B_ρ and S_ρ will respectively denote the open ball and the sphere of radius ρ and center at the origin.

It is well known that the functional

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)dx - \frac{1}{q}\int_{\mathbb{R}^N}|u|^q dx$$

is of class C^1 in the Sobolev space

$$E = \{u \in W^{1,p}(\mathbb{R}^N) : \|u\|^p = \int_{\mathbb{R}^N}(|\nabla u|^p + V^+(x)|u|^p)dx < \infty\} \tag{1.4}$$

and critical points of Φ correspond to solutions u of (1.1). Moreover, $u(x) \to 0$ as $|x| \to \infty$. It is easy to see that if

$$M = \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)dx}{\|u\|_q^p} \tag{1.5}$$

is attained at some $\overline{u}$ and M is positive, then $u = M^{\frac{1}{q-p}}\overline{u}/\|\overline{u}\|_q$ is a solution of (1.1) and $u(x) \to 0$ as $|x| \to \infty$. Such u is called a ground state. Because we have Poincaré inequality

$$\int_\Omega |u|^p dx \le C\int_\Omega |\nabla u|^p dx, \quad 1 \le p < +\infty, \quad u \in W_0^{1,p}(\Omega)$$

so E is continuously embedded in $W^{1,p}(\mathbb{R}^N)$.

Recently, there have been numerous works for the eigenvalue problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = V(x)|u|^{p-2}u \\ u \in D_0^{1,p}(\Omega), \quad u \ne 0 \end{gathered} \tag{1.6}$$

where $\Omega \subseteq \mathbb{R}^N$. We can see [3, 16, 24, 25] for different approaches. Szulkin and Willem [25] generalized several earlier results concerning the existence of an infinite sequence of eigenvalues.

Consider the quasilinear elliptic equation

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = f(x,u), \quad \text{in } \Omega \\ u \in W_0^{1,p}(\Omega), \quad u \ne 0 \end{gathered} \tag{1.7}$$

where $1 < p < N$, $N \ge 3$, λ is a parameter, Ω is an unbounded domain in $\mathbb{R}^N$. Existence of solutions to (1.7) has been investigation in the previoius decade, see for example [12, 15, 21, 22, 27, 28]. Because of the unboundedness of the domain, the Sobolev compact embedding do not hold. There are some methods to overcome this difficulty. In [28], the authors used the concentration-compactness principle posed by Lions and the mountain pass lemma to solve problem (1.3). In [27], the author use that the projection $u \mapsto f(x,u)$ is weak continuous in $W_0^{1,p}(\Omega)$ to consider the problem. In [8, 9], the authors study the problem in symmetric Sobolev spaces which possess Sobolev compact embedding. By the result and a min-max procedure formulated by Bahri and Li [5], they considered the existence of positive solutions of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^{p-1} = q(x)u^\alpha \quad \text{in} \quad \mathbb{R}^N,$$

where $q(x)$ satisfies certain conditions.

When $p = 2$, problem (1.1) has been studied in [1, 4, 6, 7, 14, 17, 17, 19]. In [20], a quasilinear problem in bounded domains was considered with Hardy type potentials. To the best of our knowledge, there is very little work on the case $p \neq 2$ for problem (1.1).

From its first appearance in the work by Lions [21, 22], the concentration-compactness principle in calculus of variations has been widely used and by many authors. In fact, one should refer to the two concentration-compactness principles, as "escape to infinity" and "concentration around points" as treated separately, originally. This seemingly harmless dichotomy however often leads to rather cumbersome and tricky calculations. To get rid of these difficulties, some authors have developed variants that encompass both possible loss of compactness in a whole; see for instance Ben-Naoum et al. [10] and Bianchi et al. [11] which seem to be the first works in this direction. When using the original principle or its variants, it is necessary beforehand to discover the so-called limiting problems that are responsible for non-compactness. Often, these are related to the invariance of the considered functional and constraint under a non-compact group; translations and dilations being the two most studied.

Motivated by the results in [6, 11, 13, 14, 17, 18, 19, 20, 22, 26, 27], we obtain the existence and the multiplicity of solutions in Theorems 3.1–3.3 by using critical point theory. By Theorems 3.4 and 3.5, we can obtain the concentration of solutions.

This paper is organized as follows. In Section 2, we state some condition and many lemmas which we need in the proof of the main Theorem. In Section 3, we give the proof of the main result of the paper.

2. Preliminaries

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset. $(u_n) \subseteq W_0^{1,p}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $p \geq 2$. Then*

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p dx \geq \lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^p dx + \int_\Omega |\nabla u|^p dx.$$

Proof. When $p = 2$ from Lieb Lemma we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx = \lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^2 dx + \int_\Omega |\nabla u|^2 dx.$$

For $3 \geq p > 2$, using the lower semi-continuity of the L^p-norm with respect to the weak convergence and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, we deduce

$$\lim_{n\to\infty} \langle |\nabla u_n|^{p-2} \nabla u_n, \nabla u_n\rangle \geq \langle |\nabla u|^{p-2} \nabla u, \nabla u\rangle$$

and

$$\begin{aligned} &\lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} (\nabla u_n - \nabla u), \nabla u_n - \nabla u\rangle \\ &= 0 \geq \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} (\nabla u - \nabla u), \nabla u - \nabla u\rangle. \end{aligned}$$

So

$$\begin{aligned} \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u_n, \nabla u_n\rangle &\geq \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u_n, \nabla u\rangle \\ &= \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u, \nabla u_n\rangle \\ &= \lim_{n\to\infty} \langle |\nabla u_n - \nabla u|^{p-2} \nabla u, \nabla u\rangle. \end{aligned}$$

Then

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^p-|\nabla u|^p)dx\\
&=\lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p-2}(|\nabla u_n|^2-|\nabla u|^2)dx+\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^{p-2}-|\nabla u|^{p-2})|\nabla u|^2dx\\
&=\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^{p-2}+|\nabla u|^{p-2})(|\nabla u_n|^2-|\nabla u|^2)dx\\
&\quad+\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^{p-2}|\nabla u|^2-|\nabla u|^{p-2}|\nabla u_n|^2)dx.
\end{aligned}$$

From $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$,

$$\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^{p-2}|\nabla u|^2-|\nabla u|^{p-2}|\nabla u_n|^2)dx=0.$$

So

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^p-|\nabla u|^p)dx&=\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^{p-2}+|\nabla u|^{p-2})(|\nabla u_n|^2-|\nabla u|^2)dx\\
&\ge\lim_{n\to\infty}\int_\Omega|\nabla u_n-\nabla u|^{p-2}(|\nabla u_n|^2-|\nabla u|^2).
\end{aligned}$$

So we have

$$\begin{aligned}
&\lim_{n\to\infty}\langle|\nabla u_n|^{p-2}\nabla u_n,\nabla u_n\rangle+\langle|\nabla u_n-\nabla u|^{p-2}\nabla u,\nabla u_n\rangle+\langle|\nabla u_n-\nabla u|^{p-2}\nabla u_n,\nabla u\rangle\\
&\ge\lim_{n\to\infty}\langle|\nabla u_n-\nabla u|^{p-2}\nabla u_n,\nabla u_n\rangle\\
&\quad+\lim_{n\to\infty}\langle|\nabla u_n-\nabla u|^{p-2}\nabla u,\nabla u\rangle+\langle|\nabla u|^{p-2}\nabla u,\nabla u\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
&\lim_{n\to\infty}\langle|\nabla u_n|^{p-2}\nabla u_n,\nabla u_n\rangle\\
&\ge\lim_{n\to\infty}\langle|\nabla u_n-\nabla u|^{p-2}\nabla u_n-\nabla u,\nabla u_n-\nabla u\rangle+\langle|\nabla u|^{p-2}\nabla u,\nabla u\rangle.
\end{aligned}$$

and

$$\lim_{n\to\infty}\int_\Omega|\nabla u_n|^pdx\ge\lim_{n\to\infty}\int_\Omega|\nabla u_n-\nabla u|^pdx+\int_\Omega|\nabla u|^pdx.$$

For $p>3$, there exist a $k\in N$ that $0<p-k\le 1$. Then, we only need to prove the inequality

$$\lim_{n\to\infty}\int_\Omega(|\nabla u_n|^p-|\nabla u|^p)dx\ge\lim_{n\to\infty}\int_\Omega|\nabla u_n-\nabla u|^{p-k}(|\nabla u_n|^k-|\nabla u|^k).$$

The proof of this iequality is similar to the above, so we omit it. Therefore, the lemma is proved. □

Let $V_b(x)=\max\{V(x),b\}$ and

$$M_b=\inf_{u\in E\setminus\{0\}}\frac{\int_{\mathbb{R}^N}(|\nabla u|^p+V_b(x)|u|^p)dx}{\|u\|_q^p}. \tag{2.1}$$

Denote the spectrum of $-\Delta_p+V$ in $L^p(\mathbb{R}^N)$ by $\sigma(-\Delta_p+V)$ and recall the definition (1.5) of M.

Lemma 2.2. *Suppose* (V1), (V2), (P1) *are satisfied and* $\sigma(-\Delta_p + V) \subset (0, \infty)$. *If* $M < M_b$, *then each minimizing sequence for* M *has a convergent subsequence. So in particular,* M *is attained at some* $u \in E \setminus \{0\}$.

Proof. Let $\{u_m\}$ be a minimizing sequence. We may assume $\|u_m\|_q = 1$. Since $V < 0$ on a set of finite measure, $\{u_m\}$ is bounded in the norm of E given by (1.4). Passing to a subsequence we may assume $u_m \rightharpoonup u$ in E and by the continuity of the embedding $E \hookrightarrow W^{1,p}(\mathbb{R}^N)$, $u_m \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$, $L^q_{\rm loc}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Let $u_m = v_m + u$. Then by Lemma 2.1, we have

$$\begin{aligned} &\lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla u_m|^p + V(x)|u_m|^p)dx \\ &\geq \lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla v_m|^p + V(x)|v_m|^p)dx + \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx, \end{aligned} \tag{2.2}$$

by the Lieb Lemma,

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} |u_m|^p dx = \lim_{m\to\infty} \int_{\mathbb{R}^N} |v_m|^p dx + \int_{\mathbb{R}^N} |u|^p dx. \tag{2.3}$$

Moreover, by (V2) and since $v_m \rightharpoonup 0$ as $m \to \infty$,

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} (V(x) - V_b(x))|v_m|^p dx \to 0. \tag{2.4}$$

Using (2.2)-(2.4) and the definitions of M, M_b, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx + \lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla v_m|^p + V(x)|v_m|^p)dx \\ &\leq M \lim_{m\to\infty} \|u_m\|_q^p \\ &= M \lim_{m\to\infty} (\|u\|_q^q + \|v_m\|_q^q)^{\frac{p}{q}} \\ &\leq M \lim_{m\to\infty} (\|u\|_q^p + \|v_m\|_q^p) \\ &\leq \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx + MM_b^{-1} \lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla v_m|^p + V_b(x)|v_m|^p)dx \\ &\leq \int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx + MM_b^{-1} \lim_{m\to\infty} \int_{\mathbb{R}^N} (|\nabla v_m|^p + V(x)|v_m|^p)dx. \end{aligned}$$

Since $MM_b^{-1} < 1$ and $\int_{\mathbb{R}^N} V^{-1}(x)|v_m|^p dx \to 0$ as $m \to \infty$, it follows that $v_m \to 0$ and therefore $u_m \to u$ as $m \to \infty$. It is clear that $u \neq 0$. $\square$

From the above lemma it follows that if $\sigma(-\Delta_p + V) \subset (0, \infty)$ and $M < M_b$, then there exists a ground state solution of (1.1). Recall that $\{u_m\}$ is called a Palais-Smale sequence at the level c (a $(PS)_c$-sequence) if $\Phi'(u_m) \to 0$ and $\Phi(u_m) \to c$. If each $(PS)_c$-sequence has a convergent subsequence, then Φ is said to satisfy the $(PS)_c$-condition.

Lemma 2.3. *If* (V1), (V2), (P1) *hold, then* Φ *satisfies* $(PS)_c$ *for all*

$$c < (\frac{1}{p} - \frac{1}{q}) M_b^{\frac{q}{(q-p)}}.$$

Proof. Let $\{u_m\}$ be a $(PS)_c$-sequence with c satisfying the inequality above. First we show that $\{u_m\}$ is bounded. We have

$$2c + d\|u_m\| \geq \Phi(u_m) - \frac{1}{p}\langle \Phi'(u_m), u_m \rangle = (\frac{1}{p} - \frac{1}{q})\|u_m\|_q^q \tag{2.5}$$

and

$$\begin{aligned} 2c + d\|u_m\| &\geq \Phi(u_m) - \frac{1}{q}\langle \Phi'(u_m), u_m\rangle \\ &= (\frac{1}{p} - \frac{1}{q})\|u_m\|^p - (\frac{1}{p} - \frac{1}{q})\int_{\mathbb{R}^N} V^-(x)|u_m|^p dx \end{aligned} \tag{2.6}$$

for some constants $d > 0$. Suppose $\|u_m\| \to \infty$ as $m \to \infty$ and let $w_m = u_m/\|u_m\|$. Dividing (2.5) by $\|u_m\|^q$ we see that $w_m \to 0$ in $L^q(\mathbb{R}^N)$ as $m \to \infty$ and therefore $w_m \rightharpoonup 0$ in E as $m \to \infty$ after passing to a subsequence. Hence $\int_{\mathbb{R}^N} V^-(x)|w_m|^p dx \to 0$ as $m \to \infty$. So dividing (2.6) by $\|u_m\|^p$, it follows that $w_m \to 0$ in E as $m \to \infty$, a contradiction. Thus $\{u_m\}$ is bounded.

As in the preceding proof, we may assume $u_m \rightharpoonup u$ in E and $u_m \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$. Set $u_m = v_m + u$. Since $\Phi'(u) = 0$ and

$$\Phi(u) = \Phi(u) - \frac{1}{p}\langle \Phi'(u), u\rangle = (\frac{1}{p} - \frac{1}{q})\|u\|_q^q \geq 0,$$

it follows from (2.2), (2.3) that

$$\lim_{m\to\infty}(|\|v_m\|^p - \|v_m\|_q^q|) \leq \lim_{m\to\infty}(|\|u_m\|^p - \|u_m\|_q^q| + |\|u\|^p - \|u\|_q^q|) = 0$$

so

$$\lim_{m\to\infty}(\|v_m\|^p - \|v_m\|_q^q) = 0 \tag{2.7}$$

and

$$c = \lim_{m\to\infty}\Phi(u_m) \geq \lim_{m\to\infty}(\Phi(v_m) + \Phi(u)) \geq \lim_{m\to\infty}\Phi(v_m). \tag{2.8}$$

By (2.7), we have

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^p + V(x)|v_m|^p)dx = \lim_{m\to\infty}\int_{\mathbb{R}^N}|v_m|^q dx = \gamma \tag{2.9}$$

possibly after passing to a subsequence, and therefore it follows from (2.8) that

$$c \geq (\frac{1}{p} - \frac{1}{q})\gamma. \tag{2.10}$$

By (2.4),

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^p + V_b(x)|v_m|^p)dx = \lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^p + V(x)|v_m|^p)dx = \gamma\,.$$

On the other hand,

$$\|v_m\|_q^p \leq M_b^- \lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^p + V_b(x)|v_m|^p)dx;$$

therefore, $\gamma^{\frac{p}{q}} \leq M_b^-\gamma$. Combining this with (2.10), we see that either $\gamma = 0$, or

$$c \geq (\frac{1}{p} - \frac{1}{q})M_b^{\frac{q}{(q-p)}}$$

hence γ must be 0 by the assumption on c. So according to (2.9), we have

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^p + V^+(x)|v_m|^p)dx = \lim_{m\to\infty}\int_{\mathbb{R}^N}(|\nabla v_m|^p + V(x)|v_m|^p)dx = 0.$$

Therefore, $v_m \to 0$ and $u_m \to u$ in E as $m \to \infty$. □

Next we recall a usual critical point theory which will be used in the below Theorem. Here $\gamma(A)$ is the Krasnoselskii genus of A.

Theorem 2.4. *Suppose $E \in C^1(M)$ is an even functional on a complete symmetric $C^{1,1}$-manifold $M \subset V \setminus \{0\}$ in some Banach space V. Also suppose E satisfies (PS) and is bounded below on M. Let $\widetilde{\gamma}(M) = \sup\{\gamma(K); K \subset M$ and symmetric$\}$. Then the functional E possesses at least $\widetilde{\gamma}(M) \le \infty$ pairs of critical points.*

3. Proof of Main Theorems

Theorem 3.1. *Suppose Assumptions* (V1), (P1) *are satisfied, $\sigma(-\Delta_p + V) \subset (0,\infty)$, $\sup_{x\in\mathbb{R}^N} V(x) = b > 0$ and the measure of the set $\{x \in \mathbb{R}^N : V(x) < b - \varepsilon\}$ is finite for all $\varepsilon > 0$. Then the infimum in* (1.5) *is attained at some $u \ge 0$. If $V \ge 0$, then $u > 0$ in $\mathbb{R}^N$.*

Proof. Since V^+ is bounded, $E = W^{1,p}(\mathbb{R}^N)$ here. Let u_b be the radially symmetric positive solution of the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + b|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N.$$

It is well known that such u_b exists, is unique and minimizes

$$N_b = \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^p + b|u|^p)dx}{\|u\|_q^p} \tag{3.1}$$

(see [12]). So if $V \equiv b$, the proof is complete. Otherwise we may assume without loss of generality that $V(0) < b$. Then

$$\begin{aligned} M &= \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)dx}{\|u\|_q^p} \\ &\le \frac{\int_{\mathbb{R}^N}(|\nabla u_b|^p + V(x)|u_b|^p)dx}{\|u_b\|_q^p} \\ &< \frac{\int_{\mathbb{R}^N}(|\nabla u_b|^p + b|u_b|^p)dx}{\|u_b\|_q^p} \\ &= N_b = M_b, \end{aligned}$$

where the last equality follows from the fact that $V_b = b$. To apply Lemma 2.2 we need to show that $M < M_{b-\varepsilon}$ for some $\varepsilon > 0$. A simple computation shows that if $\lambda > 0$, then $N_{\lambda b}$ is attained at

$$u_{\lambda b}(x) = \lambda^{\frac{1}{(q-p)}} u_b(\lambda^{\frac{1}{p}} x) \quad \text{and} \quad N_{\lambda b} = \lambda^r N_b,$$

where $r = 1 - \frac{N}{p} + \frac{N}{q}$.

Choosing $\lambda = (b-\varepsilon)/b$ we see that $N_{b-\varepsilon} < N_b$ and $N_{b-\varepsilon} \to N_b$ as $\varepsilon \to 0$. So for ε small enough we have

$$\begin{aligned} M < N_{b-\varepsilon} &= \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^p + (b-\varepsilon)|u|^p)dx}{\|u\|_q^p} \\ &\le \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^p + V_{b-\varepsilon}(x)|u|^p)dx}{\|u\|_q^p} \\ &= M_{b-\varepsilon}. \end{aligned} \tag{3.2}$$

Hence M is attained at some u. If u is replaced by $|u|$, the expression on the right-hand side of (1.5) does not change, we may assume $u \ge 0$. By the maximum principle, if $V \ge 0$, then $u > 0$ in $\mathbb{R}^N$. □

Theorem 3.2. *Suppose $V \geq 0$ and* (V1), (V2), (P1) *are satisfied. Then there exists $\Lambda > 0$ such that for each $\lambda \geq \Lambda$ the infimum in* (1.5) *is attained at some $u_\lambda > 0$. Here $V(x)$ replaced by $\lambda V(x)$.*

Proof. Here $V = V^+$. Let b be as in (V2) and

$$\begin{aligned} M^\lambda &= \inf_{u \in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x)|u|^p)dx}{\|u\|_q^p}, \\ M_b^\lambda &= \inf_{u \in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V_b(x)|u|^p)dx}{\|u\|_q^p}. \end{aligned} \tag{3.3}$$

It suffices to show that $M^\lambda < M_b^\lambda$ for all λ large enough. We may assume $V(0) < b$ and choose $\varepsilon, \delta > 0$ so that $V(x) < b - \varepsilon$ whenever $|x| < 2\delta$. Let $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a function such that $\varphi(x) = 1$ for $|x| \leq \delta$ and $\varphi(x) = 0$ for $|x| \geq 2\delta$. Set $w_{\lambda b}(x) = \varphi(x)u_{\lambda b}(x) = \lambda^{\frac{1}{q-p}} u_b(\lambda^{\frac{1}{p}}x)\varphi(x)$, where u_b is as in the proof of Theorem 3.1. Then for all sufficiently large λ and some $C_0 > 0$,

$$\begin{aligned} M^\lambda &\leq \frac{\int_{\mathbb{R}^N} (|\nabla w_{\lambda b}|^p + \lambda V(x)|w_{\lambda b}|^p)dx}{\|w_{\lambda b}\|_q^p} \\ &\leq \frac{\int_{\mathbb{R}^N} (|\nabla w_{\lambda b}|^p + \lambda(b-\varepsilon)|w_{\lambda b}|^p)dx}{\|w_{\lambda b}\|_q^p} \\ &\leq \lambda^r \Big(\frac{\int_{\mathbb{R}^N} (|\nabla u_b|^p + \lambda b|u_b|^p)dx - \varepsilon\int_{\mathbb{R}^N} |u_b|^p dx}{\|u_b\|_q^p} + \varepsilon\Big) \\ &\leq \lambda^r (N_b - C_0\varepsilon) \end{aligned}$$

where N_b is defined in (3.1) and r in (3.2). Using (3.2) and (3.3) we also see that

$$M_b^\lambda \geq \inf_{u \in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^p + \lambda b|u|^p)dx}{\|u\|_q^p} = N_{\lambda b} = \lambda^r N_b, \tag{3.4}$$

hence $M^\lambda < M_b^\lambda$. By the argument at the end of the proof of Theorem 3.1, the infimum is attained at some $u_\lambda > 0$. □

Next we consider the existence of multiple solutions under the hypothesis that $V^{-1}(0)$ has nonempty interior.

Theorem 3.3. *Suppose $V \geq 0$, $V^{-1}(0)$ has nonempty interior and* (V1), (V2), (P1) *are satisfied. For each $k \geq 1$ there exists $\Lambda_k > 0$ such that if $\lambda \geq \Lambda_k$, then* (1.2) *has at the least k pairs of nontrivial solutions in E.*

Proof. For a fixed k we can find $\varphi_1, \dots, \varphi_k \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi_j$, $1 \leq j \leq k$, is contained in the interior of $V^{-1}(0)$ and $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$ whenever $i \neq j$. Let

$$F_k = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}.$$

Since $V \geq 0$, $\Phi(u) = \frac{1}{p}\|u\|^p - \frac{1}{q}\|u\|_q^q$ and therefore there exist $\alpha, \quad \rho > 0$ such that $\Phi|_{S_\rho} \geq \alpha$. Denote the set of all symmetric (in the sense that $-A = A$) and closed subsets of E by Σ, for each $A \in \Sigma$ let $\gamma(A)$ be the Krasnoselski genus and

$$i(A) = \min_{h \in \Gamma} \gamma(h(A) \cap S_\rho)$$

where Γ is the set of all odd homeomorphisms $h \in C(E, E)$. Then i is a version of Benci's pseudoindex. Let

$$\Phi_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + \lambda V(x)|u|^p)dx - \frac{1}{q}\int_{\mathbb{R}^N}|u|^q dx, \quad \lambda \geq 1$$

and

$$c_j = \inf_{i(A)\geq j}\sup_{u\in A}\Phi_\lambda(u), \quad 1 \leq j \leq k.$$

Since $\Phi_\lambda(u) \geq \Phi(u) \geq \alpha$ for all $u \in S_\rho$ and since $\mathrm{i}(F_k) = \dim F_k = k$,

$$\alpha \leq c_1 \leq \dots \leq c_k \leq \sup_{u\in F_k}\Phi_\lambda(u) = C.$$

It is clear that C depends on k but not on λ. As in (3.4), we have

$$M_b^\lambda \geq N_{\lambda b} = \lambda^r N_b$$

where $r > 0$, and therefore $M_b^\lambda \to \infty$. Hence $C < (\frac{1}{p} - \frac{1}{q})(M_b^\lambda)^{\frac{q}{(q-p)}}$ whenever λ is large enough and it follows from Lemma 2.3 that for such λ the Palais-Smale condition is satisfied at all levels $c \leq C$. By the usual critical point theory Theorem 2.4, all c_j are critical levels and Φ_λ has at least k pairs of nontrivial critical points. □

Theorem 3.4. *Suppose* (V1), (V2), (P1) *are satisfied and* $V^{-1}(0)$ *has nonempty interior* Ω*. Let* $u_m \in E$ *be a solution of the equation*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda_m V(x)|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N. \tag{3.5}$$

If $\lambda_m \to \infty$ *and* $\|u_m\|_{\lambda_m} \leq C$ *for some* $C > 0$*, then, up to a subsequence,* $u_m \to \overline{u}$ *in* $L^q(\mathbb{R}^N)$*, where* $\overline{u}$ *is a weak solution of the equation*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-2}u, \quad x \in \Omega, \tag{3.6}$$

and $\overline{u} = 0$ *a.e. in* $\mathbb{R}^N \setminus V^{-1}(0)$*. If moreover* $V \geq 0$*, then* $u_m \to \overline{u}$ *in* E *as* $m \to \infty$*.*

Proof. Since $\lambda_m \geq 1$, $\|u_m\| \leq \|u_m\|_{\lambda_m} \leq C$. Passing to a subsequence, $u_m \rightharpoonup \overline{u}$ in E and $u_m \to \overline{u}$ in $L^q_{\rm loc}(\mathbb{R}^N)$ as $m \to \infty$. Since $\langle {\Phi_{\lambda_m}}'(u_m), \varphi\rangle = 0$, we see that

$$\lim_{m\to\infty}\int_{\mathbb{R}^N} V(x)|u_m|^{p-2}u_m\varphi dx = 0, \quad \int_{\mathbb{R}^N} V(x)|\overline{u}|^{p-2}\overline{u}\varphi dx = 0$$

and for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Therefore, $\overline{u} = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$.

We claim that $u_m \to \overline{u}$ in $L^q(\mathbb{R}^N)$ as $m \to \infty$. Assuming the contrary, it follows from Lion vanishing lemma that

$$\int_{B_\rho(x_m)} |u_m - \overline{u}|^p dx \geq \gamma$$

for some $\{x_m\} \subset \mathbb{R}^N$, ρ, $\gamma > 0$ and almost all m, where $B_\rho(x)$ denotes the open ball of radius ρ and center x.

Since $u_m \to \overline{u}$ in $L^q_{\rm loc}(\mathbb{R}^N)$, $|x_m| \to \infty$. Therefore, the measure of the set $B_\rho(x_m) \cap \{x \in \mathbb{R}^N : V(x) < b\}$ tends to 0 and

$$\begin{aligned}\lim_{m\to\infty}\|u_m\|^p_{\lambda_m} &\geq \lim_{m\to\infty}\lambda_m b\int_{B_\rho(x_m)\cap\{V\geq b\}}|u_m|^p dx\\ &= \lim_{m\to\infty}\lambda_m b(\int_{B_\rho(x_m)}|u_m - \overline{u}|^p dx) = \infty,\end{aligned}$$

which is a contradiction.

Let now $V \geq 0$. Since u_m satisfies (3.5), $\langle \Phi'_{\lambda_m}(u_m), \overline{u} \rangle = 0$ and $\overline{u} = 0$ whenever $V > 0$, it follows that

$$\|u_m\|^p \leq \|u_m\|^p_{\lambda_m} = \|u_m\|^q_q$$

and

$$\|\overline{u}\|^p = \|\overline{u}\|^p_{\lambda_m} = \|\overline{u}\|^q_q.$$

Hence $\limsup_{m\to\infty} \|u_m\|^p \leq \|\overline{u}\|^q_q = \|\overline{u}\|^p$; therefore, $u_m \to \overline{u}$ in E as $m \to \infty$. □

Theorem 3.5. *Suppose* (V1), (V2), (P1) *are satisfied and* $V^{-1}(0)$ *has nonempty interior,* $V \geq 0$*,* $u_m \in E$ *is a solution of* (3.5)*,* $\lambda_m \to \infty$ *and* $\Phi_{\lambda_m}(u_m)$ *is bounded and bounded away from* 0*. Then the conclusion of Theorem 3.4 is satisfied and* $\overline{u} \neq 0$*.*

Proof. We have

$$\Phi_{\lambda_m}(u_m) = \frac{1}{p}\|u_m\|^p_{\lambda_m} - \frac{1}{q}\|u_m\|^q_q$$

and

$$\Phi_{\lambda_m}(u_m) = \Phi_{\lambda_m}(u_m) - \frac{1}{p}\langle \Phi'_{\lambda_m}(u_m), u_m \rangle = (\frac{1}{p} - \frac{1}{q})\|u_m\|^q_q$$

Hence $\|u_m\|_q$, and therefore also $\|u_m\|_{\lambda_m}$ is bounded. So the conclusion of Theorem 3.4 holds. Moreover, as $\|u_m\|_q$ is bounded away from 0, $\overline{u} \neq 0$. □

As a consequence of this corollary, if k is fixed, then any sequence of solutions u_m of (1.2) with $\lambda = \lambda_m \to \infty$ obtained in Theorem 3.3 contains a subsequence concentrating at some $\overline{u} \neq 0$. Moreover, it is possible to obtain a positive solution for each λ, either via Theorem 3.1 or by the mountain pass theorem. It follows that each sequence $\{u_m\}$ of such solutions with $\lambda_m \to \infty$ has a subsequence concentrating at some $\overline{u}$ which is positive in Ω. Corresponding to u_m are solutions $v_m = \varepsilon_m^{p/(q-p)} u_m$ of (1.3), where $\varepsilon_m^p = \lambda_m^{-1}$. Then $v_m \to 0$ and $\varepsilon_m^{-p/(q-p)} v_m \to \overline{u}$.

subsection*Remark In the proof of Lemmas 2.2 and 2.3 and Theorems 3.1–3.3, the condition (V1) can be replaced by

(V1') $v \in L^1_{\rm loc}(\mathbb{R}^N)$ and $V^- = \max\{-V, 0\} \in L^q(\mathbb{R}^N)$, where $q = N/p$ if $N \geq p+1$, $q > 1$ if $N = p$ and $q = 1$ if $N < p$.

Meanwhile in Theorems 3.4 and 3.5 we also need $V \in L^q_{\rm loc}(\mathbb{R}^N)$.

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