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STABILITY OF DELAY DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS

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Abstract. We study the solutions to the delay differential equation equation

$$\dot{x}(t) = -a(t)x(t-h),$$

where the coefficient $a(t)$ is not necessarily positive. It is proved that this equation is exponentially stable provided that $a(t) = b + c(t)$ for some positive constant b less than $\pi/(2h)$, and the integral $\int_0^t c(s)ds$ is sufficiently small for all $t > 0$. In this case the 3/2-stability theorem is improved.

1. Introduction and preliminaries

This article concerns the equation

$$\dot{x}(t) = -a(t)x(t-h), \tag{1.1}$$

where $\dot{x} = dx/dt$, the delay h is a positive constant, and $a(t)$ a piece-wise continuous function bounded on $[0,\infty)$. We do not require that $a(t)$ be positive, and therefore, the "characteristic function" $z + a(t)e^{-zh}$ can be unstable for some $t \geq 0$.

The sharp stability condition (the so called 3/2-stability theorem) for first-order functional-differential equations with one variable delay was established by Myshkis [5] (see also [4]). A similar result was established by Lillo [3]. The 3/2-stability theorem asserts that (1.1) is uniformly stable, provided that $0 < ha(t) \leq 3/2$ for all $t \geq 0$. The upper bound 3/2 is the best possible. In fact, if $h \sup_t a(t) > 3/2$, then there are equations having unbounded solutions. The 3/2-theorem was generalized to nonlinear equations and equations with unbounded delays in the very interesting papers [6, 7, 8]. In this article, under some additional conditions we improve the 3/2-theorem.

We consider (1.1) as a perturbation of the equation

$$\dot{y}(t) = -by(t-h) \tag{1.2}$$

with a positive constant $b < \pi/(2h)$ satisfying a condition stated below. The fundamental solution to (1.2) is

$$F_b(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{zt}dz}{z + be^{-zh}}\,.$$

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For a function f defined and bounded on $[0,\infty)$ (not necessarily continuous), we introduce the norm $\|f\|_\infty = \sup_{t\ge 0}|f(t)|$. So $\|a\|_\infty = \sup_{t\ge 0}|a(t)|$. In addition, put

$$\|f\|_{L^1} = \int_0^\infty |f(t)|dt,$$

if the integral exists. Now we are in a position to formulate our main result.

Theorem 1.1. *Let there be a constant $b \in (0, \pi/(2h))$, such that*

$$w_b := \sup_{t\ge 0} |\int_0^t (a(t)-b)dt|$$

is finite and satisfies the inequality

$$w_b < \frac{1}{1+(b+\|a\|_\infty)\|F_b\|_{L^1}}. \tag{1.3}$$

Then (1.1) *is exponentially stable.*

This theorem is proved in the next section. Its assumptions are sharp: if $a(t) \equiv b$, then $w_b = 0$ and condition (1.3) is automatically fulfilled.

Furthermore, let

$$ehb < 1. \tag{1.4}$$

Then $F_b(t) \ge 0$ and (1.2) is exponentially stable, cf. [2] and references therein. Now, integrating (1.2), we have

$$1 = F_b(0) = b\int_0^\infty F_b(t-h)dt = b\int_h^\infty F_b(t-h)dt = b\|F_b\|_{L^1}.$$

Thus, Theorem 1.1 implies the following result.

Corollary 1.2. *Let* (1.4) *and*

$$w_b < \frac{b}{2b+\|a\|_\infty} \tag{1.5}$$

hold. Then (1.1) *is exponentially stable.*

Now for a positive constant ω, let

$$a(t) = b + u(\omega t), \tag{1.6}$$

where $u(t)$ is a piece-wise continuous function such that

$$\nu_u := \sup_t \Big|\int_0^t u(s)ds\Big| < \infty.$$

Then

$$w_b = \sup_t |\int_0^t u(\omega s)ds| = \nu_u/\omega.$$

For example, when $u(t) = \sin(t)$, then $\nu_u = 2$. Now Theorem 1.1 and (1.5) imply our next result.

Corollary 1.3. *Let* (1.4)*,* (1.6) *and*

$$\omega > \frac{\nu_u(3b+\|u\|_\infty)}{b} \tag{1.7}$$

hold. Then (1.1) *is exponentially stable.*

Example 1.4. *Consider the equation*

$$\dot{x} = -bx(t-1) + c_2 \sin(\omega t) x(t-1), \tag{1.8}$$

where b, c_2 *are positive constant with* $b < e^{-1}$. *Then* $\nu_u = 2c_2$ *and* (1.7) *has the form*

$$\omega > \frac{2c_2(3b + c_2)}{b}. \tag{1.9}$$

In summary, for each c_2 there exists an ω, such that (1.8) is exponentially stable. Meanwhile, the 3/2-stability theorem requires the additional condition $c_2 + b < 3/2$. Therefore, Theorem 1.1 supplements the interesting results obtained in [1].

2. Proof of Theorem 1.1

For simplicity, we put $F_b(t) = F(t)$. Due to the Variation of Constants Formula the equation

$$\dot{x}(t) = -bx(t-h) + f(t) \quad (t \ge 0),$$

with a given function f and the zero initial condition $x(t) = 0$ $(t \le 0)$ is equivalent to the equation

$$x(t) = \int_0^t F(t-s) f(s) ds. \tag{2.1}$$

Recall that a function $G(t,s)$, $(t \ge s \ge 0)$ differentiable in t, is the fundamental solution to (1.1) if it satisfies that equation in t and the initial conditions

$$G(s,s) = 1, \quad G(t,s) = 0 \quad (t < s, \ s \ge 0).$$

Put $G(t,0) = G(t)$. Subtracting (1.2) from (1.1) we have

$$\frac{d}{dt}(G(t) - F(t)) = -b(G(t-h) - F(t-h)) + c(t)G(t-h)$$

where $c(t) = -(a(t) - b)$. Now (2.1) implies

$$G(t) = F(t) + \int_0^t F(t-s) c(s) G(s-h) ds. \tag{2.2}$$

We need the following simple lemma.

Lemma 2.1. *Assume that on each finite segment of the real axis, functions* $f(t)$ *and* $v(t)$ *are boundedly differentiable and* $w(t)$ *is integrable. Then with the notation*

$$j_w(t,\tau) = \int_\tau^t w(s) ds \quad (t > \tau > -\infty),$$

the equality

$$\int_\tau^t f(s) w(s) v(s) ds = f(t) j_w(t,\tau) v(t) - \int_\tau^t [f'(s) j_w(s,\tau) v(s) + f(s) j_w(s) v'(s)] ds$$

is valid.

Proof. Clearly,

$$\frac{d}{dt} f(t) j_w(t,\tau) v(t) = f'(t) j_w(t,\tau) v(t) + f(t) w(t) v(t) + f(t) j_w(t,\tau) v'(t).$$

Integrating, this equality and taking into account that $j_w(\tau,\tau) = 0$, we arrive at the required result. □

Put $J(t) := \int_0^t c(s)ds$. By the previous lemma,

$$\begin{aligned}
&\int_0^t F(t-\tau)c(\tau)G(\tau-h)d\tau \\
&= F(0)J(t)G(t-h) - \int_0^t \Big[\frac{dF(t-\tau)}{d\tau}J(\tau)G(\tau-h) + F(t-\tau)J(\tau)\frac{dG(\tau-h)}{d\tau}\Big]d\tau.
\end{aligned}$$

However,

$$\frac{dG(\tau-h)}{d\tau} = -a(\tau-h)G(\tau-2h) \quad \text{and} \quad \frac{dF(t-\tau)}{d\tau} = -\frac{dF(t-\tau)}{dt} = bF(t-\tau-h).$$

Thus,

$$\begin{aligned}
&\int_0^t F(t-\tau)c(\tau)G(\tau-h)d\tau \\
&= J(t)G(t-h) + \int_0^t J(\tau)\Big[-bF(t-\tau-h)G(\tau-h) \\
&\quad + F(t-\tau)a(\tau-h)G(\tau-2h)\Big]d\tau.
\end{aligned}$$

Now (2.2) implies the following result.

Lemma 2.2. *It holds that*

$$\begin{aligned}
G(t) = F(t) + J(t)G(t-h) + \int_0^t J(\tau)\Big[&-bF(t-\tau-h)G(\tau-h) \\
&+ F(t-\tau)a(\tau-h)G(\tau-2h)\Big]d\tau\,.
\end{aligned}$$

From the previous lemma,

$$\|G\|_\infty \le \|F\|_\infty + \|G\|_\infty w_b[1 + (b + \|a\|_\infty)\|F\|_{L^1}]\,.$$

If condition (1.3) holds, then

$$\theta := w_b[1 + (b + \|a\|_\infty)\|F\|_{L^1}] < 1$$

and therefore,

$$\|G\|_\infty \le \frac{\|F\|_\infty}{1-\theta}. \tag{2.3}$$

So the stability of (1.1) is proved. Substituting

$$x_\epsilon(t) = e^{\epsilon t}x(t) \tag{2.4}$$

with $\epsilon > 0$ into (1.1), we have the equation

$$\dot{x}_\epsilon(t) = \epsilon x_\epsilon(t) - a(t)e^{\epsilon h}x_\epsilon(t-h). \tag{2.5}$$

If $\epsilon > 0$ is sufficiently small, then considering (2.5) as a perturbation of the equation $\dot{y}(t) = \epsilon y(t) - be^{\epsilon h}y(t-h)$, and applying our above arguments, according to (2.3) we obtain $\|x_\epsilon\|_\infty < \infty$ for any solution x_ϵ of (2.5). Hence (2.4) implies $|x(t)| \le e^{-\epsilon t}\|x_\epsilon\|_\infty$ for any solution x of (1.1).

References

[1] Berezansky, L. and Braverman, E.; On stability of some linear and nonlinear delay differential equations, *J. Math. Anal. Appl., 314*, No. 2, 391-411 (2006).

[2] Gil, M. I.; On Aizerman-Myshkis problem for systems with delay. *Automatica, 36*, 1669-1673 (2000).

[3] Lillo, J. C.; Oscillatory solutions of the equation $y'(x) = m(x)y(x-n(x))$. *J. Differ. Equations 6*, 1-35 (1969).

[4] Kolmanovskii, V. and Myshkis, A.; *Introduction to the theory and applications of functional differential equations*. Mathematics and its Applications. Kluwer, Dordrecht, 1999.

[5] Myshkis, A. D.; On solutions of linear homogeneous differential equations of the first order of stable type with a retarded argument. *Mat. Sb., N. Ser. 28(70)*, 15-54 (1951).

[6] Yoneyama, Toshiaki; On the stability for the delay-differential equation $\dot{x}(t) = -a(t)f(x(t-r(t)))$. *J. Math. Anal. Appl., 120*, 271-275 (1986).

[7] Yoneyama, Toshiaki; On the 3/2 stability theorem for one-dimensional delay-differential equations. *J. Math. Anal. Appl. 125*, 161-173 (1987).

[8] Yoneyama, Toshiaki; The 3/2 stability theorem for one-dimensional delay-differential equations with unbounded delay. *J. Math. Anal. Appl. 165*, No.1, 133-143 (1992).

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