Electronic Journal of Differential Equations, Vol. 2011(2011), No. 01, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SCHRÖDINGER-POISSON EQUATIONS WITH SUPERCRITICAL GROWTH

CLAUDIANOR O. ALVES, SÉRGIO H. M. SOARES, MARCO A. S. SOUTO

Abstract. In this article, we study a class of Schrödinger-Poisson equations in $\mathbb{R}^3$ with supercritical growth. We prove the existence of positive solutions, using variational methods combined with perturbation arguments. The solutions to subcritical Schrödinger-Poisson equations are estimated using the L^∞ norm.

1. Introduction

The system

$$\begin{gathered} -\Delta u + V(x)u + \phi u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{1.1}$$

has great importance for describing the interaction of a charged particle with an electromagnetic field; see for example [2, 3, 4, 12, 17, 18]. Recent studies have focused attention on existence, nonexistence and symmetry of solutions to (1.1). However, most of the references presented here are devoted to pure power type nonlinearities $|u|^{p-2}u$, where p is a subcritical or critical exponent. For example, Ruiz [17] studies the subcritical case and shows that if $p \leq 3$, the problem (1.1) does not admit any nontrivial solution, and if $3 < p < 6$, there exists a nontrivial radial solution of (1.1). A multiplicity result for the subcritical case has been established by Gaetano [12]. The critical case has been treated by Azzollini and Pomponio [3] and Zhao and Zhao [19].

In this article, we consider the stationary Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x)u + \phi u = f(u), \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.2}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a bounded locally Hölder continuous function satisfying:

(V0) There exists $\alpha > 0$ such that $V(x) \geq \alpha > 0$, $\forall x \in \mathbb{R}^3$.

2000 *Mathematics Subject Classification.* 35J20, 35J65.
Key words and phrases. Supercritical problems; Schrödinger-Poisson equation; variational method.

Submitted June 1, 2010. Published January 4, 2011.
C.O. Alves is supported by grants 620150/2008-4 and 303080/2009-4 from CNPq/Brazil.
S.H.M. Soares was supported by grant 313237/2009-3 from CNPq/Brazil.
M.A.S. Souto is supported by grants 2009/14218-2 from FAPESP, PROCAD 024/2007 from CAPES/Brazil and 302650/2008-3 from CNPq/Brazil.

(V1) $V(x) = V(x+y)$, for all $x \in \mathbb{R}^3$, $y \in \mathbb{Z}^3$.

The function $f \in C(\mathbb{R}, \mathbb{R})$ can be written as

$$f(s) = f_o(s) + \lambda g(s),$$

where λ is a positive real parameter, f_o and g are locally Hölder continuous functions satisfying:

(F1) $f_o(0) = g(0) = 0$ and $g(s) \geq 0$ for all s;

(F2) $\lim_{|s| \to 0^+} f_o(s)/s = 0$ and $\lim_{|s| \to 0^+} g(s)/s = 0$;

(F3) There exists $q \in (4, 2^*)$, $2^* = 6$, such that

$$|f_o(s)| \leq |s|^{q-1}, \quad \forall s \in \mathbb{R};$$

(F4) $\lim_{|s| \to +\infty} F_o(s)/s^4 = +\infty$, where $F_o(s) = \int_0^s f_o(t)dt$;

(F5) For $\alpha > 0$ given by (V_0), there exists $\sigma \in (0, \alpha)$ such that

$$sf_o(s) - 4F_o(s) \geq -\sigma s^2 \quad \text{and} \quad sg(s) - 4G(s) \geq 0$$

for all $s \neq 0$, where $G(s) = \int_0^s g(t)dt$;

(F6) There exists a sequence of positive real numbers, (M_n), converging to $+\infty$ such that

$$\frac{g(s)}{s^{q-1}} \leq \frac{g(M_n)}{M_n^{q-1}}, \quad \text{for all } s \in [0, M_n],\ n \in \mathbb{N}.$$

Since $u \equiv 0$ is a solution of (1.2), the aim of the present article is to prove the existence of nontrivial solutions for (1.2). However, it should be point out that we can not apply variational methods directly because the Euler-Lagrange functional on $H^1(\mathbb{R}^3)$ associated with (1.2) is not well defined in general. Our technique combines perturbation arguments, estimate for solutions to a subcritical Schrödinger-Poisson equation in terms of the L^∞ norm and the mountain pass theorem. Our main result is as follows.

Theorem 1.1. *Suppose that V satisfies* (V0)–(V1)*, and f satisfies* (F1)–(F6)*. Then there is a $\lambda_o > 0$ such that* (1.2) *possesses a positive solution for all $\lambda \leq \lambda_o$.*

To prove the above theorem, we argue as in Alves and Souto [1]. We first provide an estimate involving the L^∞-norm of a solution related to a subcritical problem. To do so we modify the nonlinearity obtaining a family of functionals of class C^1. Employing conditions (F1)–(F4), we show that these functionals satisfy uniformly the geometric hypotheses of the mountain pass theorem. Using this fact and the estimate, we verify the existence of a sequence in $H^1(\mathbb{R}^3)$ converging weakly to a solution of (1.2). It is important to stress that our proof does not require a growth assumption on g; consequently, on f. We observe that the condition (F6) holds if

$$\lim_{|s| \to +\infty} \frac{g(s)}{s^{q-1}} = +\infty.$$

In particular, f may be $f(s) = s^{q-1} + s^{p-1}$, for all $p > 6 > q$, or $f(s)$ may behave like e^s at infinity.

In addition, since the term $\int_{\mathbb{R}^3} \phi_u u^2\,dx$ is homogeneous of degree 4, the corresponding Ambrosetti-Rabinowitz condition on f is the following:

(AR) There exists $\theta > 4$ such that

$$0 < \theta F(s) \leq sf(s), \quad \forall s \in \mathbb{R}.$$

This condition is important not only to ensure that the functional I (see (3.2) below) has the mountain pass geometry, but also to guarantee that the Palais-Smale, or Cerami, sequences associated with I are bounded. We observe that $f(s) = s^{q-1} + \lambda(1+\cos s)s^{p-1}$, with $4 < q < 6 \le p$, satisfies the conditions (F1)–(F6), but not the Ambrosetti-Rabinowitz condition. Moreover, the function f considered here does not belong to any class of nonlinearities in the above-cited papers.

Since we intend to prove the existence of positive solutions, we consider $f : \mathbb{R} \to \mathbb{R}$ satisfying (F1)–(F6) on $[0, +\infty)$ and defined as zero on $(-\infty, 0]$.

2. A solution estimate

This section we obtain an estimate involving the L^∞ norm of a solution to a subcritical problem. This result works for any dimension $N \ge 3$.

Proposition 2.1. *Let $v \in H^1(\mathbb{R}^N)$ be a weak solution of the problem*

$$-\Delta v + b(x)v = h(x, v), \text{ in } \mathbb{R}^N, \tag{2.1}$$

where $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous functions verifying, for some $2 < q < 2^* = 2N/(N-2)$, $|h(x, s)| \le 2|s|^{q-1}$, for all $s > 0$, and b is a non-negative function in $\mathbb{R}^N$. Then, for all $C > 0$, there exists a constant $k = k(q, C) > 0$ such that if $\|v\|^2 \le C$, then $\|v\|_\infty \le k$.

Proof. For each $m \in \mathbb{N}$ and $\beta > 1$, consider

$$A_m = \{x \in \mathbb{R}^N : |v|^{\beta-1} \le m\}, \quad B_m = \mathbb{R}^N \setminus A_m,$$

$$v_m = \begin{cases} v|v|^{2(\beta-1)}, & \text{in } A_m, \\ m^2 v, & \text{in } B_m. \end{cases}$$

Observe that $v_m \in H^1(\mathbb{R}^N)$, $v_m \le |v|^{2\beta-1}$ and

$$\nabla v_m = (2\beta - 1)|v|^{2(\beta-1)}\nabla v \quad \text{in } A_m, \quad \nabla v_m = m^2 \nabla v \quad \text{in } B_m. \tag{2.2}$$

Using v_m as a test function in (2.1), we obtain

$$\int_{\mathbb{R}^N} (\nabla v \nabla v_m + b(x)vv_m)\, dx = \int_{\mathbb{R}^N} h(x, v)v_m\, dx.$$

From (2.2),

$$\int_{\mathbb{R}^N} \nabla v \nabla v_m\, dx = (2\beta - 1)\int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2\, dx + m^2 \int_{B_m} |\nabla v|^2\, dx. \tag{2.3}$$

Let

$$\omega_m = \begin{cases} v|v|^{(\beta-1)}, & \text{in } A_m, \\ mv, & \text{in } B_m. \end{cases}$$

Then

$$\omega_m^2 = vv_m \le |v|^{2\beta}, \quad 0 \le b(x)\omega_m^2 = b(x)vv_m, \quad \text{in } \mathbb{R}^N$$

and

$$\nabla \omega_m = \beta|v|^{(\beta-1)}\nabla v \quad \text{in } A_m, \quad \nabla \omega_m = m\nabla v \quad \text{in } B_m. \tag{2.4}$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla \omega_m|^2\, dx = \beta^2 \int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2\, dx + m^2 \int_{B_m} |\nabla v|^2\, dx. \tag{2.5}$$

From (2.3)-(2.5), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}(|\nabla\omega_m|^2+b(x)\omega_m^2)\,dx-\int_{\mathbb{R}^N}(\nabla v\nabla v_m+b(x)vv_m)\,dx\\&=(\beta-1)^2\int_{A_m}|v|^{2(\beta-1)}|\nabla v|^2\,dx.\end{aligned}$$

From (2.3) and $b(x)\geq 0$, we have

$$(2\beta-1)\int_{A_m}|v|^{2(\beta-1)}|\nabla v|^2\,dx\leq\int_{\mathbb{R}^N}(\nabla v\nabla v_m+b(x)vv_m)\,dx,$$

and consequently

$$\int_{\mathbb{R}^N}(|\nabla\omega_m|^2+b(x)\omega_m^2)\,dx\leq[\frac{(\beta-1)^2}{2\beta-1}+1]\int_{\mathbb{R}^N}(\nabla v\nabla v_m+b(x)vv_m)\,dx.$$

Since (2.1) holds for v, we have

$$\int_{\mathbb{R}^N}(|\nabla\omega_m|^2+b(x)\omega_m^2)\,dx\leq\frac{\beta^2}{2\beta-1}\int_{\mathbb{R}^N}h(x,v)v_m\,dx$$

and

$$\int_{\mathbb{R}^N}(|\nabla\omega_m|^2+b(x)\omega_m^2)\,dx\leq\beta^2\int_{\mathbb{R}^N}h(x,v)v_m\,dx,\tag{2.6}$$

because $2\beta-1>1$. Let E denote the Sobolev space

$$E=\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}b(x)u^2\,dx<\infty\}$$

endowed with the norm

$$\|u\|^2=\int_{\mathbb{R}^N}(|\nabla u|^2+b(x)u^2)\,dx.$$

Throughout the proof, r denotes $2^*=2N/(N-2)$. Let S be the best constant of the Sobolev immersion of $H^1(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$. Thus,

$$\|u\|_r^2\leq S\int_{\mathbb{R}^N}|\nabla u|^2\,dx$$

for every $u\in H^1(\mathbb{R}^N)$. From (2.6) , since $|h(x,s)|\leq 2|s|^{q-1}$ for all $s>0$, we have

$$\Big[\int_{A_m}|\omega_m|^r\,dx\Big]^{2/r}\leq\Big[\int_{\mathbb{R}^N}|\omega_m|^r\,dx\Big]^{2/r}\leq S\beta^2\int_{\mathbb{R}^N}h(x,v)v_m\,dx.$$

Observing that

$$h(x,v)v_m=\frac{h(x,v)}{v}vv_m=\frac{h(x,v)}{v}w_m^2,$$

we obtain

$$\Big[\int_{A_m}|\omega_m|^r\,dx\Big]^{2/r}\leq 2S\beta^2\int_{\mathbb{R}^N}|v|^{q-2}\omega_m^2\,dx.$$

For q_1 such that $1/q_1+(q-2)/r=1$, it then follows from Hölder's inequality that

$$\Big[\int_{A_m}|\omega_m|^r\,dx\Big]^{2/r}\leq S\beta^2\|v\|_r^{q-2}\Big[\int_{\mathbb{R}^N}|\omega_m|^{2q_1}\,dx\Big]^{1/q_1}.$$

Since $|\omega_m|\leq|v|^\beta$ in $\mathbb{R}^N$ and $|\omega_m|=|v|^\beta$ in A_m, we have

$$\Big[\int_{A_m}|v|^{r\beta}\Big]^{2/r}\leq S\beta^2\|v\|_r^{q-2}\Big[\int_{\mathbb{R}^N}|v|^{2q_1\beta}\,dx\Big]^{1/q_1}.$$

By the Monotone Convergence Theorem, we obtain
$$\|v\|_{r\beta} \le \beta^{1/\beta}(S\|v\|_r^{q-2})^{1/(2\beta)}\|v\|_{2\beta q_1}. \tag{2.7}$$
Since $q < r$, we have $r > 2q_1$. Set $\sigma = r/2q_1 > 1$. Setting $\beta = \sigma$ in (2.7), we obtain $2q_1\beta = r$ and
$$\|v\|_{r\sigma} \le \sigma^{1/\sigma}(S\|v\|_r^{q-2})^{1/(2\sigma)}\|v\|_r. \tag{2.8}$$
Taking $\beta = \sigma^2$ in (2.7), we obtain $2q_1\beta = r\sigma$ and
$$\|v\|_{r\sigma^2} \le \sigma^{2/\sigma^2}(S\|v\|_r^{q-2})^{1/(2\sigma^2)}\|v\|_{r\sigma}. \tag{2.9}$$
From (2.8) and (2.9), we find
$$\|v\|_{r\sigma^2} \le \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}}(S\|v\|_r^{q-2})^{\frac{1}{2}(\frac{1}{\sigma}+\frac{1}{\sigma^2})}\|v\|_r. \tag{2.10}$$
The result is obtained by iteration of the estimate (2.7). Taking $\beta = \sigma^j$, $j = 1, 2, \dots$, yields
$$\|v\|_{r\sigma^m} \le \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}+\frac{3}{\sigma^3}+\dots+\frac{j}{\sigma^j}}(S\|v\|_r^{q-2})^{\frac{1}{2}(\frac{1}{\sigma}+\frac{1}{\sigma^2}+\frac{1}{\sigma^3}+\dots+\frac{1}{\sigma^j})}\|v\|_r. \tag{2.11}$$
Since the series bellow are convergent and
$$\sum_{j=1}^{\infty}\frac{j}{\sigma^j} = \frac{\sigma}{(\sigma-1)^2}, \quad \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{\sigma^j} = \frac{1}{2(\sigma-1)},$$
from (2.11), we have
$$\|v\|_p \le \sigma^{\sigma/(\sigma-1)^2}(S\|v\|_r^{q-2})^{\frac{1}{2(\sigma-1)}}\|v\|_r,$$
for all $p \ge r$. Since $b(x)$ is nonnegative, we have $\|v\|_r \le S^{1/2}C^{1/2}$. Using that
$$\|v\|_\infty = \lim_{p\to+\infty}\|v\|_p,$$
we conclude that Proposition 2.1 is valid for
$$k = \sigma^{\frac{\sigma}{(\sigma-1)^2}}(S^{q/2}C^{(q-2)/2})^{\frac{1}{2(\sigma-1)}}S^{1/2}C^{1/2}.$$
□

3. Auxiliary problem

In this section we study the existence of a solution for the Schrödinger-Poisson system with subcritical growth. This result will be useful for obtaining our main result. More precisely, we consider the system
$$\begin{gathered} -\Delta u + V(x)u + \phi u = h(u), \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{3.1}$$
where $V : \mathbb{R}^3 \to \mathbb{R}$ is a bounded locally Hölder continuous that satisfies (V0) and (V1), and the function $h \in C(\mathbb{R}^+, \mathbb{R})$ and satisfies:

- (H1) $h(0) = 0$;
- (H2) $\lim_{s\to 0^+} h(s)/s = 0$;
- (H3) There exist $C > 0$ and $p \in (4, 6)$ such that
$$|h(s)| \le C(|s| + |s|^{p-1}), \forall s \in \mathbb{R}^+;$$
- (H4) $\lim_{s\to+\infty} H(s)/s^4 = +\infty$, where $H(s) = \int_0^s h(t)dt$;

(H5) For $\alpha > 0$ given by (V0), there exists $\sigma \in (0, \alpha)$ such that
$$sh(s) - 4H(s) \geq -\sigma s^2$$
for all $s \neq 0$.

Next we review some of the standard facts on the Schrödinger-Poisson equations (see [3, 12, 17, 19]). We begin by observing that (3.1) can be transformed into a Schrödinger equation with a nonlocal term. In fact, by Lax-Milgram theorem, given $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ such that
$$-\Delta\phi = u^2.$$
The function ϕ_u has the following properties (see [8]):

Lemma 3.1. (i) *There exists $C > 0$ such that $\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \leq C\|u\|^2$ and*
$$\int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx = \int_{\mathbb{R}^3} \phi_u u^2 \, dx \leq C\|u\|^4, \quad \forall u \in H^1(\mathbb{R}^3);$$
(ii) *$\phi_u \geq 0$, $\forall u \in H^1(\mathbb{R}^3)$;*
(iii) *$\phi_{tu} = t^2\phi_u$, $\forall t > 0, u \in H^1(\mathbb{R}^3)$.*
(iv) *If $y \in \mathbb{R}^3$ and $\tilde{u}(x) = u(x+y)$, then $\phi_{\tilde{u}}(x) = \phi_u(x+y)$ and*
$$\int_{\mathbb{R}^3} \phi_{\tilde{u}} \tilde{u}^2 \, dx = \int_{\mathbb{R}^3} \phi_u u^2 \, dx;$$
(v) *if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.*

From (V0), we can see that the $H^1(\mathbb{R}^3)$ norm is equivalent to
$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx.$$
Let I be the functional on $H^1(\mathbb{R}^3)$ defined by
$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) \, dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \, dx - \int_{\mathbb{R}^3} H(u) \, dx. \tag{3.2}$$
From the conditions on h, the functional $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and its Gateaux derivative is
$$I'(u)v = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) \, dx + \int_{\mathbb{R}^3} \phi_u uv \, dx - \int_{\mathbb{R}^3} h(u)v \, dx,$$
for every $u, v \in H^1(\mathbb{R}^3)$. Hence, corresponding to each critical point of I there is a weak solution of the Schrödinger equation with a nonlocal term:
$$-\Delta u + V(x)u + \phi_u u = h(u), \quad \text{in } \mathbb{R}^3. \tag{3.3}$$

Lemma 3.2. *Suppose that V satisfies* (V0) *and h satisfies* (H1)–(H5). *If $(u_n) \subset H^1(\mathbb{R}^3)$ is a Cerami sequence of I; i. e., $(I(u_n))$ is bounded and $(1+\|u_n\|)I'(u_n) \to 0$, then (u_n) is bounded in $H^1(\mathbb{R}^3)$.*

Proof. From (H5),
$$\begin{aligned} 4I(u_n) - I'(u_n)(u_n) &= \|u_n\|^2 + \int_{\mathbb{R}^3} [(u_n)h(u_n) - 4H(u_n)] \, dx \\ &\geq \|u_n\|^2 - \sigma \int_{\mathbb{R}^3} u_n^2 \, dx \geq \big(1 - \frac{\sigma}{\alpha}\big)\|u_n\|^2. \end{aligned}$$
Since $(4I(u_n) - I'(u_n)(u_n))$ is bounded, we conclude that (u_n) is bounded in $H^1(\mathbb{R}^3)$. ∎

Lemma 3.3. *Suppose that V satisfies* (V0) *and h satisfies* (H1)–(H4). *Then, there exist $\rho > 0$ and $e \in H^1(\mathbb{R}^3)$ with $\|e\| > \rho$, such that*

$$b \doteq \inf_{\|u\|=\rho} I(u) > I(0) = 0 \geq I(e).$$

Proof. From (H2)–(H3), for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$H(s) \leq \epsilon s^2 + C_\epsilon s^p, \quad \forall\, s \in \mathbb{R}.$$

By Sobolev inequalities, there exist positive constants α and β such that

$$I(u) \geq \big[(\frac{1}{2} - \epsilon\alpha) - \beta C_\epsilon \|u\|^{p-2}\big]\|u\|^2$$

We can assume, by decreasing ϵ if necessary, that there exist positive numbers b, ρ such that $b = \inf\{I(u), \|u\| = \rho\} > I(0) = 0$.

From (H4), for any $v \in H^1(\mathbb{R}^3)$ and $M > (1/4)\int_{\mathbb{R}^3} \phi_v v^2\, dx$, there exists $C > 0$ such that $H(s) \geq Ms^4 - Cs^2$, for all $s \in \mathbb{R}$. Hence,

$$I(tv) \leq (C + \frac{1}{2})\|v\|^2 t^2 - \Big(M - \frac{1}{4}\int_{\mathbb{R}^3} \phi_v v^2\, dx\Big)t^4 \to -\infty, \quad \text{as } t \to \infty.$$

Thus, $e = tv$ satisfies $\|e\| > \rho$ and $I(e) < 0 = I(0)$, provided t sufficiently large. $\square$

By a version of the mountain pass theorem (see [11]), there is a Cerami sequence $(u_n) \subset H^1(\mathbb{R}^3)$ such that

$$I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)I'(u_n) \to 0,$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \quad \Gamma = \{\gamma : [0,1] \to H^1(\mathbb{R}^3) : \gamma(0) = 0, \gamma(1) = e\}.$$

The main result of this section is the following.

Proposition 3.4. *Suppose that V satisfies* (V0), (V1), *and h satisfies* (H1)–(H5). *Then* (3.1) *possesses a positive solution u such that $\|u\|^2 \leq 4c\alpha/(\alpha - \sigma)$, where α and σ are given by* (V0) *and* (H4) *respectively and c is the minimax level associated with* (3.1).

Proof. By Lemma 3.2, we can assume that (u_n) is weakly convergent to u, for some $u \in H^1(\mathbb{R}^3)$. Taking $v \in C_0^\infty(\mathbb{R}^3)$, from Lemma 3.1(v), $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$, as $n \to \infty$, and so

$$\int_{\mathbb{R}^3} \phi_{u_n} uv\, dx \to \int_{\mathbb{R}^3} \phi_u uv\, dx, \quad \text{as } n \to \infty.$$

Moreover, using Hölder's inequality we obtain

$$|\int_{\mathbb{R}^3} \phi_{u_n}(u_n - u)v\, dx| \leq \|\phi_{u_n}\|_{L^{2^*}(\mathbb{R}^3)}\|u_n - u\|_{L^{12/5}(\Omega)}\|v\|_{L^{12/5}(\Omega)} = o_n(1),$$

where $\Omega = \operatorname{supp} v$. Therefore,

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n v\, dx - \int_{\mathbb{R}^3} \phi_u uv\, dx = \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)uv\, dx + \int_{\mathbb{R}^3} \phi_{u_n}(u_n - u)v\, dx = o_n(1),$$

for all $v \in C_0^\infty(\mathbb{R}^3)$, which implies

$$I'(u)v = 0, \quad \text{for all } v \in H^1(\mathbb{R}^3).$$

Consequently, u is a weak solution for (3.3). To conclude the proof, it only remains to show that $u \neq 0$. Assume by contradiction that $u \equiv 0$. By [7, Lemma 2.1] (see also [16]), we can claim that only one of the following conditions hold:

(i) For all $q \in (2, 2^*)$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} |u_n|^q \, dx = 0.$$

(ii) There are positive numbers R and η, and a sequence $(y_n) \subset \mathbb{R}^3$ such that

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} u_n^2 \, dx > \eta > 0.$$

If (i) occurs, then from (F2) and (F3), we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} h(u_n)u_n \, dx = 0.$$

By Lemma 3.1(ii),

$$\|u_n\|^2 \le \|u_n\|^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx = \int_{\mathbb{R}^3} h(u_n)u_n \, dx + o_n(1).$$

As a consequence, the sequence (u_n) is strongly convergent in $H^1(\mathbb{R}^3)$ to 0. Then $I(u_n) \to 0$, contrary to $I(u_n) \to c > 0$. Hence, (ii) is valid. From (V1) we can assume that $y_n \in \mathbb{Z}^N$. Define

$$\tilde{u}_n(x) = u_n(x + y_n).$$

From (V1) again, $(\tilde{u}_n)$ is bounded in $H^1(\mathbb{R}^3)$ and we can clearly assume that $(\tilde{u}_n)$ is weakly convergent to $\tilde{u}$ for some $\tilde{u} \in H^1(\mathbb{R}^3)$. From (ii), $\tilde{u} \neq 0$. Observing that Lemma 3.1(iv) implies that

$$I'(\tilde{u}_n)\tilde{u}_n = I'(u_n)u_n \quad \text{and} \quad I(\tilde{u}_n) = I(u_n),$$

hence that $(\tilde{u}_n)$ is a Cerami sequence of I, and finally

$$I'(\tilde{u}) = 0 \quad \text{with} \quad \tilde{u} \neq 0,$$

where we have again used Lemma 3.1(v). It follows that $\tilde{u}$ is a nontrivial solution to (3.3). Using bootstrap arguments and the maximum principle, we can conclude that the solution $\tilde{u}$ is positive.

Finally, to verify that $\tilde{u}$ satisfies inequality $\|u\|^2 \le 4c\alpha/(\alpha - \sigma)$, we observe that from (H5),

$$4I(\tilde{u}_n) - I'(\tilde{u}_n)\tilde{u}_n \ge \big(1 - \frac{\sigma}{\alpha}\big)\|\tilde{u}_n\|^2, \quad \forall n.$$

Passing to the limit we obtain

$$4c = \liminf_{n\to\infty}(4I(\tilde{u}_n) - I'(\tilde{u}_n)\tilde{u}_n) \ge \big(1 - \frac{\sigma}{\alpha}\big)\|u\|^2,$$

and the proof is complete. □

4. Preliminary results

To establish the existence of a solution to (1.2), we define a sequence of functions $\{g_n\}$ by setting

$$g_n(s) = \begin{cases} 0, & \text{if } s \le 0 \\ g(s), & \text{if } 0 \le s \le M_n \\ \frac{g(M_n)}{M_n^{q-1}} s^{q-1}, & \text{if } s \ge M_n. \end{cases}$$

From (F6), we have

$$|g_n(s)| \le \frac{g(M_n)}{M_n^{q-1}} |s|^{q-1} \quad \text{for all } s. \tag{4.1}$$

We conclude from (F3) that $f_{\lambda,n}(s) = f_o(s) + \lambda g_n(s)$ satisfies

$$|f_{\lambda,n}(s)| \le (1 + \lambda g(M_n) M_n) |s|^{q-1}, \tag{4.2}$$

which implies that the problem

$$\begin{gathered} -\Delta u + V(x)u + \phi u = f_{\lambda,n}(u), \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{4.3}$$

is variational for every $\lambda > 0$ and $n \in \mathbb{N}$. The functional associated with (4.3) is denoted by $J_{\lambda,n} : H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ and given by

$$\begin{aligned} & J_{\lambda,n}(u,\phi) \\ & = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\,dx - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2\,dx - \int_{\mathbb{R}^3} F_{\lambda,n}(u)\,dx. \end{aligned}$$

We observe that $J_{\lambda,n}$ is strongly indefinite. To overcome this difficulty, we introduce the functional $I_{\lambda,n} : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$I_{\lambda,n}(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \int_{\mathbb{R}^3} F_{\lambda,n}(u)\,dx,$$

with ϕ_u being the function defined in Section 3. By (4.2), the functional $I_{\lambda,n} \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and its Gateaux derivative is

$$I'_{\lambda,n}(u)v = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\,dx + \int_{\mathbb{R}^3} \phi_u uv\,dx - \int_{\mathbb{R}^3} f_{\lambda,n}(u)v\,dx,$$

for every $u, v \in H^1(\mathbb{R}^3)$. Hence, corresponding to each critical point of $I_{\lambda,n}$ there exists a weak solutions of

$$\begin{gathered} -\Delta u + V(x)u + \phi_u u = f_{\lambda,n}(u), \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{gathered} \tag{4.4}$$

For F_o given by (F4), we introduce an auxiliary Euler-Lagrange functional $I_o : H^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I_o(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \int_{\mathbb{R}^3} F_o(u)\,dx.$$

From (F1)–(F4) and Lemma 3.1(i,iii), it is standard to check that I_o possesses the geometric hypotheses of the mountain pass theorem (see Lemma 3.3). Then, there exist $e \in H^1(\mathbb{R}^3)$ and $c_o \in \mathbb{R}$ such that

$$c_o = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_o(\gamma(t)) > 0,$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = e\} \ne \emptyset. \tag{4.5}$$

Since $f_{\lambda,n}$ satisfies conditions (H1)–(H5) of Proposition 3.4, for every $\lambda > 0$ and $n \in \mathbb{N}$, and V satisfies (V0)–(V1), the problem (4.4) has a positive solution such that $u_{\lambda,n} \in H^1(\mathbb{R}^3)$ and

$$\|u_{\lambda,n}\|^2 \le c_{\lambda,n}\alpha/(\alpha - \sigma)$$

where $c_{\lambda,n} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{\lambda,n}(\gamma(t))$ and Γ is defined by (4.5) and is independent of λ and n. In fact, since $g(s) \geq 0$ for all s, we have $F_{\lambda,n}(s) \geq F_o(s)$. Hence

$$I_{\lambda,n}(v) \leq I_o(v), \quad \text{for all } v \in H^1(\mathbb{R}^3). \tag{4.6}$$

In particular, $I_{\lambda,n}(e) \leq I_o(e) < 0$. Thus, Γ is independent of λ and n. Moreover, from (4.6), we have

$$c_{\lambda,n} \leq c_o. \tag{4.7}$$

5. Proof of Theorem 1.1

Our proof consists in finding n, λ and a positive solution u of (4.3) such that $u(x) \leq M_n$, for all $x \in \mathbb{R}^3$. It is immediate that u and $\phi = \phi_u$ solve problem (1.2).

For $k = k(q, 4c_o\alpha/(\alpha - \sigma))$ given by Proposition 2.1, we fix n such that $M_n^2 > k$. Let $\lambda_o > 0$ be such that $\lambda_o g(M_n)M_n \leq 1$. From (4.2), we have

$$|f_{\lambda,n}(s)| \leq 2|s|^{q-1}, \quad \forall s.$$

By Proposition 3.4, there exists a solution $u = u_{\lambda,n}$ of (4.3) such that $\|u\|^2 \leq 4c_{\lambda,n}\alpha/(\alpha - \sigma)$. Combining this inequality with (4.7), yields

$$\|u\|^2 \leq 4c_{\lambda,n}\alpha/(\alpha - \sigma) \leq 4c_o\alpha/(\alpha - \sigma).$$

Invoking Lemma 3.1(ii), we conclude that $\phi_u \geq 0$ and consequently $V(x) + \phi_u$ is a non-negative function. Using Proposition 2.1, with $b(x) = V(x) + \phi_u$, we obtain that $\|u\|_\infty \leq K$, for some $K = K(q, c_o) > 0$, and the proof is complete.

Acknowledgments. This work was done while M. A. S. Souto was visiting the Universidade de São Paulo at São Carlos. He thanks all the faculty and staff of the Department of Mathematics for their support and kind hospitality.

We are grateful to the anonymous referee for a number of helpful comments that improved this article and also to Professor Alfonso Castro for your attention with us.

References

[1] Alves, C. O.; Souto, M. A. S.; On existence of solution for a class of semilinear elliptic equations wit nonlinearities that lies between two different powers, Abst. and Appl. Analysis (2008), 1–7.

[2] Ambrosetti, A.; Ruiz, D.; Multiple bound states for the Schrödinger-Poisson problem, Commun. Contemp. Math. 10 (2008), 391–404.

[3] Azzollini, A.; Pomponio, A.; Ground state solutions for the nonlinear Schrödinger-Maxwell equations, J. Math. Anal. Appl. 345 (2008), 90–108.

[4] Benci, V.; Fortunato, D.; An eigenvalue problem for the Schrödinger-Maxwell equations, Top. Meth. Nonlinear Anal. 11 (1998), 283–293.

[5] Cerami, G.; Vaira, G.; Positive solutions for some non-autonomous Schrödinger-Poisson systems, J. Differential Equations 248 (2010), 521–543.

[6] Coclite, G. M.; A multiplicity result for the nonlinear Schrödinger-Maxwell equations, Commun. Appl. Anal. 7 (2003), 417–423.

[7] Coti-Zelati, V.; Rabinowitz, P. H.; Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^3$, Comm. Pure and Appl. Math 45 (1992), 1217–1269.

[8] D'Aprile, T.; Mugnai, D.; Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), 893–906

[9] D'Aprile, T.; Mugnai, D.; Non-existence results for the coupled Klein-Gordon-Maxwell equations, Adv. Nonlinear Stud. 4 (2004), 307–322.

[10] d'Avenia, P.; Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations, Adv. Nonlinear Stud. 2 (2002), 177–192.

[11] Ekeland, I.; Convexity Methods in Hamilton Mechanics, Springer Verlag, 1990.

[12] Gaetano, S.; Multiple positive solutions for a Schrödinger-Poisson-Slater system, J. Math. Analysis and Appl. 365, (2010) 288–299.
[13] Ianni, I.; Vaira, G.; On concentration of positive bound states for the Schrödinger-Poisson problem with potentials, Adv. Nonlinear Stud. 8 (2008), 573–595.
[14] Kikuchi, H.; On the existence of a solution for elliptic system related to the Maxwell-Schrödinger equations, Nonlinear Anal. 67 (2007), 1445–1456.
[15] Kikuchi, H.; Existence of standing waves for the nonlinear Schrödinger equation with double power nonlinearity and harmonic potential, Asymptotic analysis and singularities—elliptic and parabolic PDEs and related problems, Adv. Stud. Pure Math. 47 623–633, Math. Soc. Japan, Tokyo, (2007).
[16] Lions, P. L.; The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, Analyse Nonlinéaire, I (1984), 223–283.
[17] Ruiz, D.; The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Analysis 237 (2006), 655–674.
[18] Ruiz, D.; Gaetano, S.; A note on the Schrödinger-Poisson-Slater equation on bounded domains, Adv. Nonlinear Stud. 8 (2008), 179–190.
[19] Zhao, F.; Zhao, L.; Positive solutions for Schrödinger-Poisson equations with a critical exponent, Nonlinear Anal. 70 (2009), 2150–2164.

Claudianor O. Alves
Unidade Acadêmica de Matemática e Estatística, Universidade Federal de Campina Grande, 58109-970, Campina Grande - PB, Brazil
E-mail address: `coalves@dme.ufcg.edu.br`

Sérgio H. M. Soares
Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, 13560-970, São Carlos - SP, Brazil
E-mail address: `monari@icmc.usp.br`

Marco A. S. Souto
Unidade Acadêmica de Matemática e Estatística, Universidade Federal de Campina Grande, 58109-970, Campina Grande - PB, Brazil
E-mail address: `marco@dme.ufcg.edu.br, Phone +5583-3310-1501, Fax +5583-3310-1112`