Electronic Journal of Differential Equations, Vol. 2011(2011), No. 02, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INITIAL-VALUE PROBLEMS FOR FIRST-ORDER DIFFERENTIAL RECURRENCE EQUATIONS WITH AUTO-CONVOLUTION

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Abstract. A differential recurrence equation consists of a sequence of differential equations, from which must be determined by recurrence a sequence of unknown functions. In this article, we solve two initial-value problems for some new types of nonlinear (quadratic) first order homogeneous differential recurrence equations, namely with discrete auto-convolution and with combinatorial auto-convolution of the unknown functions. In both problems, all initial values form a geometric progression, but in the second problem the first initial value is exempted and has a prescribed form. Some preliminary results showing the importance of the initial conditions are obtained by reducing the differential recurrence equations to algebraic type. Final results about solving the considered initial value problems, are shown by mathematical induction. However, they can also be shown by changing the unknown functions, or by the generating function method. So in a remark, we give a proof of the first theorem by the generating function method. Different cases of first order differential recurrence equations and their solutions are presented, including those from a previous work. Applications of the equations considered here will be given in subsequent articles.

1. Introduction

We consider *first-order differential recurrence equations* of the form

$$G_n\big(x_n'(t)\big), x_n(t), x_{n-1}(t), \dots, x_0(t)\big) = 0, \quad n = 0, 1, 2, \dots,$$

with unknowns $x_0(t), x_1(t), \dots, x_n(t), \dots$, complex-valued differentiable functions defined on an open interval I of real numbers, the functions G_n being given. For $t_0 \in I$, is called *Cauchy initial-value problem* for such an equation, the determination of its solutions $x_n(t)$, with given initial values $x_n(t_0)$, $n = 0, 1, 2, \dots$.

In this article we solve some initial-value problems for *first order homogeneous differential recurrence equations with (discrete) auto-convolution*

$$x_n'(t) = a(t)\sum_{k=0}^{n} x_k(t)x_{n-k}(t), \quad \forall t \in I,\ n = 0, 1, 2, \dots, \tag{1.1}$$

2000 *Mathematics Subject Classification.* 11B37, 34A12.
Key words and phrases. Differential recurrence equations; discrete auto-convolution; algebraic recurrence equations.

Submitted September 10, 2010. Published January 4, 2011.

and *with combinatorial auto-convolution*

$$y_n'(t) = a(t)\sum_{k=0}^{n}\binom{n}{k}y_k(t)y_{n-k}(t), \quad \forall t \in I,\ n = 0,1,2,\dots, \tag{1.2}$$

where $a(t) \not\equiv 0$ is a given integrable function.

The second type of equation reduces to the first by change of unknown functions $y_n(t) = n!x_n(t)$, $n = 0,1,2,\dots$.

Equations (1.1) and (1.2) are considered for the first time, they being different from those verified by special functions, from de Branges equation (see [8]) or other types of differential recurrence equations, studied so far (see, for example, [6, 7]).

The discret convolution or Cauchy product of numerical sequences and its inverse-discrete deconvolution (see [5] for these definitions) were formely used by the author in a series of papers [1, 2, 3] for solving numerical difference, differential and polynomial equations.

In the following we denote

$$A(t) = \int a(t)\mathrm{d}t, \quad B(t) = 1 + x_0(t)A(t_0) - x_0(t_0)A(t) \neq 0, \quad \forall t \in I\,. \tag{1.3}$$

Obviously,

$$B(t_0) = 1, \quad B'(t) = -x_0(t_0)a(t), \quad \forall t \in I. \tag{1.4}$$

2. Algebraic recurrence equation with auto-convolution

Lemma 2.1. *Let $b_n \neq 0$, $n = 0,1,2,\dots$, be a sequence of real or complex numbers. Following statements are equivalent:*

(i) $(n+1)b_0b_n = \sum_{k=0}^{n} b_kb_{n-k}$, $n = 0,1,2,\dots$;
(ii) $(n-1)b_0b_n = \sum_{k=1}^{n-1} b_kb_{n-k}$, $n = 2,3,\dots$;
(iii) $b_n = \frac{b_1}{b_0}b_{n-1}$, $n = 2,3,\dots$;
(iv) *The numbers b_n, $n = 0,1,2,\dots$, are in geometric progression, namely*

$$b_n = b_0\Big[\frac{b_1}{b_0}\Big]^n = \frac{b_1^n}{b_0^{n-1}}, \quad n = 0,1,2,\dots. \tag{2.1}$$

Proof. $(i) \Leftrightarrow (ii)$ and $(iii) \Leftrightarrow (iv)$ are obvious.

$(iv) \Rightarrow (i)$ If the sequence b_n is given by formula (2.1), we have

$$\sum_{k=0}^{n} b_kb_{n-k} = \sum_{k=0}^{n}\frac{b_1^k}{b_0^{k-1}}\frac{b_1^{n-k}}{b_0^{n-k-1}} = \sum_{k=0}^{n}\frac{b_1^n}{b_0^{n-2}} = (n+1)\frac{b_1^n}{b_0^{n-2}} = (n+1)b_0b_n,$$

$n = 0,1,2,\dots$, hence the sequence b_n satisfies (i) in Lemma 2.1.

$(ii) \Rightarrow (iv)$ (*First proof by induction*) For $n = 2$, from (ii) we obtain $b_2 = \frac{b_1^2}{b_0}$. For $n \geq 2$, we suppose that $b_k = \frac{b_1^k}{b_0^{k-1}}$, for $k = 0,1,\dots,n-1$. Then from (ii) results

$$\begin{aligned} b_n &= \frac{1}{(n-1)b_0}\sum_{k=1}^{n-1} b_kb_{n-k} = \frac{1}{(n-1)b_0}\sum_{k=1}^{n-1}\frac{b_1^k}{b_0^{k-1}}\frac{b_1^{n-k}}{b_0^{n-k-1}} \\ &= \frac{1}{(n-1)b_0}\sum_{k=1}^{n-1}\frac{b_1^n}{b_0^{n-2}} = \frac{1}{(n-1)b_0}(n-1)\frac{b_1^n}{b_0^{n-2}} = \frac{b_1^n}{b_0^{n-1}}. \end{aligned}$$

In conformity with induction axiom, formula (2.1) is true for every $n = 0,1,2,\dots$.

(i) ⇒ *(iv)* (*Second proof by the generating function method,* [9]) Denoting $G(z) = \sum_{n=0}^{\infty} b_n z^n$ generating function of the numerical sequence b_n, given by a formal series, item (i) in Lemma 2.1 gives a differential equation with the successive forms

$$b_0 \big[zG(z)\big]' = G^2(z), \quad b_0 zG'(z) + b_0 G(z) = G^2(z),$$
$$\frac{b_0 G'(z)}{G(z)\big[G(z) - b_0\big]} = \frac{1}{z}, \quad \frac{G'(z)}{G(z) - b_0} - \frac{G'(z)}{G(z)} = \frac{1}{z}.$$

Integrating results in $\ln\big|\frac{G(z)-b_0}{G(z)}\big| = \ln\big|\widetilde{k}z\big|$, hence $\frac{G(z)-b_0}{G(z)} = kz$, so $G(z) = \frac{b_0}{1-kz} = b_0 \sum_{k=0}^{\infty} k^n z^n$, where $\widetilde{k}$ and $k = \pm\widetilde{k}$ are arbitrary constants. It results $b_n = b_0 k^n$, $n = 0, 1, 2, \dots$. For $n = 1$, we have $b_1 = b_0 k$, hence $k = \frac{b_1}{b_0}$ and $b_n = \frac{b_1^n}{b_0^{n-1}}$, $n = 0, 1, 2, \dots$. □

Corollary 2.2. *For $a \neq 0$ a given number, the sequence $b_n \neq 0$, $n = 2, 3, \dots$, is solution of the equation*

$$(n-1) b_0 b_n = a \sum_{k=1}^{n-1} b_k b_{n-k}, \; n = 2, 3, \dots, \tag{2.2}$$

if and only if

$$b_n = \frac{a^{n-1} b_1^n}{b_0^{n-1}}, \; n = 1, 2, \dots. \tag{2.3}$$

Proof. Making the change of variables $b_0 = a\widetilde{b}_0$, $b_n = \widetilde{b}_n$, $n = 1, 2, \dots$, the equation (2.2) reduces to $(n-1)\widetilde{b}_0 \widetilde{b}_n = \sum_{k=1}^{n-1} \widetilde{b}_k \widetilde{b}_{n-k}$, $n = 2, 3, \dots$. In conformity with Lemma 2.1, we have $b_n = \widetilde{b}_n = \frac{\widetilde{b}_1^n}{\widetilde{b}_0^{n-1}} = \frac{a^{n-1} b_1^n}{b_0^{n-1}}$, $n = 1, 2, \dots$. □

Corollary 2.3. *The sequence $b_n \neq 0$, $n = 0, 1, 2, \dots$, is solution of the equation*

$$(n+1) b_0 b_n = \sum_{k=0}^{n} \binom{n}{k} b_k b_{n-k}, \quad n = 0, 1, 2, \dots,$$

if and only if

$$b_n = \frac{n! b_1^n}{b_0^{n-1}}, \quad n = 0, 1, 2, \dots.$$

Proof. Making the change of variables $b_n = n!\widetilde{b}_n$, $n = 0, 1, 2, \dots$, the considered equation reduces to $(n+1)\widetilde{b}_0 \widetilde{b}_n = \sum_{k=0}^{n} \widetilde{b}_k \widetilde{b}_{n-k}$, $n = 0, 1, 2, \dots$, and Corollary 2.3 follows from Lemma 2.1. □

Remark. Other results, related to those of Lemma 2.1 and its Corollary 2.3, were given in [4, Theorem 1.1 and Corollary 1.2] .

3. First initial-value problem

Lemma 3.1. *The functions*

$$x_n(t) = \frac{x_n(t_0)}{B^{n+1}(t)}, \quad \forall t \in I, \; n = 0, 1, 2, \dots, \tag{3.1}$$

are solutions of (1.1) *if and only if their initial values $x_n(t_0) \neq 0$ $n = 0, 1, 2, \dots$, are in geometric progression.*

Proof. If the functions $x_n(t)$ are given by formula (3.1), the equation (1.1) takes successively the form

$$-\frac{(n+1)x_n(t_0)B'(t)}{B^{n+2}(t)} = a(t)\sum_{k=0}^{n}\frac{x_k(t_0)}{B^{k+1}(t)}\frac{x_{n-k}(t_0)}{B^{n-k+1}(t)},$$

$$\frac{(n+1)x_n(t_0)x_0(t_0)a(t)}{B^{n+2}(t)} = \frac{a(t)}{B^{n+2}(t)}\sum_{k=0}^{n}x_k(t_0)x_{n-k}(t_0),$$

$$(n+1)x_0(t_0)x_n(t_0) = \sum_{k=0}^{n}x_k(t_0)x_{n-k}(t_0)\,.$$

In conformity with Lemma 2.1, the last equality is true if and only if the values $x_n(t_0)$, $n=0,1,2,\dots$, are in geometric progression. □

Theorem 3.2. *The differentiable functions $x_n(t)$, $n=0,1,2,\dots$, with initial values $x_n(t_0)\neq 0$, $n=0,1,2,\dots$, in geometric progression, are solutions of* (1.1) *if and only if they are given by* (3.1).

Proof. We suppose that the functions $x_n(t)$ are solutions of (1.1) and have their initial values $x_n(t_0)$, $n=0,1,2,\dots$, are in geometric progression, hence

$$x_n(t_0) = x_0(t_0)\Big[\frac{x_1(t_0)}{x_0(t_0)}\Big]^n = \frac{x_1^n(t_0)}{x_0^{n-1}(t_0)},\ n=0,1,2,\dots\,. \tag{3.2}$$

For $n=0$, the equation (1.1) has the form $x_0'(t)=a(t)x_0^2(t)$, hence $\frac{x_0'(t)}{x_0^2(t)}=a(t)$. By integration, we obtain $-\frac{1}{x_0(t)}=A(t)+C_0$, hence $x_0(t)=-\frac{1}{A(t)+C_0}$, where C_0 is an arbitrary constant. For $t=t_0$, it results in $C_0=-\frac{1+x_0(t_0)A(t_0)}{x_0(t_0)}$; therefore,

$$x_0(t) = -\frac{1}{A(t)-\frac{1+x_0(t_0)A(t_0)}{x_0(t_0)}} = \frac{x_0(t_0)}{1+x_0(t_0)A(t_0)-x_0(t_0)A(t)}\,.$$

In conformity with (1.3),

$$x_0(t) = \frac{x_0(t_0)}{B(t)}\,. \tag{3.3}$$

For $n=1$, the equation (1.1) has the form $x_1'(t)=2a(t)x_0(t)x_1(t)$, hence

$$\frac{x_1'(t)}{x_1(t)} = -2\frac{B'(t)}{B(t)}.$$

By integration, we obtain $x_1(t)=\frac{C_1}{B^2(t)}$, with C_1 an arbitrary constant. For $t=t_0$, it results $C_1=x_1(t_0)$, hence

$$x_1(t) = \frac{x_1(t_0)}{B^2(t)}\,. \tag{3.4}$$

For $n\geq 2$, equation (1.1) has the form

$$x_n'(t) = 2a(t)x_0(t)x_n(t) + a(t)\sum_{k=1}^{n-1}x_k(t)x_{n-k}(t),$$

hence, using the relation $a(t)=-\frac{B'(t)}{x_0(t_0)}$, obtained from (1.4),

$$x_n'(t) + 2\frac{B'(t)}{B(t)}x_n(t) = a(t)\sum_{k=1}^{n-1}x_k(t)x_{n-k}(t),$$

with general solution

$$\begin{aligned} x_n(t) &= \exp\Big(-2\int \frac{B'(t)}{B(t)}\mathrm{d}t\Big)\Big[\int \exp\Big(2\int \frac{B'(t)}{B(t)}\mathrm{d}t\Big)a(t)\sum_{k=1}^{n-1} x_n(t)x_{n-k}(t)\mathrm{d}t + C_n\Big] \\ &= \frac{1}{B^2(t)}\Big[C_n - \frac{1}{x_0(t_0)}\int B^2(t)B'(t)\sum_{k=1}^{n-1} x_k(t)x_{n-k}(t)\mathrm{d}t\Big], \end{aligned} \tag{3.5}$$

where C_n is an arbitrary constant.

For $n = 2$, from (3.4) and (3.5), it results

$$\begin{aligned} x_2(t) &= \frac{1}{B^2(t)}\Big[C_2 - \frac{1}{x_0(t_0)}\int B^2(t)B'(t)x_1^2(t)\mathrm{d}t\Big] \\ &= \frac{1}{B^2(t)}\Big[C_2 - \frac{x_1^2(t_0)}{x_0(t_0)}\int \frac{B'(t)}{B^2(t)}\mathrm{d}t\Big] = \frac{1}{B^2(t)}\Big[C_2 + \frac{x_1^2(t_0)}{x_0(t_0)B(t)}\Big]. \end{aligned} \tag{3.6}$$

From (3.6) for $t = t_0$ and (3.2) for $n = 2$, it results

$$x_2(t_0) = C_2 + \frac{x_1^2(t_0)}{x_0(t_0)} = \frac{x_1^2(t_0)}{x_0(t_0)},$$

hence $C_2 = 0$, and (3.6) becomes

$$x_2(t) = \frac{x_1^2(t_0)}{x_0(t_0)B^3(t)}. \tag{3.7}$$

For $n \geq 2$ fixed and $k = 0, 1, 2, \dots, n-1$, we suppose that

$$x_k(t) = \frac{x_1^k(t_0)}{x_0^{k-1}(t_0)B^{k+1}(t)}\,.$$

Then

$$\begin{aligned} \sum_{k=1}^{n-1} x_k(t)x_{n-k}(t) &= \sum_{k=1}^{n-1} \frac{x_1^k(t_0)}{x_0^{k-1}(t_0)B^{k+1}(t)}\frac{x_1^{n-k}(t_0)}{x_0^{n-k-1}(t_0)B^{n-k+1}(t)} \\ &= \sum_{k=1}^{n-1} \frac{x_1^n(t_0)}{x_0^{n-2}(t_0)B^{n+2}(t)} = \frac{(n-1)x_1^n(t_0)}{x_0^{n-2}(t_0)B^{n+2}(t)} \end{aligned}$$

and (3.5) becomes

$$\begin{aligned} x_n(t) &= \frac{1}{B^2(t)}\Big[C_n - \frac{(n-1)x_1^n(t_0)}{x_0^{n-1}(t_0)}\int \frac{B'(t)}{B^n(t)}\mathrm{d}t\Big] \\ &= \frac{1}{B^2(t)}\Big[C_n + \frac{x_1^n(t_0)}{x_0^{n-1}(t_0)B^{n-1}(t)}\Big]\,. \end{aligned} \tag{3.8}$$

From which, for $t = t_0$ and (3.2), results $x_n(t_0) = C_n + \frac{x_1^n(t_0)}{x_0^{n-1}(t_0)} = \frac{x_1^n(t_0)}{x_0^{n-1}(t_0)}$, hence $C_n = 0$, and (3.8) becomes

$$x_n(t) = \frac{x_1^n(t_0)}{x_0^{n-1}(t_0)B^{n+1}(t)}, \quad n = 0, 1, 2, \dots\,. \tag{3.9}$$

According to induction axiom, (3.9) is satisfied for any natural number n. From (3.2) and (3.9), it results (3.1).

Reciprocally, if the functions $x_n(t)$ are given by (3.1), with the initial values in geometric progression, then we have (3.9), hence

$$\begin{aligned} x_n'(t) &= -\frac{(n+1)x_1^n(t_0)B'(t)}{x_0^{n-1}(t_0)B^{n+2}(t)} = \frac{(n+1)x_1^n(t_0)a(t)}{x_0^{n-2}(t_0)B^{n+2}(t)} = a(t)\sum_{k=0}^{n}\frac{x_1^n(t_0)}{x_0^{n-2}(t_0)B^{n+2}(t)} \\ &= a(t)\sum_{k=0}^{n}\frac{x_1^k(t_0)}{x_0^{k-1}(t_0)B^{k+1}(t)}\frac{x_1^{n-k}(t_0)}{x_0^{n-k-1}(t_0)B^{n-k+1}(t)} = a(t)\sum_{k=0}^{n}x_k(t)x_{n-k}(t); \end{aligned}$$

therefore, the functions $x_n(t)$ satisfy (1.1). This also results by Lemma 3.1. □

Remark. Theorem 3.2 can also be demonstrated using the generating function $G(t,z)=\sum_{n=0}^{\infty}x_n(t)z^n$, of the sequence of functions $x_n(t)$, $n=0,1,2,\dots$, given by a formal series. Then (1.1) is equivalent to equation $\frac{\partial}{\partial t}G(t,z)=a(t)G^2(t,z)$, with solution $G(t,z)=\frac{1}{u(z)-A(t)}$, where $u(z)$ is an arbitrary function. For $z=0$, we obtain $x_0(t)=G(t,0)=\frac{1}{u(0)-A(t)}$. Let $v(z)=u(0)-u(z)$, with $v(0)=0$. Using geometric series, we have

$$\begin{aligned} G(t,z) &= \frac{1}{u(0)-A(t)-v(z)} \\ &= \frac{1}{u(0)-A(t)}\frac{1}{1-\frac{v(z)}{u(0)-A(t)}} = \sum_{n=0}^{\infty}\frac{v^n(z)}{\big(u(0)-A(t)\big)^{n+1}}\,. \end{aligned}$$

The choice $v(z)=Cz$, where C is an arbitrary constant, gives

$$G(t,z)=\sum_{n=0}^{\infty}\frac{C^n}{\big(u(0)-A(t)\big)^{n+1}}z^n,$$

hence

$$x_n(t)=\frac{C^n}{\big(u(0)-A(t)\big)^{n+1}},\quad n=0,1,2,\dots. \tag{3.10}$$

For $t=t_0$ and $n=0$, from (3.10) it results $x_0(t_0)=\frac{1}{u(0)-A(t_0)}$, hence $u(0)=\frac{1+x_0(t_0)A(t_0)}{x_0(t_0)}$, and

$$u(0)-A(t)=\frac{1+x_0(t_0)A(t_0)}{x_0(t_0)}-A(t)=\frac{B(t)}{x_0(t_0)}. \tag{3.11}$$

For $t=t_0$ and $n=1$, from (3.10) results $x_1(t_0)=\frac{C}{\big(u(0)-A(t_0)\big)^2}=Cx_0^2(t_0)$, hence

$$C=\frac{x_1(t_0)}{x_0^2(t_0)}. \tag{3.12}$$

From (3.10), (3.11) and (3.12) it results

$$x_n(t)=\frac{x_1^n(t_0)}{x_0^{2n}(t_0)}\frac{x_0^{n+1}(t_0)}{B^{n+1}(t)}=\frac{x_1^n(t_0)}{x_0^{n-1}(t_0)B^{n+1}(t)},\quad n=0,1,2,\dots. \tag{3.13}$$

For $t=t_0$, from (3.13) we obtain (3.2), hence (3.13) takes the form (3.1) and the initial values $x_n(t_0)$, $n=0,1,2,\dots$, are in geometric progression. This last statement also follows by Lemma 3.1. The reciprocal affirmation results both by direct calculation as above or by Lemma 3.1.

Corollary 3.3. *The differentiable functions $y_n(t)$, $n = 0, 1, 2, \dots$, whose initial values are*

$$y_n(t_0) = \frac{n! y_1^n(t_0)}{y_0^{n-1}(t_0)}, \quad n = 0, 1, 2, \dots,$$

are solutions of the differential recurrence equation with combinatorial auto-convolution (1.2) *if and only if they are given by*

$$y_n(t) = \frac{y_n(t_0)}{B^{n+1}(t)}, \quad \forall t \in I, \ n = 0, 1, 2, \dots .$$

4. Second initial-value problem

Lemma 4.1. *If $A(t_0) \neq 0$ and*

$$x_0(t_0) = \frac{x_1^2(t_0)}{x_2(t_0)} - \frac{1}{A(t_0)} \neq 0, \tag{4.1}$$

then the functions

$$x_0(t) = \frac{x_0(t_0)}{B(t)}, \quad x_n(t) = \frac{x_n(t_0) A^{n-1}(t)}{A^{n-1}(t_0) B^{n+1}(t)}, \quad \forall t \in I, \ n = 1, 2, \dots . \tag{4.2}$$

are solutions of (1.1) *if and only if the initial values $x_n(t_0) \neq 0$, $n = 1, 2, \dots$, are in geometric progression.*

Proof. If the functions $x_n(t)$ are given by formula (4.2), then (1.1) is obviously satisfied for $n = 0$ and $n = 1$. For $n = 2, 3, \dots$, it takes successively the form

$$\begin{aligned}
&\frac{x_n(t_0)}{A^{n-1}(t_0)} \frac{(n-1)A^{n-2}(t)A'(t)B^{n+1}(t) - (n+1)B^n(t)B'(t)A^{n-1}(t)}{B^{2n+2}(t)} \\
&= \frac{2a(t)x_0(t_0)x_n(t_0)A^{n-1}(t)}{A^{n-1}(t_0)B^{n+2}(t)} \\
&\quad + a(t) \sum_{k=1}^{n-1} \frac{x_k(t_0)A^{k-1}(t)}{A^{k-1}(t_0)B^{k+1}(t)} \frac{x_{n-k}(t_0)A^{n-k-1}(t)}{A^{n-k-1}(t_0)B^{n-k+1}(t)},
\end{aligned}$$

$$\begin{aligned}
&\frac{x_n(t_0)}{A^{n-1}(t_0)} \frac{(n-1)A^{n-2}(t)a(t)B^{n+1}(t) + (n+1)x_0(t_0)B^n(t)a(t)A^{n-1}(t)}{B^{2n+2}(t)} \\
&= \frac{2a(t)x_0(t_0)x_n(t_0)A^{n-1}(t)}{A^{n-1}(t_0)B^{n+2}(t)} + \frac{a(t)A^{n-2}(t)}{A^{n-2}(t_0)B^{n+2}(t)} \sum_{k=1}^{n-1} x_k(t_0)x_{n-k}(t_0),
\end{aligned}$$

$$\begin{aligned}
&\frac{x_n(t_0)}{A(t_0)} \big[(n-1)B(t) + (n+1)x_0(t_0)A(t)\big] \\
&= \frac{2x_0(t_0)x_n(t_0)A(t)}{A(t_0)} + \sum_{k=1}^{n-1} x_k(t_0)x_{n-k}(t_0),
\end{aligned}$$

$$(n-1)\frac{x_n(t_0)}{A(t_0)} \big[B(t) + x_0(t_0)A(t)\big] = \sum_{k=1}^{n-1} x_k(t_0)x_{n-k}(t_0).$$

From (1.4) and (4.1) it results

$$B(t) + x_0(t_0)A(t) = 1 + x_0(t_0)A(t_0) = \frac{x_1^2(t_0)A(t_0)}{x_2(t_0)}, \quad \forall t \in I; \tag{4.3}$$

therefore, (1.1) is still equivalent to the relations

$$(n-1)\frac{x_n(t_0)}{A(t_0)}\frac{x_1^2(t_0)A(t_0)}{x_2(t_0)} = \sum_{k=1}^{n-1} x_k(t_0)x_{n-k}(t_0),$$

$$(n-1)x_0(t_0)x_n(t_0) = \frac{x_0(t_0)x_2(t_0)}{x_1^2(t_0)}\sum_{k=1}^{n-1} x_k(t_0)x_{n-k}(t_0).$$

By Corollary 2.2, for $b_n = x_n(t_0)$, $n = 0,1,2,\dots$ and $a = \frac{x_0(t_0)x_2(t_0)}{x_1^2(t_0)}$, the last equality is equivalent to the relation

$$x_n(t_0) = \frac{x_0^{n-1}(t_0)x_2^{n-1}(t_0)}{x_1^{2n-2}(t_0)}\frac{x_1^n(t_0)}{x_0^{n-1}(t_0)} = \frac{x_2^{n-1}(t_0)}{x_1^{n-2}(t_0)} = x_1(t_0)\Big[\frac{x_2(t_0)}{x_1(t_0)}\Big]^{n-1},$$

where $n = 1,2,\dots$; hence with the fact that the initial values $x_n(t_0)$, $n = 1,2,\dots$, are in geometric progression. □

Theorem 4.2. *If $A(t_0) \neq 0$, while $x_0(t_0)$ is given by* (4.1)*, then the differentiable functions $x_n(t)$, $n = 0,1,2,\dots$, with initial values $x_n(t_0)$, $n = 1,2,\dots$, in geometric progression, are solutions of* (1.1) *if and only if they are given by* (4.2)*.*

Proof. We suppose that the functions $x_n(t)$, $n = 0,1,2,\dots$, are solutions of (1.1) and the initial values $x_n(t_0)$, $n = 1,2,\dots$, are in geometric progression, hence

$$x_n(t_0) = x_1(t_0)\Big[\frac{x_2(t_0)}{x_1(t_0)}\Big]^{n-1} = \frac{x_2^{n-1}(t_0)}{x_1^{n-2}(t_0)}, \quad n = 1,2,\dots. \tag{4.4}$$

As was shown in the proof of Theorem 3.2, the functions $x_0(t), x_1(t), x_n(t)$ for $n \geq 2$, and $x_2(t)$ are given by (3.3), (3.4), (3.5) and (3.6). From (3.6) and (1.3) we have

$$\begin{aligned} x_2(t) &= \frac{C_2x_0(t_0)B(t) + x_1^2(t_0)}{x_0(t_0)B^3(t)} \\ &= \frac{C_2x_0(t_0)\big[1 + x_0(t_0)A(t_0)\big] - C_2x_0^2(t_0)A(t) + x_1^2(t_0)}{x_0(t_0)B^3(t)}. \end{aligned} \tag{4.5}$$

We take

$$C_2 = -\frac{x_1^2(t_0)}{x_0(t_0)\big[1 + x_0(t_0)A(t_0)\big]} = -\frac{x_2(t_0)}{x_0(t_0)A(t_0)}. \tag{4.6}$$

the above equality resulting from (4.3), which in turn resulted from (4.1). From (4.5) and (4.6) it results

$$x_2(t) = -\frac{C_2x_0(t_0)A(t)}{B^3(t)} = \frac{x_2(t_0)A(t)}{A(t_0)B^3(t)}. \tag{4.7}$$

From (3.5), for $n = 3$, (3.4) and (4.7), it results

$$\begin{aligned} x_3(t) &= \frac{1}{B^2(t)}\Big[C_3 - \frac{2}{x_0(t_0)}\int B^2(t)B'(t)x_1(t)x_2(t)dt\Big] \\ &= \frac{1}{B^2(t)}\Big[C_3 - \frac{2x_1(t_0)x_2(t_0)}{x_0(t_0)A(t_0)}\int \frac{A(t)B'(t)}{B^3(t)}dt\Big]. \end{aligned} \tag{4.8}$$

Using (1.3) and (1.4), we obtain $A'(t) = a(t) = -\frac{B'(t)}{x_0(t_0)}$; hence

$$\Big[\frac{A(t)}{B(t)}\Big]' = \frac{A'(t)B(t) - A(t)B'(t)}{B^2(t)} = -\frac{[B(t) + x_0(t_0)A(t)]B'(t)}{x_0(t_0)B^2(t)}. \tag{4.9}$$

Using again (4.3), formula (4.9) takes the form

$$\Big[\frac{A(t)}{B(t)}\Big]' = -\frac{x_1^2(t_0)A(t_0)B'(t)}{x_0(t_0)x_2(t_0)B^2(t)},$$

from which we obtain

$$\frac{B'(t)}{B^2(t)} = -\frac{x_0(t_0)x_2(t_0)}{x_1^2(t_0)A(t_0)}\Big[\frac{A(t)}{B(t)}\Big]'. \tag{4.10}$$

From (4.8) and (4.10), it results

$$\begin{aligned} x_3(t) &= \frac{1}{B^2(t)}\Big[C_3 + \frac{2x_2^2(t_0)}{x_1(t_0)A^2(t_0)}\int \frac{A(t)}{B(t)}\Big[\frac{A(t)}{B(t)}\Big]' dt\Big] \\ &= \frac{1}{B^2(t)}\Big[C_3 + \frac{x_2^2(t_0)A^2(t)}{x_1(t_0)A^2(t_0)B^2(t)}\Big]. \end{aligned} \tag{4.11}$$

From which for $t = t_0$ and (4.4) for $n = 3$, it results

$$x_3(t_0) = C_3 + \frac{x_2^2(t_0)}{x_1(t_0)} = \frac{x_2^2(t_0)}{x_1(t_0)},$$

hence $C_3 = 0$ and formula (4.11) becomes

$$x_3(t) = \frac{x_2^2(t_0)A^2(t)}{x_1(t_0)A^2(t_0)B^4(t)}. \tag{4.12}$$

For $n \geq 3$ fixed and $k = 1, 2, \dots, n-1$, we suppose that

$$x_k(t) = \frac{x_2^{k-1}(t_0)A^{k-1}(t)}{x_1^{k-2}(t_0)A^{k-1}(t_0)B^{k+1}(t)}. \tag{4.13}$$

Then

$$\begin{aligned} &\sum_{k=1}^{n-1} x_k(t)x_{n-k}(t) \\ &= \sum_{k=1}^{n-1} \frac{x_2^{k-1}(t_0)A^{k-1}(t)}{x_1^{k-2}(t_0)A^{k-1}(t_0)B^{k+1}(t)} \frac{x_2^{n-k-1}(t_0)A^{n-k-1}(t)}{x_1^{n-k-2}(t_0)A^{n-k-1}(t_0)B^{n-k+1}(t)} \\ &= \frac{(n-1)x_2^{n-2}(t_0)A^{n-2}(t)}{x_1^{n-4}(t_0)A^{n-2}(t_0)B^{n+2}(t)}. \end{aligned} \tag{4.14}$$

From (3.5) and (4.14) it results

$$x_n(t) = \frac{1}{B^2(t)}\Big[C_n - \frac{(n-1)x_2^{n-2}(t_0)}{x_0(t_0)x_1^{n-4}(t_0)A^{n-2}(t_0)}\int \frac{A^{n-2}(t)B'(t)}{B^n(t)}\mathrm{d}t\Big]. \tag{4.15}$$

From (4.10) and (4.15), it results

$$\begin{aligned} x_n(t) &= \frac{1}{B^2(t)}\Big[C_n + \frac{(n-1)x_2^{n-1}(t_0)}{x_1^{n-2}(t_0)A^{n-1}(t_0)}\int \Big[\frac{A(t)}{B(t)}\Big]^{n-2}\Big[\frac{A(t)}{B(t)}\Big]'\mathrm{d}t\Big] \\ &= \frac{1}{B^2(t)}\Big[C_n + \frac{x_2^{n-1}(t_0)A^{n-1}(t)}{x_1^{n-2}(t_0)A^{n-1}(t_0)B^{n-1}(t)}\Big]. \end{aligned} \tag{4.16}$$

From this euqality for $t = t_0$ and (4.4), it results $x_n(t_0) = C_n + \frac{x_2^{n-1}(t_0)}{x_1^{n-2}(t_0)} = \frac{x_2^{n-1}(t_0)}{x_1^{n-2}(t_0)}$, hence $C_n = 0$ and formula (4.16) becomes

$$x_n(t) = \frac{x_2^{n-1}(t_0)A^{n-1}(t)}{x_1^{n-2}(t_0)A^{n-1}(t_0)B^{n+1}(t)}. \tag{4.17}$$

In conformity with induction axiom, formula (4.17) is satisfied for every natural number $n \geq 1$. For $t = t_0$, from (4.17) it results $x_n(t_0) = \frac{x_2^{n-1}(t_0)}{x_1^{n-2}(t_0)}$, hence (4.17) reduces to second formula (4.2).

Reciprocally, if the functions $x_n(t)$, $n = 0, 1, 2, \dots$, are given by (4.2), and the initial values $x_n(t_0)$, $n = 1, 2, \dots$, are in geometric progression, then we have $x_0(t) = \frac{x_0(t_0)}{B(t)}$ and

$$x_n(t) = \frac{x_n(t_0)A^{n-1}(t)}{A^{n-1}(t_0)B^{n+1}(t)} = \frac{x_2^{n-1}(t_0)A^{n-1}(t)}{x_1^{n-2}(t_0)A^{n-1}(t_0)B^{n+1}(t)},$$

hence for $n = 1, 2, \dots$, using (4.3), we have

$$\begin{aligned}
&x_n'(t) - 2a(t)x_0(t)x_n(t)\\
&= \frac{x_n(t_0)\big[(n-1)A^{n-2}(t)A'(t)B^{n+1}(t) - (n+1)A^{n-1}(t)B^n(t)B'(t)\big]}{A^{n-1}(t_0)B^{2n+2}(t)}\\
&\quad - \frac{2x_0(t_0)x_n(t_0)a(t)A^{n-1}(t)}{A^{n-1}(t_0)B^{n+2}(t)}\\
&= \frac{x_n(t_0)a(t)A^{n-2}(t)}{A^{n-1}(t_0)B^{n+2}(t)}\big[(n-1)B(t) + (n+1)x_0(t_0)A(t) - 2x_0(t_0)A(t)\big]\\
&= \frac{(n-1)x_2^{n-1}(t_0)a(t)A^{n-2}(t)}{x_1^{n-2}(t_0)A^{n-1}(t_0)B^{n+2}(t)}\big[B(t) + x_0(t_0)A(t)\big]\\
&= \frac{(n-1)x_2^{n-2}(t_0)a(t)A^{n-2}(t)}{x_1^{n-4}(t_0)A^{n-2}(t_0)B^{n+2}(t)}\\
&= a(t)\sum_{k=1}^{n-1} \frac{x_2^{k-1}(t_0)A^{k-1}(t)}{x_1^{k-2}(t_0)A^{k-1}(t_0)B^{k+1}(t)} \frac{x_2^{n-k-1}(t_0)A^{n-k-1}(t)}{x_1^{n-k-2}(t_0)A^{n-k-1}(t_0)B^{n-k+1}(t)}\\
&= a(t)\sum_{k=1}^{n-1} x_k(t)x_{n-k}(t)\,;
\end{aligned}$$

therefore, the functions $x_n(t)$ given by (4.2) satisfy (1.1). This also results by Lemma 4.1. □

Remark. From (4.1) it results that in hypotheses of the Theorem 4.2, those of Theorem 3.2 are not satisfied. The solutions of the second initial values problem are different from those of the first problem, because in the proof of Theorem 3.2 all arbitrary constants C_n, $n \geq 2$, that arise in solving the differential equations are zero, while in the proof of Theorem 4.2, the constant C_2 is non-zero, given by formula (4.6).

Corollary 4.3. *If $A(t_0) \neq 0$, and $y_0(t_0) = \frac{y_1^2(t_0)}{y_2(t_0)} - \frac{1}{A(t_0)} \neq 0$, then the differentiable functions $y_n(t)$, $n = 0, 1, 2, \dots$, with $y_n(t_0) = \frac{n! y_2^{n-1}(t_0)}{y_1^{n-2}(t_0)}$, $n = 1, 2, \dots$, are solutions* *of the recurrence equation with combinatorial auto-convolution* (1.2) *if and only if they are*

$$y_0(t) = \frac{y_0(t_0)}{B(t)}, \quad y_n(t) = \frac{y_n(t_0)A^{n-1}(t)}{A^{n-1}(t_0)B^{n+1}(t)}, \quad t \in I,\ n = 1, 2, \dots .$$

5. Examples

We give some examples that illustrate the above results, including those from [4].

(1) $x_n'(t) = \sum_{k=0}^{n} x_k(t)x_{n-k}(t)$, $n = 1, 2, \dots$. Here $a(t) = 1$, so $A(t) = t$.

(a) For the initial values $x_n(0) = 1$, $n = 0, 1, 2, \dots$, we have $B(t) = 1 + x_0(0)A(0) - x_0(0)A(t) = 1 - t$, hence we obtain the solutions $x_n(t) = \frac{x_n(0)}{B^{n+1}(t)} = \frac{1}{(1-t)^{n+1}}$, $n = 0, 1, 2, \dots$. Because, $A(0) = 0$, second initial values problem can not be considered.

(b) If $x_n(1) = \frac{1}{2^n}$, we obtain $x_n(t) = \frac{1}{2^n(2-t)^{n+1}}$, $n = 0, 1, 2, \dots$. The second initial values problem can not be considered when $x_n(1) = \frac{1}{2^n}$, $n = 1, 2, \dots$, because $x_0(1) = \frac{x_1^2(1)}{x_2(1)} - \frac{1}{A(t)} = 0$.

(c) If $x_n(1) = \frac{1}{2^{n-1}}$, $n = 0, 1, 2, \dots$, then $x_0(1) = 2$, $B(t) = 1 + x_0(t)A(t) - x_0(t)A(t) = 3 - 2t$, hence $x_n(t) = \frac{1}{2^{n-1}(3-2t)^{n+1}}$, $n = 0, 1, 2, \dots$. The second initial values problem when $x_n(t) = \frac{1}{2^{n-1}}$, $n = 1, 2, \dots$, can also be considered and will be given in the next example.

(d) Let $x_n(1) = \frac{1}{2^{n-1}}$, $n = 1, 2, \dots$, and $x_0(1) = \frac{x_1^2(1)}{x_2(1)} - \frac{1}{A(1)} = 1$. Then $B(t) = 2 - t$, hence $x_0(t) = \frac{x_0(1)}{B(t)} = \frac{1}{2-t}$ and $x_n(t) = \frac{x_n(1)A^{n-1}(t)}{A^{n-1}(1)B^{n+1}(t)} = \frac{t^{n-1}}{2^{n-1}(2-t)^{n+1}}$, $n = 1, 2, \dots$.

(2) $y_n'(t) = \sum_{k=0}^{n} \binom{n}{k} y_k(t)y_{n-k}(t)$, $n = 0, 1, 2, \dots$.

(a) If $y_n(0) = n!$, then $y_n(t) = \frac{n!}{(1-t)^{n+1}}$, $n = 0, 1, 2, \dots$.

(b) If $y_n(1) = \frac{n!}{2^n}$, then $y_n(t) = \frac{n!}{2^n(2-t)^{n+1}}$, $n = 0, 1, 2, \dots$.

(c) If $y_n(1) = \frac{n!}{2^{n-1}}$, then $y_n(t) = \frac{n!}{2^{n-1}(3-2t)^{n+1}}$, $n = 0, 1, 2, \dots$.

(d) If $y_0(1) = 1$ and $y_n(1) = \frac{n!}{2^{n-1}}$, $n = 1, 2, \dots$, then $y_0(t) = \frac{1}{2-t}$ and $y_n(t) = \frac{n!t^{n-1}}{2^{n-1}(2-t)^{n+1}}$, $n = 1, 2, \dots$.

(3) $x_n'(t) = e^t \sum_{k=o}^{n} x_k(t)x_{n-k}(t)$, $n = 0, 1, 2, \dots$. Here $a(t) = A(t) = e^t$.

If $x_n(0) = 1$, $n = 0, 1, 2, \dots$, we obtain the solutions $x_n(t) = \frac{1}{(2-e^t)^{n+1}}$, $n = 0, 1, 2, \dots$. The second problem can not be considered when $x_n(0) = 1$, $n = 1, 2, \dots$, because $x_0(0) = \frac{x_1^2(0)}{x_2(0)} - \frac{1}{A(0)} = 0$.

(4) $x_n'(t) = \sin t \sum_{k=0}^{n} x_k(t)x_{n-k}(t)$, $n = 0, 1, 2, \dots$. Here $A(t) = -\cos t$.

(a) Let $x_n(0) = 1$, $n = 0, 1, 2, \dots$. Then $B(t) = 1 + x_0(0)A(0) - x_0(0)A(t) = \cos t$, hence $x_n(t) = \frac{1}{\cos^{n+1} t}$, $n = 0, 1, 2, \dots$.

(b) If $x_n(0) = 1$, $n = 1, 2, \dots$, and $x_0(0) = \frac{x_1^2(0)}{x_2(0)} - \frac{1}{A(0)} = 2$, then $B(t) = 2\cos t - 1$, hence $x_0(t) = \frac{x_0(0)}{B(t)} = \frac{2}{2\cos t - 1}$ and

$$x_n(t) = \frac{x_n(0)A^{n-1}(t)}{A^{n-1}(0)B^{n+1}(t)} = \frac{\cos^{n-1} t}{(2\cos t - 1)^{n+1}}, \quad n = 1, 2, \dots .$$

(c) For $x_n\left(\frac{\pi}{2}\right) = 1$, $n = 0, 1, 2, \dots$, we obtain $x_n(t) = \frac{1}{(1+\cos t)^{n+1}}$, $n = 0, 1, 2, \dots$. The second problem can not be considered, because $A\left(\frac{\pi}{2}\right) = 0$.

(5) $x_n'(t) = \frac{1}{t}\sum_{k=0}^n x_k(t)x_{n-k}(t)$, $x_n(1) = 1$, $n = 0,1,2,\dots$. Then $x_n(t) = \frac{1}{(1-\ln t)^{n+1}}$, $n = 0,1,2,\dots$. The second problem can not be considered, because $A(1) = 0$.

(6) $x_n'(t) = \frac{1}{t^2}\sum_{k=0}^n x_k(t)x_{n-k}(t)$, $n = 0,1,2,\dots$. Here $a(t) = \frac{1}{t^2}$, so $A(t) = -\frac{1}{t}$.

(a) If $x_n(1) = 1$, then $B(t) = \frac{1}{t}$, hence $x_n(t) = t^{n+1}$, $n = 0,1,2,\dots$.

(b) Let $x_n(1) = 1$, $n = 1,2,\dots$, and $x_0(1) = \frac{x_1^2(1)}{x_2(1)} - \frac{1}{A(1)} = 2$. Then $B(t) = \frac{2-t}{t}$, hence the solutions are $x_0(t) = \frac{x_0(1)}{B(t)} = \frac{2t}{2-t}$, $x_n(t) = \frac{x_n(1)A^{n-1}(t)}{A^{n-1}(1)B^{n+1}(t)} = \frac{t^2}{(2-t)^{n+1}}$, $n = 1,2,\dots$.

(c) If $x_n(2) = (-1)^{n+1}4$, then $B(t) = \frac{3t-4}{t}$, hence $x_n(t) = \frac{(-1)^{n+1}4t^{n+1}}{(3t-4)^{n+1}}$, $n = 0,1,2,\dots$.

(d) Let $x_n(2) = (-1)^{n+1}4$, $n = 1,2,\dots$, $x_0(2) = \frac{x_1^2(2)}{x_2(2)} - \frac{1}{A(2)} = -2$. Then $B(t) = \frac{2(t-1)}{t}$, hence the solutions are $x_0(t) = \frac{t}{1-t}$, $x_n(t) = \frac{t^2}{(1-t)^{n+1}}$, $n = 1,2,\dots$.

(7) $y_n'(t) = \frac{1}{t^2}\sum_{k=0}^n \binom{n}{k} y_k(t)y_{n-k}(t)$, $n = 0,1,2,\dots$.

(a) If $y_n(1) = n!$, then $y_n(t) = n!t^{n+1}$, $n = 0,1,2,\dots$.

(b) If $y_0(1) = 2$, and $y_n(1) = n!$, $n = 1,2,\dots$, then $y_0(t) = \frac{2t}{2-t}$, and $y_n(t) = \frac{n!t^2}{(2-t)^{n+1}}$, $n = 1,2,\dots$.

(c) If $y_n(2) = (-1)^{n+1}n!4$, then $y_n(t) = \frac{(-1)^{n+1}n!4t^{n+1}}{(3t-4)^{n+1}}$, $n = 0,1,2,\dots$.

(d) If $y_0(2) = -2$, and $y_n(2) = (-1)^{n+1}n!4$, $n = 1,2,\dots$, then $y_0(t) = \frac{t}{1-t}$, and $y_n(t) = \frac{n!t^2}{(1-t)^{n+1}}$, $n = 1,2,\dots$.

(8) $tz_n'(t) + z_n(t) = \sum_{k=0}^n z_k(t)z_{n-k}(t)$, $n = 0,1,2,\dots$. We make the change of unknown functions $x_n(t) = tz_n(t)$, $n = 0,1,2,\dots$.

(a) If $z_n(1) = 1$, then $z_n(t) = t^n$, $n = 0,1,\dots$. This example was given in [4, Theorem 4.1 (a)].

(b) Let $z_0(1) = 2$, $z_n(1) = 1$, $n = 1,2,\dots$. Then $z_0(t) = \frac{2}{2-t}$, $z_n(t) = \frac{t}{(2-t)^{n+1}}$, $n = 1,2,\dots$.

(c) If $z_n(2) = (-1)^{n+1}2$, then $z_n(t) = \frac{(-1)^{n+1}4t^n}{(3t-4)^{n+1}}$, $n = 0,1,2,\dots$.

(d) Let $z_0(2) = -1$, and $z_n(2) = (-1)^{n+1}2$, $n = 1,2,\dots$. Then $z_0(t) = \frac{1}{1-t}$ and $z_n(t) = \frac{t}{(1-t)^{n+1}}$, $n = 1,2,\dots$. This example was given in [4, theorem 4.1 (b)].

(9) $tu_n'(t) + u_n(t) = \sum_{k=0}^n \binom{n}{k} u_k(t)u_{n-k}(t)$, $n = 0,1,2,\dots$.

(a) If $u_n(1) = n!$, $n = 0,1,2,\dots$. Then $u_n(t) = n!t^n$, $n = 0,1,2,\dots$. This example was given in [4, corollary of theorem 4.1 (a)].

(b) If $u_0(1) = 2$, $u_n(1) = n!$, $n = 1,2,\dots$, then $u_0(t) = \frac{2}{2-t}$, $u_n(t) = \frac{n!t}{(2-t)^{n+1}}$, $n = 1,2,\dots$.

(c) If $u_n(2) = (-1)^{n+1}n!2$, $n = 0,1,2,\dots$, then $u_n(t) = \frac{(-1)^{n+1}n!4t^n}{(3t-4)^{n+1}}$, $n = 0,1,2,\dots$.

(d) If $u_0(2) = -1$ and $u_n(2) = (-1)^{n+1}n!2$, $n = 1,2,\dots$, then $u_0(t) = \frac{1}{1-t}$, and $u_n(t) = \frac{n!t}{(1-t)^{n+1}}$, $n = 1,2,\dots$. This example was given in [4, corollary of theorem 4.1 (b)], with some mistakes corrected here.

Acknowledgments. The proof of the Theorem 3.2 by the generating function method, included in the remark in section 3, was suggested by the anonymous referee, to whom I want to give many thanks.

References

[1] M. Cîrnu; *Solving difference and differential equations by discrete deconvolution*, UPB Scientific Bulletin, Series A, vol. 69, no. 1, pp. 13-26, 2007.

[2] M. Cîrnu; *Determination of particular solutions of nonhomogeneous linear differential equations by discrete deconvolution*, UPB Scientific Bulletin, Series A, vol. 69, no. 2, pp. 3-16, 2007.

[3] M. Cîrnu; *Approximate calculus by deconvolution of the polynomial roots*, UPB Scientific Bulletin, Series A, no. 4, pp. 9-22, 2007.

[4] M. Cîrnu; *First order differential recurrence equations with discrete auto-convolution*, International Journal of Mathematics and Computation, vol. 4, no. S09, pp. 124-128, 2009.

[5] M. A. Cuenod; *Introduction a l'Analyse Impulsionelle. Principes et Applications*, Dunod, Paris, 1970.

[6] I. P. Dimitrov; *The stability of the solutions of differential-recurrence equations*, (Russian), Differencialnye Uravnenia, vol. 8 (1972), Part I, pp. 1168-1175, Part II, pp. 1372-1376.

[7] N. M. Flaisher; *A certain differential recurrence equation*, (Russian) Arch. Math., vol. 4 1968, pp. 237-239.

[8] W. Koepf, D. Schmersau; *Solution properties of the de Branges differential recurrence equation*, Complex Variable Theory and Applications, vol. 50 (2005), no. 7-11, pp. 585-595.

[9] H. S. Wilf; *Generating functionology*, Academic Press, 1990.

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