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SOLVABILITY OF DEGENERATED PARABOLIC EQUATIONS WITHOUT SIGN CONDITION AND THREE UNBOUNDED NONLINEARITIES

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Abstract. In this article, we study the problem

$$\begin{gathered}\frac{\partial}{\partial t}b(x,u)-\operatorname{div}(a(x,t,u,Du))+H(x,t,u,Du)=f\quad\text{in }\Omega\times]0,T[,\\ b(x,u)(t=0)=b(x,u_0)\quad\text{in }\Omega,\\ u=0\quad\text{in }\partial\Omega\times]0,T[\end{gathered}$$

in the framework of weighted Sobolev spaces, with $b(x,u)$ unbounded function on u. The main contribution of our work is to prove the existence of a renormalized solution without the sign condition and the coercivity condition on $H(x,t,u,Du)$. The critical growth condition on H is with respect to Du and no growth condition with respect to u. The second term f belongs to $L^1(Q)$, and $b(x,u_0)\in L^1(\Omega)$.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2<p<\infty$, $Q=\Omega\times[0,T]$ and $w=\{w_i(x):0\le i\le N\}$ be a vector of weight functions (i.e., every component $w_i(x)$ is a measurable almost everywhere strictly positive function on Ω), satisfying some integrability conditions (see Section 2). And let $Au=-\operatorname{div}(a(x,t,u,Du))$ be a Leray-Lions operator defined from the weighted Sobolev space $L^p(0,T;W_0^{1,p}(\Omega,w))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$.

Now, we consider the degenerated parabolic problem associated for the differential equation

$$\begin{gathered}\frac{\partial b(x,u)}{\partial t}+Au+H(x,t,u,Du)=f\quad\text{in }Q,\\ u=0\quad\text{on }\partial\Omega\times]0,T[,\\ b(x,u)(t=0)=b(x,u_0)\quad\text{on }\Omega\end{gathered}\tag{1.1}$$

where $b(x,u)$ is a unbounded function on u, H is a nonlinear lower order term. Problem (1.1) is studied in [2] with $f\in L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ and under the strong hypothesis relatively to H, more precisely they supposed that $b(x,u)=u$

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and the nonlinearity H satisfying the sign condition

$$H(x,t,s,\xi)s \geq 0 \tag{1.2}$$

and the growth condition of the form

$$|H(x,t,s,\xi)| \leq b(s)\Big(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x,t)\Big). \tag{1.3}$$

In the case where the second membre $f \in L^1(Q)$, (1.1) is studied in [3].

It is our purpose to prove the existence of renormalized solution for (1.1) in the setting of the weighted Sobolev space without the sign condition (1.2), and without the following coercivity condition

$$|H(x,t,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i(x)|\xi_i|^p \quad \text{for} |s| \geq \gamma, \tag{1.4}$$

our growth condition on H is simpler than (1.3) it is a growth with respect to Du and no growth condition with respect to u (see assumption (H3) below), the second term f belongs to $L^1(Q)$. Note that our paper generalizes [2, 3]. The case $H(x,t,u,Du) = \operatorname{div}(\phi(u))$ is studied by Redwane in the classical Sobolev spaces $W^{1,p}(\Omega)$ and in Orlicz spaces; see [15, 16].

The notion of renormalized solution was introduced by Diperna and Lions [8] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (1.1) by Boccardo et al [5] when the right hand side is in $W^{-1,p'}(\Omega)$, by Rakotoson [14] when the right hand side is in $L^1(\Omega)$, and finally by Dal Maso, Murat, Orsina and Prignet [7] for the case of right hand side is general measure data.

Our article can be see as a continuation of [4] in the case where $b(x,u) = u$, $a(x,t,s,\xi)$ is independent of s and $H = 0$. The plan of the article is as follows. In Section 2 we give some preliminaries and the definition of weighted Sobolev spaces. In Section 3 we make precise all the assumptions on $b, a, H, f, b(x,u_0)$. In section 4 we give some technical results. In Section 5 we give the definition of a renormalized solution of (1.1) and we establish the existence of such a solution (Theorem 5.3). Section 6 is devoted to an example which illustrates our abstract result, and finally an appendix in section 7.

2. Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$ and $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions; i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that , there exits

$$r_0 > \max(N,p) \quad \text{such that } w_i^{\frac{-r_0}{r_0-p}} \in L^1_{\rm loc}(\Omega), \tag{2.1}$$

$$w_i \in L^1_{\rm loc}(\Omega), \tag{2.2}$$

$$w_i^{\frac{-1}{p-1}} \in L^1_{\rm loc}(\Omega), \tag{2.3}$$

for any $0 \le i \le N$. We denote by $W^{1,p}(\Omega, w)$ the space of real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for } i = 1, \dots, N.$$

Which is a Banach space under the norm

$$\|u\|_{1,p,w} = \Big[\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \tag{2.4}$$

Condition (2.2) implies that $C_0^\infty(\Omega)$ is a space of $W^{1,p}(\Omega, w)$ and consequently, we can introduce the subspace $V = W_0^{1,p}(\Omega, w)$ of $W^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4). Moreover, condition (2.3) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0, \dots, N\}$ and where p' is the conjugate of p; i.e., $p' = \frac{p}{p-1}$, (see [11]).

3. Basic assumptions

Assumption (H1). For $2 \le p < \infty$, we assume that the expression

$$|||u|||_V = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{3.1}$$

is a norm defined on V which is equivalent to the norm (2.4), and there exists a weight function σ on Ω such that,

$$\sigma \in L^1(\Omega) \quad \text{and } \sigma^{-1} \in L^1(\Omega).$$

We assume also the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^p \sigma\,dx\Big)^{1/q} \le c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}, \tag{3.2}$$

holds for every $u \in V$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$W^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^p(\Omega, \sigma), \tag{3.3}$$

expressed by the inequality (3.2) is compact. Notice that $(V, |||\cdot|||_V)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 3.1. If we assume that $w_0(x) \equiv 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{p}, +\infty[\cap [\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega) \quad \text{and} \quad w_i^{\frac{N}{N-1}} \in L^1_{\rm loc}(\Omega) \quad \text{for all } i = 1, \dots, N. \tag{3.4}$$

Notice that the assumptions (2.2) and (3.4) imply

$$|||u||| = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}, \tag{3.5}$$

which is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to (2.4) and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega), \tag{3.6}$$

is compact for all $1 \le q \le p_1^*$ if $p\nu < N(\nu+1)$ and for all $q \ge 1$ if $p\nu \ge N(\nu+1)$ where $p_1 = \frac{p\nu}{\nu+1}$ and p_1^* is the Sobolev conjugate of p_1; see [10, pp 30-31].

Assumption (H2).

$$b : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function.} \tag{3.7}$$

such that for every $x \in \Omega$, $b(x,.)$ is a strictly increasing C^1-function with $b(x,0) = 0$. Next, for any $k > 0$, there exists $\lambda_k > 0$ and functions $A_k \in L^1(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \le \frac{\partial b(x,s)}{\partial s} \le A_k(x) \quad \text{and} \quad \Big|D_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \le B_k(x) \tag{3.8}$$

for almost every $x \in \Omega$, for every s such that $|s| \le k$, we denote by $D_x\big(\frac{\partial b(x,s)}{\partial s}\big)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions. For $i = 1,\dots,N$,

$$|a_i(x,t,s,\xi)| \le \beta w_i^{1/p}(x)[k(x,t) + \sigma^{1/p'}|s|^{q/p'} + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}], \tag{3.9}$$

for a.e. $(x,t) \in Q$,all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, some function $k(x,t) \in L^{p'}(Q)$ and $\beta > 0$. Here σ and q are as in (H1).

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0 \quad \text{for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.10}$$

$$a(x,t,s,\xi).\xi \ge \alpha \sum_{i=1}^{N} w_i|\xi_i|^p, \tag{3.11}$$

Where α is a strictly positive constant.

Assumption (H3). Furthermore, let $H(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e $(x,t) \in Q$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \le \gamma(x,t) + g(s)\sum_{i=1}^{N} w_i(x)|\xi_i|^p \tag{3.12}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x,t)$ belongs to $L^1(Q)$.

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

4. Some technical results

Characterization of the time mollification of a function u. To deal with time derivative, we introduce a time mollification of a function u belonging to a some weighted Lebesgue space. Thus we define for all $\mu \ge 0$ and all $(x,t) \in Q$,

$$u_\mu = \mu \int_{\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))ds$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$.

Proposition 4.1 ([2]). *(1) If $u \in L^p(Q, w_i)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and,*

$$\|u_\mu\|_{L^p(Q,w_i)} \leq \|u\|_{L^p(Q,w_i)}.$$

(2) If $u \in W_0^{1,p}(Q, w)$, then $u_\mu \to u$ in $W_0^{1,p}(Q, w)$ as $\mu \to \infty$.

(3) If $u_n \to u$ in $W_0^{1,p}(Q, w)$, then $(u_n)_\mu \to u_\mu$ in $W_0^{1,p}(Q, w)$.

Some weighted embedding and compactness results. In this section we establish some embedding and compactness results in weighted Sobolev spaces, some trace results, Aubin's and Simon's results [17]. Let $V = W_0^{1,p}(\Omega, w)$, $H = L^2(\Omega, \sigma)$ and let $V^* = W^{-1,p'}$, with $(2 \leq p < \infty)$. Let $X = L^p(0, T; W_0^{1,p}(\Omega, w))$. The dual space of X is $X^* = L^{p'}(0, T, V^*)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and denoting the space $W_p^1(0, T, V, H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*},$$

which is a Banach space. Here u' stands for the generalized derivative of u; i.e.,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad \text{for all } \varphi \in C_0^\infty(0, T).$$

Lemma 4.2 ([18]). *(1) The evolution triple $V \subseteq H \subseteq V^*$ is satisfied.*

(2) The imbedding $W_p^1(0, T, V, H) \subseteq C(0, T, H)$ is continuous.

(3) The imbedding $W_p^1(0, T, V, H) \subseteq L^p(Q, \sigma)$ is compact.

Lemma 4.3 ([2]). *Let $g \in L^r(Q, \gamma)$ and let $g_n \in L^r(Q, \gamma)$, with $\|g_n\|_{L^r(Q,\gamma)} \leq C$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in Q, then $g_n \rightharpoonup g$ in $L^r(Q, \gamma)$ where $n \to \infty$.*

Lemma 4.4 ([2]). *Assume that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad \text{in } D'(Q)$$

where α_n and β_n are bounded respectively in X^* and in $L^1(Q)$. If v_n is bounded in $L^p(0, T; W_0^{1,p}(\Omega, w))$, then $v_n \to u$ in $L^p_{\rm loc}(Q, \sigma)$. Further $v_n \to v$ strongly in $L^1(Q)$ where $n \to \infty$.

Lemma 4.5 ([2]). *Assume that* (H1) *and* (H2) *are satisfied and let (u_n) be a sequence in $L^p(0, T; W_0^{1,p}(\Omega, w))$ such that $u_n \rightharpoonup u$ weakly in $L^p(0, T; W_0^{1,p}(\Omega, w))$ and*

$$\int_Q [a(x, t, u_n, Du_n) - a(x, t, u, Du)][Du_n - Du]\, dx\, dt \to 0. \tag{4.1}$$

Then, $u_n \to u$ in $L^p(0, T; W_0^{1,p}(\Omega, w))$.

Definition 4.6. A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph

$$G(T) = \{(u, T(u)) \in X \times X^* \quad \text{for all } u \in D(T)\}$$

is not a proper subset of any monotone set in $X \times X^*$. Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map L from the subset $D(L) = \{v \in X : v' \in X^*, v(0) = 0\}$ of X into X^* by

$$\langle Lu, v\rangle_X = \int_0^T \langle u'(t), v(t)_V dt\rangle \quad u \in D(L),\ v \in X$$

Lemma 4.7 ([18]). *L is a closed linear maximal monotone map.*

In our study we deal with mappings of the form $F = L + S$ where L is a given linear densely defined maximal monotone map from $D(L) \subset X$ to X^* and S is a bounded demicontinuous map of monotone type from X to X^*.

Definition 4.8. A mapping S is called pseudo-monotone with $u_n \rightharpoonup u$, $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle \leq 0$, we have

$$\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle = 0$$

and $S(u_n) \rightharpoonup S(u)$ as $n \to \infty$.

5. Main results

Consider the problem

$$\begin{gathered}
b(x,u_0) \in L^1(\Omega), \quad f \in L^1(Q) \\
\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,Du)) + H(x,t,u,Du) = f \quad \text{in } Q \\
u = 0 \quad \text{on } \partial\Omega\times]0,T[, \\
b(x,u)(t=0) = b(x,u_0) \quad \text{on } \Omega.
\end{gathered} \tag{5.1}$$

Definition 5.1. Let $f \in L^1(Q)$ and $b(x,u_0) \in L^1(\Omega)$. A real-valued function u defined on Q is a renormalized solution of problem 5.1 if

$$T_k(u) \in L^p(0,T; W_0^{1,p}(\Omega,w)) \quad \text{for all } k \geq 0 \text{ and } b(x,u) \in L^\infty(0,T;L^1(\Omega));, \tag{5.2}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Du\,dx\,dt \to 0 \quad \text{as } m \to +\infty; \tag{5.3}$$

$$\begin{aligned}
&\frac{\partial B_S(x,u)}{\partial t} - \operatorname{div}\left(S'(u)a(x,t,u,Du)\right) \\
&+ S''(u)a(x,t,u,Du)Du + H(x,t,u,Du)S'(u) \\
&= fS'(u) \quad \text{in } D'(Q);
\end{aligned} \tag{5.4}$$

for all functions $S \in W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where $B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ and

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad \text{in } \Omega. \tag{5.5}$$

Remark 5.2. Equation (5.4) is formally obtained through pointwise multiplication of (5.1) by $S'(u)$. However, while $a(x,t,u,Du)$ and $H(x,t,u,Du)$ does not in general make sense in (5.1), all the terms in (5.1) have a meaning in $D'(Q)$.
Indeed, if M is such that $suppS' \subset [-M,M]$, the following identifications are made in (5.4):

- $S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function.
- $S'(u)a(x,t,u,Du)$ identifies with $S'(u)a(x,t,T_M(u),DT_M(u))$ a.e. in Q. Since $|T_M(u)| \leq M$ a.e. in Q and $S'(u) \in L^\infty(Q)$, we obtain from (3.9) and (5.2) that
$$S'(u)a(x,t,T_M(u),DT_M(u)) \in \prod_{i=1}^N L^{p'}(Q,w_i^*)$$
- $S''(u)a(x,t,u,Du)Du$ identifies with $S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u)$ and
$$S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \in L^1(Q).$$

- $S'(u)H(x,t,u,Du)$ identifies with $S'(u)H(x,t,T_M(u),DT_M(u))$ a.e in Q. Since $|T_M(u)| \le M$ a.e in Q and $S'(u) \in L^\infty(Q)$, we obtain from (3.9) and (3.12) that
$$S'(u)H(x,t,T_M(u),DT_M(u)) \in L^1(Q)\,.$$
- $S'(u)f$ belongs to $L^1(Q)$.

The above considerations show that (5.4) holds in $D'(Q)$ and that
$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w_i^*)) + L^1(Q).$$
Due to the properties of S and (5.4), $\frac{\partial S(u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w_i^*)) + L^1(Q)$, which implies that $S(u) \in C^0([0,T];L^1(\Omega))$ so that the initial condition (5.5) makes sense, since, due to the properties of S (increasing) and (6.1), we have
$$\big|B_S(x,r) - B_S(x,r')\big| \le A_k(x)\big|S(r) - S(r')\big| \quad \text{for all } r,r' \in \mathbb{R}. \tag{5.6}$$

Theorem 5.3. *Let $f \in L^1(Q)$ and $b(x,u_0) \in L^1(\Omega)$. Assume that* (H1)–(H3) *hold. Then, there exists at least one renormalized solution u of problem* (5.1) *(in the sense of Definition 5.1).*

The proof of this theorem is done in four steps.

Step 1: Approximate problem and a priori estimates. For $n > 0$, let us define the following approximation of b, H, f and u_0;
$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r \quad \text{for } n > 0, \tag{5.7}$$
In view of (5.7), b_n is a Carathéodory function and satisfies (6.1), there exist $\lambda_n > 0$ and functions $A_n \in L^1(\Omega)$ and $B_n \in L^p(\Omega)$ such that
$$\lambda_n \le \frac{\partial b_n(x,s)}{\partial s} \le A_n(x) \quad \text{and} \quad \Big|D_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\Big| \le B_n(x)$$
a.e. in Ω, $s \in \mathbb{R}$.
$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1 + \frac{1}{n}|H(x,t,s,\xi)|}\chi_{\Omega_n}.$$
Note that Ω_n is a sequence of compacts covering the bounded open set Ω and χ_{Ω_n} is its characteristic function.
$$f_n \in L^{p'}(Q), \quad \text{and} \quad f_n \to f \quad \text{a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to +\infty, \tag{5.8}$$
$$u_{0n} \in D(\Omega), \quad \|b_n(x,u_{0n})\|_{L^1} \le \|b(x,u_0)\|_{L^1}, \tag{5.9}$$
$$b_n(x,u_{0n}) \to b(x,u_0) \quad \text{a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{5.10}$$

Let us now consider the approximate problem:
$$\begin{gathered}\frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,Du_n)) + H_n(x,t,u_n,Du_n) = f_n \quad \text{in } D'(Q),\\ u_n = 0 \quad \text{in } (0,T)\times\partial\Omega,\\ b_n(x,u_n(t=0)) = b_n(x,u_{0n}).\end{gathered} \tag{5.11}$$
Note that $H_n(x,t,s,\xi)$ satisfies the following conditions
$$|H_n(x,t,s,\xi)| \le H(x,t,s,\xi) \quad \text{and} \quad |H_n(x,t,s,\xi)| \le n.$$

For all $u, v \in L^p(0,T;W_0^{1,p}(\Omega,w))$,

$$
\begin{aligned}
&\Big|\int_Q H_n(x,t,u,Du)v\,dx\,dt\Big| \\
&\le \Big(\int_Q |H_n(x,t,u,Du)|^{q'}\sigma^{-\frac{q'}{q}}\,dx\,dt\Big)^{1/q'}\Big(\int_Q |v|^q\sigma\,dx\,dt\Big)^{1/q} \\
&\le n\int_0^T\Big(\int_{\Omega_n}\sigma^{1-q'}dx\Big)^{1/q'}dt\|v\|_{L^q(Q,\sigma)} \\
&\le C_n\|v\|_{L^p(0,T;W_0^{1,p}(\Omega,w))}.
\end{aligned}
$$

Moreover, since $f_n \in L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$, proving existence of a weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega,w))$ of (5.11) is an easy task (see e.g. [13],[2]).

Let $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q)$ with $\varphi > 0$, choosing $v = \exp(G(u_n))\varphi$ as test function in 5.11 where $G(s) = \int_0^s \frac{g(r)}{\alpha}dr$ (the function g appears in (3.12)). We have

$$
\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi\,dx\,dt + \int_Q a(x,t,u_n,Du_n)D(\exp(G(u_n))\varphi)\,dx\,dt \\
&= \int_Q H_n(x,t,u_n,Du_n)\exp(G(u_n))\varphi\,dx\,dt + \int_Q f_n\exp(G(u_n))\varphi\,dx\,dt.
\end{aligned}
$$

In view of (3.12), we obtain

$$
\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi\,dx\,dt \\
&+ \int_Q a(x,t,u_n,Du_n)Du_n\frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi\,dx\,dt \\
&+ \int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi\,dx\,dt \\
&\le \int_Q \gamma(x,t)\exp(G(u_n))\varphi\,dx\,dt + \int_Q g(u_n)\sum_{i=1}^N\Big|\frac{\partial u_n}{\partial x_i}\Big|w_i\exp(G(u_n))\varphi dxdt \\
&\quad + \int_Q f_n\exp(G(u_n))\varphi\,dx\,dt.
\end{aligned}
$$

By (3.11), we obtain

$$
\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi\,dx\,dt + \int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi\,dx\,dt \\
&\le \int_Q \gamma(x,t)\exp(G(u_n))\varphi\,dx\,dt + \int_Q f_n\exp(G(u_n))\varphi\,dx\,dt,
\end{aligned}
\tag{5.12}
$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q), \varphi > 0$. On the other hand, taking $v = \exp(-G(u_n))\varphi$ as test function in (5.11) we deduce, as in (5.12), that

$$
\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi\,dx\,dt + \int_Q a(x,t,u_n,Du_n)\exp(-G(u_n))D\varphi\,dx\,dt \\
&+ \int_Q \gamma(x,t)\exp(-G(u_n))\varphi\,dx\,dt
\end{aligned}
$$

$$\geq \int_Q f_n \exp(-G(u_n))\varphi \,dx\,dt, \tag{5.13}$$

for all $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q), \varphi > 0$. Let $\varphi = T_k(u_n)^+\chi_{(0,\tau)}$, for every $\tau \in [0,T]$, in (5.12) we have,

$$\begin{aligned}
&\int_\Omega B_k^n(x,u_n(\tau))\exp(G(u_n))dx + \int_{Q_\tau} a(x,t,u_n,Du_n)\exp(G(u_n))DT_k(u_n)^+\,dx\,dt \\
&\leq \int_{Q_\tau} \gamma(x,t)\exp(G(u_n))T_k(u_n)^+\,dx\,dt + \int_{Q_\tau} f_n \exp(G(u_n))T_k(u_n)^+\,dx\,dt \\
&\quad + \int_\Omega B_k^n(x,u_{0n})dx,
\end{aligned} \tag{5.14}$$

where $B_k^n(x,r) = \int_0^r T_k(s)^+ \frac{\partial b_n(x,s)}{\partial s} ds$. Due to this definition, we have

$$0 \leq \int_\Omega B_k^n(x,u_{0n})dx \leq k\int_\Omega |b_n(x,u_{0n})|dx \leq k\|b(x,u_0)\|_{L^1(\Omega)}. \tag{5.15}$$

Using this inequality, $B_k^n(x,u_n) \geq 0$ and $G(u_n) \leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we deduce

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,DT_k(u_n)^+)DT_k(u_n)^+\exp(G(u_n))\,dx\,dt \\
&\leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big(\|u_{0n}\|_{L^1(\Omega)} + \|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x,u_{0n})\|_{L^1(\Omega)}\Big) \\
&\leq c_1 k.
\end{aligned}$$

Thanks to (3.11), we have

$$\alpha\int_{Q_\tau}\sum_{i=1}^N w_i(x)\Big|\frac{\partial T_k(u_n)^+}{\partial x_i}\Big|^p \exp(G(u_n))\,dx\,dt \leq c_1 k. \tag{5.16}$$

We deduce that

$$\alpha\int_Q\sum_{i=1}^N w_i(x)\Big|\frac{\partial T_k(u_n)^+}{\partial x_i}\Big|^p\,dx\,dt \leq c_1 k. \tag{5.17}$$

Similarly to (5.17), we take $\varphi = T_k(u_n)^-\chi_{(0,\tau)}$ in (5.13) we deduce that

$$\alpha\int_Q\sum_{i=1}^N w_i(x)\Big|\frac{\partial T_k(u_n)^-}{\partial x_i}\Big|^p\,dx\,dt \leq c_2 k \tag{5.18}$$

where c_2 is a positive constant. Combining (5.17) and (5.18) we conclude that

$$\|T_k(u_n)\|^p_{L^p(0,T;W_0^{1,p}(\Omega,w))} \leq ck. \tag{5.19}$$

We deduce from the above inequality, (5.14) and (5.15), that

$$\int_\Omega B_k^n(x,u_n)dx \leq k(\|f\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}) \equiv Ck. \tag{5.20}$$

Then, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega,w))$, and $T_k(u_n) \rightharpoonup v_k$ in the space $L^p(0,T;W_0^{1,p}(\Omega,w))$, and by the compact imbedding (3.6) gives

$$T_k(u_n) \to v_k \quad \text{strongly in } L^p(Q,\sigma) \text{ and a.e. in } Q.$$

Let $k > 0$ be large enough and B_R be a ball of Ω, we have

$$k\,\text{meas}(\{|u_n| > k\} \cap B_R \times [0,T])$$

$$
\begin{aligned}
&= \int_0^T \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\,dx\,dt \\
&\le \int_0^T \int_{B_R} |T_k(u_n)|\,dx\,dt \\
&\le \Big(\int_Q |T_k(u_n)|^p \sigma\,dx\,dt\Big)^{1/p} \Big(\int_0^T \int_{B_R} \sigma^{1-p'}\,dx\,dt\Big)^{1/p'} \\
&\le T c_R \Big(\int_Q \sum_{i=1}^N w_i(x)\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p\,dx\,dt\Big)^{1/p} \\
&\le c k^{1/p},
\end{aligned}
$$

which implies

$$
\text{meas}(\{|u_n|>k\}\cap B_R\times[0,T]) \le \frac{c_1}{k^{1-\frac{1}{p}}}, \quad \forall k\ge 1.
$$

So, we have

$$
\lim_{k\to+\infty}(\text{meas}(\{|u_n|>k\}\cap B_R\times[0,T]))=0.
$$

Now we turn to prove the almost every convergence of u_n and $b_n(x,u_n)$. Consider now a function non decreasing $g_k\in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\le\frac{k}{2}$ and $g_k(s)=k$ for $|s|\ge k$. Multiplying the approximate equation by $g_k'(b_n(x,u_n))$, we obtain

$$
\begin{aligned}
&\frac{\partial g_k(b_n(x,u_n))}{\partial t} - \operatorname{div}(a(x,t,u_n,Du_n)g_k'(b_n(x,u_n))) \\
&+ a(x,t,u_n,Du_n)g_k''(b_n(x,u_n))D_x\Big(\frac{\partial b_n(x,u_n)}{\partial s}\Big)Du_n \\
&+ H_n(x,t,u_n,Du_n)g_k'(b_n(x,u_n)) \\
&= f_n g_k'(b_n(x,u_n))
\end{aligned} \tag{5.21}
$$

in the sense of distributions, which implies that

$$
g_k(b_n(x,u_n)) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega,w)), \tag{5.22}
$$

$$
\frac{\partial g_k(b_n(x,u_n))}{\partial t} \text{ is bounded in } X^*+L^1(Q), \tag{5.23}
$$

independent of n as long as $k<n$. Due to Definition (3.7) and (5.7) of b_n, it is clear that

$$
\{|b_n(x,u_n)|\le k\}\subset\{|u_n|\le k^*\}
$$

as long as $k<n$ and k^* is a constant independent of n. As a first consequence we have

$$
Dg_k(b_n(x,u_n)) = g_k'(x,b_n(u_n))D_x\Big(\frac{\partial b_n(x,T_{k^*}(u_n))}{\partial s}\Big)DT_{k^*}(u_n) \quad \text{a.e in } Q \tag{5.24}
$$

as long as $k<n$. Secondly, the following estimate holds

$$
\begin{aligned}
&\Big\|g_k'(b_n(x,u_n))D_x\Big(\frac{\partial b_n(x,T_{k^*}(u_n))}{\partial s}\Big)\Big\|_{L^\infty(Q)} \\
&\le \|g_k'\|_{L^\infty(Q)}\Big(\max_{|r|\le k^*}\Big(D_x\big(\frac{\partial b_n(x,s)}{\partial s}\big)\Big)+1\Big).
\end{aligned}
$$

As a consequence of (5.19), (5.24) we then obtain (5.22). To show that (5.23) holds, due to (5.21) we obtain

$$\begin{aligned}\frac{\partial g_k(b_n(x,u_n))}{\partial t} &= \operatorname{div}(a(x,t,u_n,Du_n)g_k'(b_n(x,u_n)))\\ &\quad - a(x,t,u_n,Du_n)g_k''(b_n(u_n))D_x\Big(\frac{\partial b_n(x,u_n)}{\partial s}\Big)\\ &\quad + H_n(x,t,u_n,Du_n)g_k'(b_n(x,u_n)) + f_n g_k'(b_n(x,u_n)).\end{aligned} \tag{5.25}$$

Since support of g_k' and support of g_k'' are both included in $[-k,k]$, u_n may be replaced by $T_{k^*}(u_n)$ in each of these terms. As a consequence, each term on the right-hand side of (5.25) is bounded either in $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ or in $L^1(Q)$. Hence lemma 4.4 allows us to conclude that $g_k(b_n(x,u_n))$ is compact in $L^p_{\rm loc}(Q,\sigma)$. Thus, for a subsequence, it also converges in measure and almost every where in Q, due to the choice of g_k, we conclude that for each k, the sequence $T_k(b_n(x,u_n))$ converges almost everywhere in Q (since we have, for every $\lambda>0$,)

$$\begin{aligned}&\operatorname{meas}(\{\big|b_n(x,u_n)-b_m(x,u_m)\big|>\lambda\}\cap B_R\times[0,T])\\ &\le \operatorname{meas}(\{|b_n(x,u_n)|>k\}\cap B_R\times[0,T]) + \operatorname{meas}(\{|b_m(x,u_m)|>k\}\cap B_R\times[0,T])\\ &\quad + \operatorname{meas}(\{\big|g_k(b_n(x,u_n))-g_k(b_m(x,u_m))\big|>\lambda\}).\end{aligned}$$

Let $\varepsilon>0$, then there exist $k(\varepsilon)>0$ such that

$$\operatorname{meas}(\{\big|b_n(x,u_n)-b_m(x,u_m)\big|>\lambda\}\cap B_R\times[0,T]) \le \varepsilon$$

for all $n,m\ge n_0(k(\varepsilon),\lambda,R)$. This proves that $(b_n(x,u_n))$ is a Cauchy sequence in measure in $B_R\times[0,T]$, thus converges almost everywhere to some measurable function v. Then for a subsequence denoted again u_n,

$$u_n\to u \quad \text{a.e. in } Q, \tag{5.26}$$

$$b_n(x,u_n)\to b(x,u) \quad \text{a.e. in } Q. \tag{5.27}$$

We can deduce from (5.19) that

$$T_k(u_n)\rightharpoonup T_k(u) \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega,w)) \tag{5.28}$$

and then, the compact imbedding (3.3) gives

$$T_k(u_n)\to T_k(u) \quad \text{strongly in } L^q(Q,\sigma) \text{ and a.e. in } Q.$$

Which implies, by using (3.9), for all $k>0$ that there exists a function $h_k\in\prod_{i=1}^N L^{p'}(Q,w_i^*)$, such that

$$a(x,t,T_k(u_n),DT_k(u_n))\rightharpoonup h_k \quad \text{weakly in } \prod_{i=1}^N L^{p'}(Q,w_i^*). \tag{5.29}$$

We now establish that $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Using (5.26) and passing to the limit-inf in (5.20) as n tends to $+\infty$, we obtain that

$$\frac{1}{k}\int_\Omega B_k(x,u)(\tau)dx \le [\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}] \equiv C,$$

for almost any τ in $(0,T)$. Due to the definition of $B_k(x,s)$ and the fact that $\frac{1}{k}B_k(x,u)$ converges pointwise to $b(x,u)$, as k tends to $+\infty$, shows that $b(x,u)$ belong to $L^\infty(0,T;L^1(\Omega))$.

Lemma 5.4. *Let u_n be a solution of the approximate problem* (5.11). *Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt=0 \tag{5.30}$$

Proof. Considering the function $\varphi=T_1(u_n-T_m(u_n))^-:=\alpha_m(u_n)$ in (5.13) this function is admissible since $\varphi\in L^p(0,T;W_0^{1,p}(\Omega,w))$ and $\varphi\ge 0$. Then, we have

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\alpha_m(u_n)\,dx\,dt+\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,Du_n)Du_n\alpha_m'(u_n)\,dx\,dt\\
&\quad+\int_Q f_n\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt\\
&\le\int_Q \gamma(x,t)\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt.
\end{aligned}$$

Which, by setting $B_n^m(x,r)=\int_0^r\frac{\partial b_n(x,s)}{\partial s}\alpha_m(s)ds$, gives

$$\begin{aligned}
&\int_\Omega B_n^m(x,u_n)(T)dx+\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,Du_n)Du_n\alpha_m'(u_n)\,dx\,dt\\
&+\int_Q f_n\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt\\
&\le\int_Q \gamma(x,t)\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt+\int_\Omega B_n^m(x,u_{0n})dx.
\end{aligned}$$

Since $B_n^m(x,u_n)(T)\ge 0$ and by Lebesgue's theorem, we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_Q f_n\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt=0. \tag{5.31}$$

Similarly, since $\gamma\in L^1(\Omega)$, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_Q \gamma\exp(-G(u_n))\alpha_m(u_n)\,dx\,dt=0. \tag{5.32}$$

We conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt=0. \tag{5.33}$$

On the other hand, let $\varphi=T_1(u_n-T_m(u_n))^+$ as test function in (5.12) and reasoning as in the proof of (5.33) we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m)\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt=0. \tag{5.34}$$

Thus (5.30) follows from (5.33) and (5.34). □

Step 2: Almost everywhere convergence of the gradients. This step is devoted to introduce for $k\ge 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes (see Lemma 6 and proposition 3, p.230, and proposition 4, p.231, in[12]).

Let $\psi_i \in D(\Omega)$ be a sequence which converge strongly to u_0 in $L^1(\Omega)$. Set $w_\mu^i = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w_μ^i is a smooth function having the following properties:

$$\frac{\partial w_\mu^i}{\partial t} = \mu(T_k(u) - w_\mu^i), \quad w_\mu^i(0) = T_k(\psi_i), \quad \left|w_\mu^i\right| \le k, \tag{5.35}$$

$$w_\mu^i \to T_k(u) \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega,w)), \tag{5.36}$$

as $\mu \to \infty$. We introduce the following function of one real:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le m \\ 0 & \text{if } |s| \ge m+1 \\ m+1-s & \text{if } m \le s \le m+1 \\ m+1+s & \text{if } -(m+1) \le s \le -m \end{cases}$$

where $m > k$.

Let $\varphi = (T_k(u_n) - w_\mu^i)^+ h_m(u_n) \in L^p(0,T;W_0^{1,p}(\Omega,w)) \cap L^\infty(Q)$ and $\varphi \ge 0$, then we take this function in (5.12), we obtain

$$\begin{aligned} &\int_{\{T_k(u_n)-w_\mu^i \ge 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w_\mu^i)h_m(u_n)\,dx\,dt \\ &+ \int_{\{T_k(u_n)-w_\mu^i \ge 0\}} a(x,t,u_n,Du_n)D(T_k(u_n) - w_\mu^i)h_m(u_n)\,dx\,dt \\ &- \int_{\{m \le u_n \le m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w_\mu^i)^+\,dx\,dt \\ &\le \int_Q \gamma(x,t)\exp(G(u_n))(T_k(u_n) - w_\mu^i)^+ h_m(u_n)\,dx\,dt \\ &\quad + \int_Q f_n \exp(G(u_n))(T_k(u_n) - w_\mu^i)^+ h_m(u_n)\,dx\,dt. \end{aligned} \tag{5.37}$$

Observe that

$$\begin{aligned} &\int_{\{m \le u_n \le m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w_\mu^i)^+\,dx\,dt \\ &\le 2k\int_{\{m \le u_n \le m+1\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt. \end{aligned}$$

Thanks to (5.30) the third integral tend to zero as n and m tend to infinity, and by Lebesgue's theorem, we deduce that the right hand side converge to zero as n, m and μ tend to infinity. Since

$$(T_k(u_n) - w_\mu^i)^+ h_m(u_n) \rightharpoonup (T_k(u) - w_\mu^i)^+ h_m(u) \quad \text{weakly* in } L^\infty(Q), \text{ as } n \to \infty,$$
$$\text{and } (T_k(u) - w_\mu^i)^+ h_m(u) \rightharpoonup 0 \quad \text{weakly* in } L^\infty(Q) \text{ as } \mu \to \infty.$$

Let $\varepsilon_l(n,m,\mu,i)$ $l = 1,\dots,n$ various functions tend to zero as n, m, i and μ tend to infinity.

The definition of the sequence w_μ^i makes it possible to establish the following lemma, which will be proved in the Appendix.

Lemma 5.5. [14] *For $k \geq 0$ we have*

$$\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt \geq \varepsilon(n,m,\mu,i) \tag{5.38}$$

On the other hand, the second term of left hand side of (5.37) reads as follows

$$\begin{aligned}
&\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt \\
&= \int_{\{T_k(u_n)-w^i_\mu \geq 0, |u_n|\leq k\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt \\
&\quad - \int_{\{T_k(u_n)-w^i_\mu \geq 0, |u_n|\geq k\}} a(x,t,u_n,Du_n)Dw^i_\mu h_m(u_n)\,dx\,dt.
\end{aligned}$$

Since $m > k$, $h_m(u_n) = 0$ on $\{|u_n| \geq m+1\}$, One has

$$\begin{aligned}
&\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt \\
&= \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt \\
&\quad - \int_{\{T_k(u_n)-w^i_\mu \geq 0, |u_n|\geq k\}} a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))Dw^i_\mu h_m(u_n)\,dx\,dt \\
&= J_1 + J_2
\end{aligned} \tag{5.39}$$

In the following we pass to the limit in (5.39): first we let n tend to $+\infty$, then μ and finally m, tend to $+\infty$. Since $a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))$ is bounded in $\prod_{i=1}^N L^{p'}(Q,w^*_i)$, we have that

$$a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))h_m(u_n)\chi_{\{|u_n|>k\}} \to h_m h_m(u)\chi_{\{|u|>k\}}$$

strongly in $\prod_{i=1}^N L^{p'}(Q,w^*_i)$ as n tends to infinity, it follows that

$$\begin{aligned}
J_2 &= \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} h_m Dw^i_\mu h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \varepsilon(n) \\
&= \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} h_m(DT_k(u)_\mu - e^{-\mu t}DT_k(\psi_i))h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \varepsilon(n).
\end{aligned}$$

By letting $\mu \to +\infty$,

$$J_2 = \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} h_m DT_k(u)\,dx\,dt + \varepsilon(n,\mu).$$

Using now the term J_1 of (5.39) one can easily show that

$$
\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\quad\times[DT_k(u_n)-DT_k(u)]\,h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u))(DT_k(u_n)-DT_k(u))h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))DT_k(u)h_m(u_n)\,dx\,dt\\
&\quad-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))Dw_\mu^i h_m(u_n)\,dx\,dt\\
&=K_1+K_2+K_3+K_4.
\end{aligned}
\tag{5.40}
$$

We shall go to the limit as n and $\mu\to+\infty$ in the three integrals of the right-hand side. Starting with K_2, we have by letting $n\to+\infty$,

$$
K_2=\varepsilon(n). \tag{5.41}
$$

About K_3, we have by letting $n\to+\infty$ and using (5.29),

$$
K_3=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k DT_k(u)h_m(u)\chi_{\{|u|>k\}}\,dx\,dt+\varepsilon(n)
$$

By letting $\mu\to+\infty$,

$$
K_3=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k DT_k(u)\,dx\,dt+\varepsilon(n,\mu). \tag{5.42}
$$

For K_4 we can write

$$
K_4=-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k Dw_\mu^i h_m(u)\,dx\,dt+\varepsilon(n),
$$

By letting $\mu\to+\infty$,

$$
K_4=-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} h_k DT_k(u)\,dx\,dt+\varepsilon(n,\mu). \tag{5.43}
$$

We then conclude that

$$
\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt\\
&=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\quad\times[DT_k(u_n)-DT_k(u)]\,h_m(u_n)\,dx\,dt+\varepsilon(n,\mu).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\quad\times[DT_k(u_n)-DT_k(u)]\,dx\,dt\\
&=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\quad\times[DT_k(u_n)-DT_k(u)]\,h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))(DT_k(u_n)-DT_k(u))\\
&\quad\times(1-h_m(u_n))\,dx\,dt\\
&\quad-\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} a(x,t,T_k(u_n),DT_k(u))(DT_k(u_n)-DT_k(u))\\
&\quad\times(1-h_m(u_n))\,dx\,dt.
\end{aligned}
\tag{5.44}
$$

Since $h_m(u_n)=1$ in $\{|u_n|\leq m\}$ and $\{|u_n|\leq k\}\subset\{|u_n|\leq m\}$ for m large enough, we deduce from (5.44) that

$$
\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\times[DT_k(u_n)-DT_k(u)]\,dx\,dt\\
&=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\quad\times[DT_k(u_n)-DT_k(u)]\,h_m(u_n)\,dx\,dt\\
&\quad+\int_{\{T_k(u_n)-w_\mu^i\geq 0,|u_n|>k\}} a(x,t,T_k(u_n),DT_k(u))DT_k(u)(1-h_m(u_n))\,dx\,dt.
\end{aligned}
$$

It is easy to see that the last terms of the last equality tend to zero as $n\to+\infty$, which implies

$$
\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\times[DT_k(u_n)-DT_k(u)]\,dx\,dt\\
&=\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\quad\times[DT_k(u_n)-DT_k(u)]\,h_m(u_n)\,dx\,dt+\varepsilon(n)
\end{aligned}
$$

Combining (5.38), (5.40), (5.41), (5.42), (5.43) and (5.44), we obtain

$$
\begin{aligned}
&\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\times[DT_k(u_n)-DT_k(u)]\,dx\,dt\leq\varepsilon(n,\mu,m)
\end{aligned}
\tag{5.45}
$$

Passing to the limit in (5.45) as n and m tend to infinity, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n))-a(x,t,T_k(u_n),DT_k(u))]\\
&\times[DT_k(u_n)-DT_k(u)]\,dx\,dt=0.
\end{aligned}
\tag{5.46}
$$

On the other hand, taking $\varphi = (T_k(u_n) - w_\mu^i)^- h_m(u_n)$ in (5.13), we deduce as in (5.46) that

$$\begin{aligned} &\lim_{n\to\infty} \int_{\{T_k(u_n)-w_\mu^i\le 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\ &\times [DT_k(u_n) - DT_k(u)]\,dx\,dt = 0. \end{aligned} \tag{5.47}$$

Combining (5.46) and (5.47), we conclude

$$\begin{aligned} &\lim_{n\to\infty} \int_Q [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\ &\times [DT_k(u_n) - DT_k(u)]\,dx\,dt = 0. \end{aligned} \tag{5.48}$$

Which, by lemma (4.5), implies

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega,w)) \text{ for all } k. \tag{5.49}$$

Now, observe that for every $\sigma > 0$,

$$\begin{aligned} &\operatorname{meas}\{(x,t)\in\Omega\times[0,T] : |Du_n - Du| > \sigma\} \\ &\le \operatorname{meas}\{(x,t)\in\Omega\times[0,T] : |Du_n| > k\} \\ &\quad + \operatorname{meas}\{(x,t)\in\Omega\times[0,T] : |u| > k\} \\ &\quad + \operatorname{meas}\{(x,t)\in\Omega\times[0,T] : \big|DT_k(u_n) - DT_k(u)\big| > \sigma\} \end{aligned}$$

then as a consequence of (5.49) we have that Du_n converges to Du in measure and therefore, always reasoning for a subsequence,

$$Du_n \to Du \quad \text{a. e. in } Q. \tag{5.50}$$

Which implies

$$a(x,t,T_k(u_n),DT_k(u_n)) \rightharpoonup a(x,t,T_k(u),DT_k(u)) \quad \text{in } \prod_{i=1}^N L^{p'}(Q,w_i^*). \tag{5.51}$$

Step 3: Equi-integrability of the nonlinearity sequence. We shall now prove that $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ a.e. in Q, Consider a function $\rho_h(s) = \int_0^s g(\nu)\chi_{\{\nu>h\}}d\nu$, take $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as test function in (5.12), we obtain

$$\begin{aligned} &\Big[\int_\Omega B_h^n(x,u_n)dx\Big]_0^T + \int_Q a(x,t,u_n,Du_n)Du_n g(u_n)\chi_{\{u_n>h\}}\,dx\,dt \\ &\le \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\Big), \end{aligned}$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)ds$, which implies, since $B_h^n(x,r) \ge 0$,

$$\begin{aligned} &\int_Q a(x,t,u_n,Du_n)Du_n g(u_n)\chi_{\{u_n>h\}}\,dx\,dt \\ &\le \Big(\int_h^\infty g(s)ds\Big)\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\big(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\big) + \int_\Omega B_h^n(x,u_{0n})dx. \end{aligned}$$

Using (3.11), we have

$$\int_{\{u_n>h\}} g(u_n)\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p\,dx\,dt \le C\int_h^\infty g(s)\,ds.$$

Since $g\in L^1(\mathbb{R})$, we have

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} g(u_n)\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p\,dx\,dt=0.$$

Similarly, let $\varphi=\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (5.13), we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} g(u_n)\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p\,dx\,dt=0.$$

Consequently,

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} g(u_n)\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p\,dx\,dt=0,$$

which, for h large enough, implies

$$\begin{aligned}\int_Q g(u_n)\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p\,dx\,dt &\le \int_{\{|u_n|<h\}} g(u_n)\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p\,dx\,dt+1\\ &\le \int_Q g(T_k(u_n))\sum_{i=1}^N w_i\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p\,dx\,dt+1.\end{aligned}$$

Then by (5.49) and Vitali's theorem, we can deduce that $g(u_n)\sum_{i=1}^N w_i\big|\frac{\partial u_n}{\partial x_i}\big|^p$ converges to $g(u)\sum_{i=1}^N w_i\big|\frac{\partial u}{\partial x_i}\big|^p$ strongly in $L^1(Q)$. Consequently, using (3.12), we conclude that

$$H_n(x,t,u_n,Du_n)\to H(x,t,u,Du)\quad\text{strongly in } L^1(Q). \tag{5.52}$$

Step 4. In this step we prove that u satisfies (5.3), (5.4) and (5.5).

Lemma 5.6. *The limit u of the approximate solution u_n of* (5.11) *satisfies*

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,Du)\,Du\,dx\,dt=0.$$

Proof. Note that for any fixed $m\ge 0$,

$$\begin{aligned}&\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,Du_n)Du_n\\ &=\int_Q a(x,t,u_n,Du_n)(DT_{m+1}(u_n)-DT_m(u_n))\\ &=\int_Q a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))DT_{m+1}(u_n)\\ &\quad-\int_Q a(x,t,T_m(u_n),DT_m(u_n))DT_m(u_n).\end{aligned}$$

According to (5.51) and (5.49), one is alloed to pass to the limit as $n \to +\infty$ for fixed $m \geq 0$, and to obtain

$$\begin{aligned} &\lim_{n\to+\infty} \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt \\ &= \int_Q a(x,t,T_{m+1}(u),DT_{m+1}(u))DT_{m+1}(u)\,dx\,dt \\ &\quad - \int_Q a(x,t,T_m(u),DT_m(u))DT_m(u_n)\,dx\,dt. \\ &= \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u,Du)Du\,dx\,dt. \end{aligned} \tag{5.53}$$

Taking the limit as $m \to +\infty$ in (5.53) and using the estimate (5.30) show that u satisfies (5.4) and the proof is complete. □

Now, we show that u satisfies (5.4) and (5.5). Let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support. Let M be a positive real number such that support of (S') is a subset of $[-M,M]$. Pointwise multiplication of the approximate equation (5.11) by $S'(u_n)$ leads to

$$\begin{aligned} &\frac{\partial B_S^n(x,u_n)}{\partial t} - \operatorname{div}[S'(u_n)a(u_n,Du_n)] + S''(u_n)a(u_n,Du_n)Du_n \\ &+ S'(u_n)H_n(u_n,Du_n) \\ &= fS'(u_n) \quad \text{in } D'(Q). \end{aligned} \tag{5.54}$$

Passing to the limit, as n tends to $+\infty$, we have

• Since S is bounded and continuous, $u_n \to u$ a.e. in Q implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e. in Q and L^∞ weak-*. Then

$$\frac{\partial B_S^n(x,u_n)}{\partial t} \quad \text{converges to} \quad \frac{\partial B_S(x,u)}{\partial t}$$

in $D'(Q)$ as n tends to $+\infty$.

• Since $\operatorname{supp}(S') \subset [-M,M]$, we have for $n \geq M$,

$$S'(u_n)a_n(u_n,Du_n) = S'(u_n)a(T_M(u_n),DT_M(u_n)) \quad \text{a.e. in } Q.$$

The pointwise convergence of $u_n tou$ and (5.51) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(u_n,Du_n) \rightharpoonup S'(u)a(T_M(u),DT_M(u)) \quad \text{in } \prod_{i=1}^N L^{p'}(Q,w_i^*), \tag{5.55}$$

as n tends to $+\infty$. $S'(u)a(T_M(u),DT_M(u))$ has been denoted by $S'(u)a(u,Du)$ in equation (5.4).

• Regarding the 'energy' term, we have

$$S''(u_n)a(u_n,Du_n)Du_n = S''(u_n)a(T_M(u_n),DT_M(u_n))DT_M(u_n) \quad \text{a.e. in } Q.$$

The pointwise convergence of $S'(u_n)toS'(u)$ and (5.51) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a_n(u_n,Du_n)Du_n \rightharpoonup S''(u)a(T_M(u),DT_M(u))DT_M(u) \;\; \textit{weakly} \quad \text{in } L^1(Q). \tag{5.56}$$

Recall that

$$S''(u)a(T_M(u), DT_M(u))DT_M(u) = S''(u)a(u, Du)Du \quad \text{a.e. in } Q.$$

- Since $\operatorname{supp}(S') \subset [-M, M]$, by (5.52), we have

$$S'(u_n)H_n(x, t, u_n, Du_n) \to S'(u)H(x, t, u, Du) \quad \text{strongly in } L^1(Q), \tag{5.57}$$

as n tends to $+\infty$.

- Due to (5.8) and ($u_n \to u$ a.e in Q), we have

$$S'(u_n)f_n \to S'(u)f \quad \text{strongly in } L^1(Q) \text{ as } n \to +\infty.$$

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (5.54) and to conclude that u satisfies (5.4).

It remains to show that $B_S(x, u)$ satisfies the initial condition (5.5). To this end, firstly remark that, S being bounded, $B_S^n(x, u_n)$ is bounded in $L^\infty(Q)$. Secondly, (5.54) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega, w^*))$. As a consequence, an Aubin's type lemma (see, e.g, [17]) implies that $B_S^n(x, u_n)$ lies in a compact set of $C^0([0, T], L^1(\Omega))$. It follows that on the one hand, $B_S^n(x, u_n)(t = 0) = B_S^n(x, u_0^n)$ converges to $B_S(x, u)(t = 0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that

$$B_S(x, u)(t = 0) = B_S(x, u_0) \quad \text{in } \Omega.$$

As a conclusion, steps 1–5 complete the proof of theorem 5.3.

6. Example

Let us consider the special case

$$b(x, r) = \sigma(x)|s|^{q(x)-2}s,$$

and $q : \Omega \to]1, +\infty[$ with $q(x) \le -|x|^2 + 2$. Then $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, Such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1-function with $b(x, 0) = 0$. Next, for any $k > 0$, there exist $\lambda_k > 0$ and functions $A_k \in L^1(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \le \frac{\partial b(x, s)}{\partial s} \le A_k(x), \quad \left|D_x\Big(\frac{\partial b(x, s)}{\partial s}\Big)\right| \le B_k(x), \tag{6.1}$$

$$H(x, t, s, \xi) = \rho \sin(s) \exp(s^{-2}) \sum_{i=1}^N w_i(x)|\xi_i|^p, \quad \rho \in \mathbb{R}, \tag{6.2}$$

$$a_i(x, t, s, d) = w_i(x)|d_i|^{p-1} \operatorname{sgn}(d_i), \quad i = 1, \dots, N, \tag{6.3}$$

with $w_i(x)$, ($i = 1, \dots, N$), a weight function strictly positive, $x \in Q$. Then, we can consider the Hardy inequality in the form

$$\Big(\int_\Omega |u(x)|^p \sigma(x) dx\Big)^{1/p} \le c\Big(\int_\Omega |Du(x)|^p w(x) dx\Big)^{1/p}.$$

It is easy to show that the $a_i(t, x, s, d)$ are Caratheodory functions satisfying the growth condition (3.9) and the coercivity (3.11). On the order hand the monotonicity condition is verified. In fact,

$$\sum_{i=1}^N (a_i(x, t, d) - a(x, t, d'))\,(d_i - d_i')$$

$$= w(x)\sum_{i=1}^{N-1}\big(|d_i|^{p-1}\operatorname{sgn}(d_i)-|d_i'|^{p-1}\operatorname{sgn}(d_i')\big)(d_i-d_i')>0,$$

for almost all $x\in\Omega$ and for all $d,d'\in\mathbb{R}^N$. This last inequality can not be strict, since for $d\neq d'$, since $w>0$ a.e. in Ω.

While the Carathéodory function $H(x,t,s,\xi)$ satisfies the condition (3.12) indeed

$$|H(x,t,s,\xi)|\le|\rho|\exp(s^{-2})\sum_{i=1}^{N}w_i(x)|\xi_i|^p=g(s)\sum_{i=1}^{N}w_i(x)|\xi_i|^p$$

where $g(s)=|\rho|\exp(s^{-2}$ is a function positive continuous which belongs to $L^1(\mathbb{R})$. Note that $H(x,t,s,\xi)$ does not satisfy the sign condition (1.2) and the coercivity condition (1.4).

In particular, let us use special weight function, w, expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x)=\operatorname{dist}(x,\partial\Omega)$ and set $w(x)=d^\lambda(x)$, $\sigma(x)=d^\mu(x)$.

Finally, the hypotheses of Theorem 5.3 are satisfied. Therefore, for all $f\in L^1(Q)$, the problem

$$\begin{gathered}
b(x,u)\in L^\infty([0,T];L^1(\Omega));\quad T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega,w)),\\
\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}}d^\lambda(x)\sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p-1}\operatorname{sgn}(\frac{\partial u}{\partial x_i})\frac{\partial u}{\partial x_i}\,dx\,dt=0;\\
B_S(x,r)=\int_0^r\frac{\partial b(x,\sigma)}{\partial\sigma}S'(\sigma)d\sigma,\\
\int_\Omega B_S(x,u(T))\varphi(T)dx-\int_Q B_S(x,u)\frac{\partial\varphi}{\partial t}\,dx\,dt\\
+\int_Q S'(u)d^\lambda(x)\sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p-1}\operatorname{sgn}(\frac{\partial u}{\partial x_i})\frac{\partial\varphi}{\partial x_i}\,dx\,dt\\
+\int_Q S''(u)d^\lambda(x)\sum_{i=1}^{N}\big|\frac{\partial u}{\partial x_i}\big|^{p-1}\operatorname{sgn}(\frac{\partial u}{\partial x_i})\frac{\partial u}{\partial x_i}\varphi\,dx\,dt\\
+\int_Q\rho S'(u)\sin(u)\exp(u^{-2})\sum_{i=1}^{N}w_i\big|\frac{\partial u}{\partial x_i}\big|^{p-1}\varphi\,dx\,dt\\
=\int_Q fS'(u)\varphi\,dx\,dt+\int_\Omega B_S(x,u_0)\varphi(0)dx,\\
B_S(x,u)(t=0)=B_S(x,u_0)\quad\text{in }\Omega,
\end{gathered}$$

for all $\varphi\in C_0^\infty(Q)$ and $S\in W^{1,\infty}(\mathbb{R})$ with $S'\in C_0^\infty(\mathbb{R})$, has at least one renormalised solution.

7. Appendix

Proof of Lemma 5.5. (see also [15]) Integration by parts and the use of the properties of $(w)^i_\mu$ yield

$$\begin{aligned}
&\int_0^T \int_{\{x\in\Omega; T_k(u_n)-w^i_\mu\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)\exp(G(u_n))(T_k(u_n)-w^i_\mu)\,dx\,dt \\
&= \int_0^T \int_{\{x\in\Omega; T_k(u_n)-w^i_\mu\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)T_k(u_n)\exp(G(u_n)),dx\,dt \\
&- \int_0^T \int_{\{x\in\Omega; T_k(u_n)-w^i_\mu\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)\exp(G(u_n))w^i_\mu dxdt \\
&= I_1^n + I_2^{n,\mu}.
\end{aligned} \tag{7.1}$$

We denote

$$B^n_{m,k}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} h_m(s)T_k(s)\exp(G(s))ds,$$
$$B^n_m(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} h_m(s)\exp(G(s))ds.$$

By a standard argument we can write the first term on the right-hand side of (7.1) as

$$\begin{aligned}
I_1^n &= \Big[\int_{\{x\in\Omega;\ T_k(u_n)-w^i_\mu\geq 0\}} B^n_{m,k}(x,u_n)dx\Big]_0^T \\
&= \int_{\{x\in\Omega;\ T_k(u_n)(T)-w^i_\mu(T)\geq 0\}} B^n_{m,k}(x,T_m(u_n)(T))dx \\
&- \int_{\{x\in\Omega;\ T_k(u_n)(0)-w^i_\mu(0)\geq 0\}} B^n_{m,k}(x,T_m(u_n)(0))dx.
\end{aligned} \tag{7.2}$$

We observe that

$$\frac{\partial b_n(x,T_m(u_n))}{\partial s} h_m(u_n) = \Big(\frac{\partial b_n(x,T_m(u_n))}{\partial s} + \frac{1}{n}\Big) h_m(u_n)$$

for $n > m$ with $\operatorname{supp} h_m \subset [-m;m]$. Passing to the limit in (7.2) as $n \to +\infty$, we deduce that

$$\begin{aligned}
I_1^n &= \int_{\{x\in\Omega;\ T_k(u)(T)-w^i_\mu(T)\geq 0\}} B_{m,k}(x,T_m(u(T)))dx \\
&- \int_{\{x\in\Omega;\ T_k(u)(0)-w^i_\mu(0)\geq 0\}} B_{m,k}(x,T_m(u_0))dx + \varepsilon(n).
\end{aligned} \tag{7.3}$$

where $B_{m,k}(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} h_m(s)T_k(s)\exp(G(s))ds$. Passing to the limit in (7.3) as $i \to +\infty$ and $\mu \to +\infty$, we have

$$I_1^n = \int_\Omega [B_{m,k}(x,u(T)) - B_{m,k}(x,u_0)]dx + \varepsilon(n,\mu,i). \tag{7.4}$$

The second term on the right-hand side of (7.1) can be written as

$$
\begin{aligned}
I_2^{n,\mu} &= -\int_0^T \int_{\{x\in\Omega;/T_k(u_n)-w_\mu^i\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)\exp(G(u_n))w_\mu^i dxdt \\
&= -\Big[\int_{\{x\in\Omega;\ T_k(u_n)-w_\mu^i\geq 0\}} B_m^n(x,u_n)w_\mu^i dx\Big]_0^T \\
&\int_0^T \int_{\{x\in\Omega;\ T_k(u_n)-w_\mu^i\geq 0\}} B_m^n(x,u_n)\frac{\partial w_\mu^i}{\partial t}\,dx\,dt \\
&= -\int_{\{x\in\Omega;\ T_k(u_n)(T)-w_\mu^i(T)\geq 0\}} B_m^n(x,T_m(u_n(T)))w_\mu^i(T)dx \\
&+ \int_{\{x\in\Omega;\ T_k(u_n)(0)-w_\mu^i(0)\geq 0\}} B_m^n(x,u_{0n})w_\mu^i(0)dx \\
&\quad + \mu\int_0^T \int_{\{x\in\Omega;\ T_k(u_n)-w_\mu^i\geq 0\}} B_m^n(x,u_n)(T_k(u)-w_\mu^i)\,dx\,dt.
\end{aligned}
\tag{7.5}
$$

By passing to the limit as n tends to infinity in (7.5), we obtain

$$
\begin{aligned}
I_2^{n,\mu} &= -\int_{\{x\in\Omega;\ T_k(u)-w_\mu^i\geq 0\}} [B_m(x,u(T))w_\mu^i(T) - B_m(x,u_0)w_\mu^i(0)dx \\
&\quad + \mu\int_{\{x\in\Omega;\ T_k(u)-w_\mu^i\geq 0\}} \int_0^T B_m(x,u)(T_k(u)-w_\mu^i)\,dx\,dt + \varepsilon(n),
\end{aligned}
$$

where $B_m(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} h_m(s)\exp(G(s))ds$. Therefore, passing to the limit, in i and μ , in the first terms on the right-hand side of the last equality, we deduce that

$$
\begin{aligned}
&\int_{\{x\in\Omega;\ T_k(u)-w_\mu^i\geq 0\}} [B_m(x,u(T))w_\mu^i(T) - B_m(x,u_0)w_\mu^i(0)dx \\
&= \int_\Omega [B_m(x,u(T))(T_k(u(T)) - B_m(x,u_0)T_k(u_0))dx + \varepsilon(n,\mu,i).
\end{aligned}
\tag{7.6}
$$

The second term on the right-hand side of (7.5) can be rewritten as

$$
\begin{aligned}
&\mu\int_0^T \int_{\{x\in\Omega;\ T_k(u)-w_\mu^i\geq 0\}} B_m(x,u)(T_k(u)-w_\mu^i)\,dx\,dt \\
&= \mu\int_0^T \int_{\{x\in\Omega;\ T_k(u)-w_\mu^i\geq 0\}} (B_m(x,u)-B_m(x,T_k(u)))(T_k(u)-w_\mu^i)\,dx\,dt \\
&+ \mu\int_0^T \int_{\{x\in\Omega;\ T_k(u)-w_\mu^i\geq 0\}} (B_m(x,T_k(u))-B_m(x,w_\mu^i)(T_k(u)-w_\mu^i)\,dx\,dt \\
&+ \mu\int_0^T \int_{\{x\in\Omega;\ T_k(u)-w_\mu^i\geq 0\}} B_m(x,w_\mu^i)(T_k(u)-w_\mu^i)\,dx\,dt \\
&= J_1+J_2+J_3,
\end{aligned}
\tag{7.7}
$$

where

$$\begin{aligned} J_1 &= \mu \int_0^T \int_{\{x\in\Omega;\ T_k(u)-w^i_\mu\geq 0; u>k\}} (B_m(x,u)-B_m(x,k))(k-w^i_\mu)\,dx\,dt \\ &\quad + \mu \int_0^T \int_{\{x\in\Omega;\ T_k(u)-w^i_\mu\geq 0; u<-k\}} (B_m(x,u)-B_m(x,-k))(-k-w^i_\mu)\,dx\,dt \\ &\geq 0. \end{aligned} \tag{7.8}$$

As $B_m(x,z)$ is non-decreasing for z and $-k\leq w^i_\mu \leq k$, it follows that

$$J_2 \geq 0. \tag{7.9}$$

Moreover,

$$\begin{aligned} J_3 &= \mu \int_0^T \int_{\{x\in\Omega;\ T_k(u)-w^i_\mu\geq 0\}} B_m(x,w^i_\mu)(T_k(u)-w^i_\mu)\,dx\,dt \\ &= \int_0^T \int_{\{x\in\Omega;\ T_k(u)-w^i_\mu\geq 0\}} B_m(x,w^i_\mu)\frac{\partial (w)^i_\mu}{\partial t}dxdt \\ &= \int_{\{x\in\Omega;\ T_k(u)(T)-w^i_\mu(T)\geq 0\}} \overline{B}(x,w^i_\mu(T))dx \\ &\quad - \int_{\{x\in\Omega;\ T_k(u)(0)-w^i_\mu(0)\geq 0\}} \overline{B}((x,w^i_\mu(0))dx, \end{aligned} \tag{7.10}$$

where $\overline{B}(x,z)=\int_0^z B_m(x,r)dr$. Also $w^i_\mu \to T_k(u)$ a.e. in Q as i and μ tends to $+\infty$ and $|w^i_\mu|\leq k$. Then Lebegue's convergence theorem shows that

$$J_3 = \int_\Omega (\overline{B}(x,T_k(u(T))) - \overline{B}(x,T_k(u_0)))dx + \varepsilon(n,\mu,i). \tag{7.11}$$

In view of (7.6)-(7.11), one has

$$\begin{aligned} I_2^{n,\mu} &\geq -\int_\Omega [B_m(x,u(T))T_k(u(T)) - B_m(x,u_0)T_k(u_0)]dx \\ &\quad + \int_\Omega (\overline{B}(x,T_k(u(T))) - \overline{B}(x,T_k(u_0)))dx + \varepsilon(n,\mu,i). \end{aligned} \tag{7.12}$$

As a consequence of (7.1), (7.4) and (7.12), we deduce that

$$\begin{aligned} &\int_{\{(x,t)\in\Omega\times(0,T);\ \ T_k(u)-w^i_\mu\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} h_m(u_n)\exp(G(u_n))(T_k(u_n)-w^i_\mu)dxdt \geq \\ &\geq \int_\Omega [B_{m,k}(x,u(T)) - B_{m,k}(x,u_0)]dx \\ &\quad - \int_\Omega [B_m(x,u(T))T_k(u(T)) - B_m(x,u_0)T_k(u_0)]dx \\ &\quad + \int_\Omega (\overline{B}(x,T_k(u(T))) - \overline{B}(x,T_k(u_0)))dx + \varepsilon(n,\mu,i). \end{aligned} \tag{7.13}$$

Observe that for any $z\in\mathbb{R}$ and for almost every $x\in\Omega$, we have

$$\overline{B}(x,T_k(z)) = B_m(x,z)T_k(z) - B_{m,k}(x,z).$$

Indeed,

$$
\begin{aligned}
\overline{B}(x,T_k(z)) &= \int_0^{T_k(z)} B_m(x,r)dr \\
&= \Big[r\int_0^r \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma\Big]_0^{T_k(z)} \\
&\quad -\int_0^{T_k(z)} r\frac{\partial b(x,r)}{\partial r}h_m(r)\exp(G(r))dr \\
&= T_k(z)\int_0^{T_k(z)} \frac{\partial b(x,r)}{\partial r}h_m(r)\exp(G(r))dr \\
&\quad -\int_0^{T_k(z)} T_k(r)\frac{\partial b(x,r)}{\partial r}h_m(r)\exp(G(r))dr \\
&= T_k(z)B_m(x,T_k(z)) - B_{m,k}(x,T_k(z)).
\end{aligned}
\tag{7.14}
$$

This is due to the fact that for $|r|<k$, we have

$$
\overline{B}(x,T_k(r)) = T_k(r)B_m(x,r) - B_{m,k}(x,r),
$$

and if $r>k$ we have

$$
\begin{aligned}
&B_{m,k}(x,r)\\
&= \int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\sigma\exp(G(\sigma))d\sigma + k\int_k^r \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma,\\
&-T_k(r)B_m(x,r)\\
&= -k\int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma - k\int_k^r \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma,
\end{aligned}
$$

and

$$
\overline{B}(x,k) = k\int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))d\sigma - k\int_0^k \frac{\partial b(x,\sigma)}{\partial\sigma}h_m(\sigma)\exp(G(\sigma))\sigma\,d\sigma.
$$

The case $r<-k$ is similar to the previous one. This conclude the proof. □

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