

A GENERALIZATION OF OSGOOD'S TEST AND A COMPARISON CRITERION FOR INTEGRAL EQUATIONS WITH NOISE

MARCOS J. CEBALLOS-LIRA, JORGE E. MACÍAS-DÍAZ, JOSÉ VILLA

ABSTRACT. In this article, we prove a generalization of Osgood's test for the explosion of the solutions of initial-value problems. We also establish a comparison criterion for the solution of integral equations with noise, and provide estimations of the time of explosion of problems arising in the investigation of crack failures where the noise is the absolute value of the Brownian motion.

1. INTRODUCTION

Let x_0 be a positive, real number, let b be a positive, real-valued function defined on $[0, \infty)$, and suppose that y is an extended real-valued function with the same domain as b . The present work is motivated by a criterion for the explosion of the solutions of ordinary differential equations of the form

$$\begin{aligned} \frac{dy(t)}{dt} &= b(y(t)), \quad t > 0, \\ y(0) &= x_0. \end{aligned} \tag{1.1}$$

More precisely, the time of explosion of the solution of this initial-value problem is the nonnegative, extended real number $t_e = \sup\{t \geq 0 : y(t) < \infty\}$. The above-mentioned criterion is called *Osgood's test* after its author [9], and it states that t_e is finite if and only if $\int_{x_0}^{\infty} ds/b(s) < \infty$. In such case, $t_e = \int_{x_0}^{\infty} ds/b(s)$.

A natural question readily arises about the possibility to extend Osgood's test to more general, initial-value problems, say, to problems in which the drift function b in the ordinary differential equation of (1.1) is multiplied by a suitable, nonnegative function of t . Another direction of investigation would be to investigate conditions under which the solutions of the integral form of such equation with a noise function added, explode in a finite time. Evidently the consideration of these two problems as a single one is an interesting topic of study *per se*. In fact, the purpose of this paper is to provide a generalization of Osgood's test to integral equations with noise, which generalize the problem presented in (1.1). Important, as it is in the recent literature [5, 7], the problem of establishing analytical conditions under which the time of explosion of the problem under investigation is finite, is tackled here. In

2000 *Mathematics Subject Classification*. 45G10, 45R05, 92F05, 74R10, 74R15.

Key words and phrases. Osgood's test; comparison criterion; time of explosion; integral equations with noise; crack failure.

©2011 Texas State University - San Marcos.

Submitted December 7, 2010. Published January 12, 2011.

the way, we establish a comparison criterion for the solutions of integral equations with noise, and show some applications to the spread of cracks in rigid surfaces.

Our manuscript is divided in the following way: Section 2 introduces the integral equation with noise that motivates this manuscript, along with a convenient simplification for its study; a generalization of Osgood's test is presented in this stage for the associated initial-value problem for both scenarios: noiseless and noisy systems. Section 3 establishes a comparison criterion for the solutions of two noiseless systems with comparable initial conditions. A necessary condition for the explosion of the solutions of the problem under investigation is provided in this section, together with an illustrative counterexample and a partial converse. In Section 4, we give upper and lower bounds for the value of the time of explosion of our integral equation. Finally, Section 5 provides estimates of probabilities associated to the time of explosion of a system in which the noise is the absolute value of the Brownian motion.

2. OSGOOD'S TEST

Let $\overline{\mathbb{R}}$ denote the set of extended real numbers. Throughout, $a, b : [0, \infty) \rightarrow \mathbb{R}$ will represent positive, continuous functions, while the function $g : [0, \infty) \rightarrow \mathbb{R}$ will be continuous and nonnegative. For physical reasons, the function g is called a *noise*. In this work, x_0 will denote a positive, real number, and $X : [0, \infty) \rightarrow \overline{\mathbb{R}}$ will be a nonnegative function whose dependency on $t \geq 0$ is represented by X_t . We are interested in establishing conditions under which the solutions of the integral equation

$$X_t = x_0 + \int_0^t a(s)b(X_s)ds + g(t), \quad t \geq 0, \quad (2.1)$$

explode in finite time. More precisely, we define the *time of explosion* of X as the nonnegative, extended real number $T_e^X = \sup\{t \geq 0 : X_t < \infty\}$. In this manuscript, we investigate conditions under which the time of explosion of X is a real number.

Letting $Y_t = X_t - g(t)$, one sees immediately that the problem under consideration is equivalent to finding the time of explosion of the solution Y of the equation

$$Y_t = x_0 + \int_0^t a(s)b(Y_s + g(s))ds, \quad t \geq 0. \quad (2.2)$$

As a matter of fact, $T_e^X = T_e^Y$. From this point on, this common, extended real number will be denoted simply by T_e for the sake of brevity.

Remark 2.1. It is worth noticing that (2.2) can be presented in differential form as the equivalent, initial-value problem

$$\begin{aligned} \frac{dY_t}{dt} &= a(t)b(Y_t + g(t)), \quad t > 0, \\ Y_0 &= x_0, \end{aligned} \quad (2.3)$$

a problem for which the existence of solutions is guaranteed, for instance, when b is locally Lipschitzian and a is regulated (see [4, (10.4.6)])

Let r be a real number such that $0 < r \leq x_0$. We define the functions $A : [0, \infty) \rightarrow \mathbb{R}$ and $B_r : [x_0 - r, \infty) \rightarrow \mathbb{R}$ by

$$A(t) = \int_0^t a(s)ds \quad \text{and} \quad B_r(x) = \int_{x_0-r}^x \frac{ds}{b(s)}.$$

For the sake of convenience, we let B be the function B_0 . Evidently, both of these functions are nonnegative, increasing and continuous, and so are their inverses. On the other hand, if $r \geq -x_0$, we let $\tilde{B}_r : [x_0 + r, \infty) \rightarrow \mathbb{R}$ be given by $\tilde{B}_r = B_{-r}$. For every fixed $x \geq x_0$, we define $\tilde{B}^x : [-x_0, x - x_0] \rightarrow \mathbb{R}$ by $\tilde{B}^x(r) = \tilde{B}_r(x)$; we prefer this second notation in either case. Additionally, we define $\beta : [-x_0, \infty) \rightarrow \mathbb{R}$ by $\beta(t) = \tilde{B}^\infty(t)$. All of these functions and their inverses are nonnegative, continuous and decreasing in their domains.

Lemma 2.2 (Generalized Osgood's test). *The initial-value problem*

$$\begin{aligned} \frac{dy(t)}{dt} &= a(t)b(y(t)), \quad t > 0, \\ y(0) &= x_0, \end{aligned} \tag{2.4}$$

has a unique solution given by $y(t) = B^{-1}(A(t))$, for $t < A^{-1}(B(\infty))$. The solution explodes in finite time if and only if $B(\infty) < A(\infty)$, in which case, $T_e^y = A^{-1}(B(\infty))$.

Proof. The function $y(t) = B^{-1}(A(t))$ is evidently a solution of (2.4). Additionally, expressing the differential equation in (2.4) as $y'(s)/b(y(s)) = a(s)$, integrating both sides over $[0, t]$ and performing a suitable substitution, we obtain that $B(y(t)) = A(t)$, whence the uniqueness follows. Moreover, $y(t)$ is real if and only if $t < A^{-1}(B(\infty))$.

Now, if the solution of (2.4) explodes at the time $t_e < \infty$, then $B(\infty) = A(t_e) < A(\infty)$. Conversely, the number $A^{-1}(B(\infty))$ is real, so that

$$B(y(A^{-1}(B(\infty)))) = A(A^{-1}(B(\infty))) = B(\infty).$$

This implies that $T_e^y \leq A^{-1}(B(\infty))$, and the opposite inequality follows from the fact that the solution of (2.4) exists for $t < A^{-1}(B(\infty))$. \square

As a consequence, the solution of (2.4) is nonnegative, continuous and increasing on $[0, T_e^y)$, and so is its inverse on $[x_0, \infty)$. Likewise, the function $\bar{B} : [x_0, \infty) \rightarrow \mathbb{R}$, given by the formula

$$\bar{B}(y) = \int_{x_0}^y \frac{ds}{b(s + g(Y^{-1}(s)))},$$

is nonnegative, continuous and increasing.

Corollary 2.3. *The solution of (2.3) can be expressed as $Y_t = \bar{B}^{-1}(A(t))$, for every $t < A^{-1}(\bar{B}(\infty))$.*

The proof of the above corollary follows as in Osgood's test.

3. A COMPARISON THEOREM

Theorem 3.1 (Comparison criterion). *Let $0 < x_0 \leq x_1$, let b be non-decreasing, and assume that the functions $u, v : [0, \infty) \rightarrow \bar{\mathbb{R}}$ satisfy*

$$v(t) \geq x_1 + \int_0^t a(s)b(v(s))ds \quad \text{and} \quad u(t) = x_0 + \int_0^t a(s)b(u(s))ds, \quad t \geq 0.$$

Then, $v(t) \geq u(t)$ for every $t \geq 0$, and $T_e^v \leq A^{-1}(B(\infty))$.

Proof. It is sufficient to show that $v \geq u$ because, in such case, $T_e^v \leq T_e^u = A^{-1}(B(\infty))$. Assume first that $x_0 < x_1$, and let $N = \{t \geq 0 : u(s) \leq v(s), s \in [0, t]\}$. The set N is nonempty, so $\tilde{T} = \sup N$ exists in $\overline{\mathbb{R}}$. If \tilde{T} were a real number, then

$$L = \lim_{\epsilon \rightarrow 0^+} (v(\tilde{T} + \epsilon) - u(\tilde{T} + \epsilon)) \geq x_1 - x_0 + \lim_{\epsilon \rightarrow 0^+} \int_{\tilde{T}}^{\tilde{T} + \epsilon} a(s)[b(v(s)) - b(u(s))] ds$$

by the fact that $v(s) - u(s) \geq 0$, for every $s \in [0, \tilde{T}]$. It follows that $L \geq x_1 - x_0$. By definition, there exists $\delta > 0$ such that $v(\tilde{T} + s) - u(\tilde{T} + s) > 0$ for every $s \in [0, \delta)$, whence it follows that $\tilde{T} + \frac{\delta}{2} \in N$, a contradiction. Consequently, $u(t) \leq v(t)$ for every $t \geq 0$. Now, in case that $x_0 = x_1$, the solution of the equation

$$u_r(t) = x_0 - r + \int_0^t a(s)b(u_r(s))ds, \quad 0 < r < x_0,$$

satisfies $v(t) \geq u_r(t)$, for every $t \geq 0$. Using Osgood's test and the continuity of B_r^{-1} , we obtain

$$v(t) \geq \lim_{r \rightarrow 0^+} u_r(t) = \lim_{r \rightarrow 0^+} B_r^{-1}(A(t)) = B^{-1}(A(t)) = u(t).$$

□

Theorem 3.2. *Suppose that b is non-decreasing, and $B(\infty) < A(\infty)$. Then the solution of (2.3) explodes in finite time. The time of explosion of Y is $t_e = A^{-1}(\overline{B}(\infty))$.*

Proof. The fact that b is non-decreasing yields

$$Y_t = x_0 + \int_0^t a(s)b(Y_s + g(s))ds \geq x_0 + \int_0^t a(s)b(Y_s)ds.$$

Theorem 3.1 gives that $T_e \leq A^{-1}(B(\infty))$ when we compare Y with the solution of $\tilde{Y}_t = x_0 + \int_0^t a(s)b(\tilde{Y}_s)ds$. On the other hand, $A(t_e) = \overline{B}(\infty) \leq B(\infty) < A(\infty)$, which implies that t_e is real. The expression of t_e and Corollary 2.3 yield $A(t_e) = \overline{B}(Y_{T_e}) = A(T_e)$. We conclude that $T_e = t_e$. □

Intuitively, it is not generally true that the explosion of the solutions of (2.3) in finite time is a sufficient condition for the inequality $B(\infty) < A(\infty)$ to be satisfied. This assertion follows after noticing that A and B do not depend of the noise function; however, the time of explosion does. We will establish our claim precisely through the following counter-example.

Example 3.3. Let $x_0 = 1$, and let a , b and g be given by the expressions $a(t) = e^{-t}$, $b(t) = \frac{1}{4}t^3$, and $g(t) = e^t$, for every $t > 0$. Expanding the expression $(Y_s + e^s)^3$ in (2.2), we obtain $Y_t \geq 1 + \frac{1}{4} \int_0^t Y_s^2 ds$. Then $Y_t \geq (1 - \frac{1}{4}t)^{-1}$, which implies that Y explodes in finite time. However, $B(\infty) = 2 > 1 = A(\infty)$.

The following result is a partial converse of Theorem 3.2. We let $\hat{g}(t) = \sup\{g(s) : s \in [0, t]\}$, for every $t \geq 0$.

Proposition 3.4. *Suppose that b is non-decreasing, and that*

$$\hat{g}(t) < b(x_0) \int_t^\infty a(s)ds.$$

If the solution Y of (2.2) explodes in finite time, then $B(\infty) < A(\infty)$.

Proof. By Corollary 2.3, $\overline{B}(\infty) = \overline{B}(Y_{T_e}) = A(T_e)$. Since b is non-decreasing and $g(Y_s^{-1}) \leq \widehat{g}(T_e)$ for every $s \in [x_0, \infty)$, we obtain

$$\int_{x_0}^{\infty} \frac{ds}{b(s + \widehat{g}(T_e))} \leq A(T_e).$$

Separating the integral in the definition of $B(\infty)$ as the sum of the integrals over the intervals $[x_0, x_0 + \widehat{g}(T_e)]$ and $[x_0 + \widehat{g}(T_e), \infty)$, using the facts that b is positive and non-decreasing, and employing the last inequality and the hypothesis, we obtain

$$\begin{aligned} B(\infty) &\leq \int_{x_0}^{x_0 + \widehat{g}(T_e)} \frac{ds}{b(x_0)} + \int_{x_0}^{\infty} \frac{ds}{b(s + \widehat{g}(T_e))} \\ &\leq \frac{\widehat{g}(T_e)}{b(x_0)} + A(T_e) < A(\infty). \end{aligned}$$

□

4. APPROXIMATION OF THE EXPLOSION TIME

It is important to notice that the time of explosion of Y , as given by the Theorem 3.2, presents the disadvantage of depending on the solution Y itself. In this section, we will derive some approximations to T_e which do not present this shortcoming. For the remainder of this manuscript and for the sake of convenience, we let $T = A^{-1}(B(\infty))$. Throughout this section, b will be a non-decreasing function.

The Comparison criterion and Osgood's test yield that the time of explosion of the solution Y of (2.2) satisfies $T_e \leq T$. On the other hand,

$$Y_t \leq x_0 + \int_0^t a(s)b(Y_s + \widehat{g}(T))ds,$$

and the Comparison criterion leads us to conclude that

$$A^{-1}(\beta(\widehat{g}(T))) \leq T_e \leq T. \quad (4.1)$$

In general, the function $b : [0, \infty) \rightarrow \mathbb{R}$ is *sub-multiplicative* if there exists a positive constant c such that $b(xy) \leq cb(x)b(y)$, for every $x, y \geq 0$. Evidently, exponential and power functions are sub-multiplicative.

Suppose that b is a sub-multiplicative function, and let c be the positive number provided by the definition of sub-multiplicativity. In the following, it will be convenient to define the function $\tilde{A} : [0, \infty) \rightarrow \mathbb{R}$ by

$$\tilde{A}(t) = c \int_0^t a(s)b\left(\frac{1}{x_0}g(s) + 1\right)ds.$$

This function is nonnegative, continuous and increasing and, thus, it is invertible, and has a continuous and increasing inverse.

Proposition 4.1. *Let b be a sub-multiplicative function. Then $T_e \geq \tilde{A}^{-1}(B(\infty))$.*

Proof. Since $Y_s \geq x_0$ for every $s \geq 0$, we obtain

$$\begin{aligned} g(s) + Y_s &= Y_s g(s) \left(\frac{1}{Y_s} + \frac{1}{g(s)} \right) \\ &\leq Y_s g(s) \left(\frac{1}{x_0} + \frac{1}{g(s)} \right) = Y_s \left(\frac{1}{x_0} g(s) + 1 \right). \end{aligned}$$

Monotonicity and sub-multiplicativity of b , along with (2.2), yield

$$Y_t \leq x_0 + c \int_0^t a(s) b\left(\frac{1}{x_0} g(s) + 1\right) b(Y_s) ds.$$

The conclusion of the proposition follows now from the Comparison criterion and Osgood's test. \square

5. AN APPLICATION

Throughout this section, we consider the stochastic differential equation (2.1) with noise function $|W_t|$, where W is the Brownian motion. The noise function is taken in absolute value in view of physical considerations on the dynamics of cracks growth under fatigue loading. In fact, it has been established experimentally that cracks in the subcritical stage grow with a velocity that increases with the crack length [10]. The governing equation is called *Paris' equation*, and it is a power law (which is a sub-multiplicative function) in which the exponent is determined empirically. As a matter of fact, it has been established that Paris' law is valid for a wide range of materials [1, 2, 8, 11].

For the remainder of this work, we let $\Phi(x)$ represent the probability that a random variable with standard normal distribution assumes values in $[-x, x]$, for every $x \geq 0$.

Proposition 5.1. *Let $0 \leq t < T$. Then*

$$P(T_e \leq t) \leq 1 - \Phi\left(\frac{\beta^{-1}(A(t))}{\sqrt{T}}\right). \quad (5.1)$$

Proof. We use here the first inequality of (4.1). Notice that

$$P(T_e \leq t) \leq P(A^{-1}(\beta(|\widehat{W}_T|)) \leq t) = P(|\widehat{W}_T| \geq \beta^{-1}(A(t))).$$

Then [6, Section 2.8, Eq. (8.4)] completes the proof. \square

For every nonnegative, real number r , we let $T_r = \inf\{t > 0 : |W_t| = r\}$. Evidently, $|W_s| \leq r$, for every $s \in [0, T_r]$.

Proposition 5.2. *Let $0 \leq t \leq T$. For every $r \geq 0$,*

$$P(T_e \leq t | T_r < T) \leq \frac{1 - \Phi(r/\sqrt{A^{-1}(B(\tilde{B}_r^{-1}(A(t))))})}{1 - \Phi(r/\sqrt{T})}. \quad (5.2)$$

Proof. Notice that $|\widehat{W}_T| \geq r$ whenever $T_r < T$. Moreover, Osgood's test and the Comparison criterion imply that $Y_t \geq B^{-1}(A(t))$, for every $t \geq 0$. Using (4.1), we obtain $A(T_e) \geq \tilde{B}_{|\widehat{W}_T|}(\infty) \geq \tilde{B}_r(\infty) \geq \tilde{B}_r(B^{-1}(A(T_r)))$. Therefore,

$$\begin{aligned} P(T_e \leq t | T_r < T) &\leq \frac{P(\tilde{B}_r(B^{-1}(A(T_r))) \leq A(t))}{P(T_r < T)} \\ &= \frac{P(T_r < A^{-1}(B(\tilde{B}_r^{-1}(A(t))))}{P(T_r < T)}. \end{aligned}$$

The conclusion follows now from [6] as in Proposition 5.1. \square

On physical grounds, the function Y may represent the temporal behavior of the transversal length of a crack failure on some material [10]. In this context, the parameter x_0 represents the initial, transversal length of the crack, and L is the transversal length of the material. For practical purposes, one may think of the wing of an airplane which has a fixed transversal length, on which there is a crack with known initial length. In such case, one investigates the dynamics of the length of the crack with respect to time, in order to conduct preventive maintenance on the wing and avoid possible accidents [3].

Proposition 5.3. *If $L > x_0$, then*

$$P(Y_L^{-1} \leq t) \leq 1 - \Phi\left(\frac{\tilde{B}_{A(t)}^{-1}(L)}{\sqrt{T}}\right).$$

Proof. Let \tilde{Y} the solution of $\tilde{Y}_t = x_0 + \int_0^t a(s)b(\tilde{Y}_s + |\widehat{W}_T|)ds$, for every $0 \leq t < T$. By Osgood's test and the Comparison criterion, $\tilde{B}_{|\widehat{W}_T|}^{-1}(A(t)) = \tilde{Y}_t \geq Y_t$. Once again, the conclusion is reached using [6] in the right-most end of the chain of identities and inequalities

$$\begin{aligned} P(Y_L^{-1} \leq t) &\leq P(\tilde{Y}_L^{-1} \leq t) \\ &= P(\tilde{B}_{|\widehat{W}_T|}(L) \leq A(t)) \\ &= 1 - P(|\widehat{W}_T| \leq \tilde{B}_{A(t)}^{-1}(L)). \end{aligned}$$

□

Example 5.4. Let x_0 , a_0 and α be positive numbers, and let $a(t) = a_0$ and $b(t) = t^{1+\alpha}$, for every $t \geq 0$. Observe that $A(t) = a_0 t$ and, for every $r \in [0, x_0]$ and every $x \geq x_0$,

$$B_r(x) = \frac{1}{\alpha} \left[\frac{1}{(x_0 - r)^\alpha} - \frac{1}{x^\alpha} \right],$$

so that $T = (\alpha a_0 x_0^\alpha)^{-1}$. By (5.1),

$$P(T_e \leq t) \leq 1 - \Phi\left(\frac{(\alpha a_0 t)^{-1/\alpha} - x_0}{\sqrt{T}}\right), \quad (5.3)$$

for every $0 \leq t < T$. In order to estimate the value of t for which $T_e \leq t$ with a probability of at most 0.05, Equation (5.3) yields

$$1 - \Phi\left(\frac{(\alpha a_0 t)^{-1/\alpha} - x_0}{\sqrt{T}}\right) \leq 0.05$$

whence it follows that $t = \frac{1}{\alpha a_0} [x_0 + \sqrt{T} \Phi^{-1}(0.95)]^{-\alpha}$. Proposition 4.1 and monotonicity on the integrand imply that

$$\begin{aligned} P(T_e \leq t) &\leq P(B(\infty) \leq \tilde{A}(t)) \\ &\leq P\left(\frac{1}{\alpha x_0^\alpha} \leq \int_0^t a_0 \left(\frac{1}{x_0} |\widehat{W}_t| + 1\right)^{1+\alpha} ds\right) \\ &\leq P\left(\frac{1}{\alpha x_0^\alpha} \leq a_0 t \left(\frac{1}{x_0} |\widehat{W}_t| + 1\right)^{1+\alpha}\right) \\ &= 1 - \Phi\left(\frac{x_0}{\sqrt{t}} \left((\alpha a_0 x_0^\alpha t)^{-1/(1+\alpha)} - 1\right)\right). \end{aligned}$$

This last estimate of $P(T_e \leq t)$ is better than that given by (5.3), in view of the fact that $\alpha a_0 x_0^\alpha t > 1$.

Acknowledgments. M. J. Ceballos-Lira wishes to acknowledge the financial support of the Mexican Council for Science and Technology (CONACYT) to pursue postgraduate studies in Universidad Juárez Autónoma de Tabasco (UJAT); he also wishes to thank UJAT and the Universidad Autónoma de Aguascalientes (UAA) for additional, partial, financial support. J. Villa acknowledges the partial support of CONACYT grant 118294, and grant PIM08-2 at UAA.

REFERENCES

- [1] R. J. Allen, G. S. Booth, T. Jutla; *A review of fatigue crack growth characterisation by linear elastic fracture mechanics (LEFM). Part I – Principles and methods of data generation*, Fatigue & Fracture Eng. Mater. & Struct. 11 (1988) 45–69.
- [2] R. J. Allen, G. S. Booth, T. Jutla; *A review of fatigue crack growth characterisation by linear elastic fracture mechanics (LEFM). Part II – Advisory documents and applications within national standards*, Fatigue & Fracture Eng. Mater. & Struct. 11 (1988) 71–108.
- [3] P. Augustin; *Simulation of Fatigue Crack Growth in Integrally Stiffened Panels Under the Constant Amplitude and Spectrum Loading*, Fatigue Aircraft Struct. 1 (2009) 5–19.
- [4] J. Dieudonné; *Foundations of Modern Analysis*, 2nd Edition, Pure and Applied Mathematics, Academic Press, Inc., New York, 1969.
- [5] P. Groisman, J. D. Rossi; *Explosion time in stochastic differential equations with small diffusion*, Electron. J. Diff. Eqns., Vol. 2007 No. 140 (2007) 1-9.
- [6] I. Karatzas, S. E. Shreve; *Brownian Motion and Stochastic Calculus*, 2nd Edition, Graduate Texts in Mathematics, Springer, USA, 1998.
- [7] J. A. León, J. Villa; *An Osgood criterion for integral equations with applications to stochastic differential equations with an additive noise*, Stat. & Prob. Letters. In press, doi:10.1016/j.spl.2010.10.001.
- [8] K. J. Mach, B. B. Hale, M. W. Denny, D. V. Nelson; *Death by small forces: a fracture and fatigue analysis of wave-swept macroalgae.*, J. Experim. Biol. 210 (2007) 2231.
- [9] W. F. Osgood; *Beweis der Existenz einer Lösung der Differentialgleichung $dy/dx = f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitz'schen Bedingung*, Monatshefte für Mathematik 9 (1898) 331–345.
- [10] P. Paris, F. Erdogan; *A critical analysis of crack propagation laws*, J. Basic Eng. 85 (1963) 528–534.
- [11] R. P. Wei; *Fracture mechanics approach to fatigue analysis in design*, J. Eng. Mater. Tech. 100 (1978) 113.

MARCOS JOSIAS CEBALLOS-LIRA

DIVISIÓN ACADÉMICA DE CIENCIAS BÁSICAS, UNIVERSIDAD JUÁREZ AUTÓNOMA DE TABASCO,
KM. 1 CARRETERA CUNDUACÁN-JALPA DE MÉNDEZ, CUNDUACÁN, TAB. 86690, MEXICO

E-mail address: marjoce1.81@hotmail.com

JORGE EDUARDO MACÍAS-DÍAZ

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA, UNIVERSIDAD AUTÓNOMA DE AGUASCALIENTES,
AVENIDA UNIVERSIDAD 940, CIUDAD UNIVERSITARIA, AGUASCALIENTES, AGS. 20131, MEXICO

E-mail address: jemacias@correo.uaa.mx

JOSÉ VILLA MORALES

DEPARTAMENTO DE MATEMÁTICAS Y FÍSICA, UNIVERSIDAD AUTÓNOMA DE AGUASCALIENTES,
AVENIDA UNIVERSIDAD 940, CIUDAD UNIVERSITARIA, AGUASCALIENTES, AGS. 20131, MEXICO

E-mail address: jvilla@correo.uaa.mx