Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 06, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A REGULARITY CRITERION FOR THE NAVIER-STOKES EQUATIONS IN TERMS OF THE HORIZONTAL DERIVATIVES OF THE TWO VELOCITY COMPONENTS

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Abstract. In this article, we consider the regularity for weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. It is proved that if the horizontal derivatives of the two velocity components

$$\nabla_h \widetilde{u} \in L^{2/(2-r)}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)), \quad \text{for } 0<r<1,$$

then the weak solution is actually strong, where $\dot{\mathcal{M}}_{2,3/r}$ is the critical Morrey-Campanato space and $\widetilde{u}=(u_1,u_2,0)$, $\nabla_h\widetilde{u}=(\partial_1 u_1,\partial_2 u_2,0)$.

1. Introduction

We consider the following Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3\times(0,T)$,

$$\begin{gathered}\partial_t u+(u\cdot\nabla)u+\nabla p=\Delta u,\\ \nabla.u=0,\\ u(x,0)=u_0(x),\end{gathered}\tag{1.1}$$

where $u=(u_1,u_2,u_3)$ is the velocity field, $p(x,t)$ is a scalar pressure, and $u_0(x)$ with $\nabla.u_0=0$ in the sense of distribution is the initial velocity field.

Although a global weak solution of (1.1) was first constructed by Leray [11] in 1934, the fundamental problem on uniqueness and regularity of weak solutions still remains open, although huge contributions have been made in an effort to understand regularities of the weak solution. It is well-known that regularity can be persistent under certain condition, which was introduced in the celebrated work of Serrin [14], and can be described as follows (see also Struwe [15]).

A *weak solution* u is regular if it satisfies the growth condition

$$u\in L^p(0,T;L^q(\mathbb{R}^3))\equiv L_t^pL_x^q,\quad \text{for } \frac{2}{p}+\frac{3}{q}=1,\ 3<q\le\infty\,. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35Q30, 76F65.
Key words and phrases. Navier-Stokes equations; Leray-Hopf weak solutions; regularity criterion.

Submitted October 18, 2010. Published January 12, 2011.

Regularity was also extended by Beirão da Veiga [1] with (1.2) replaced by the velocity gradient growth condition:

$$\nabla u \in L_t^p L_x^q, \quad \text{for } \frac{2}{p} + \frac{3}{q} = 2,\ \frac{3}{2} < q \leq \infty. \tag{1.3}$$

We recall that the condition (1.2) is important from the point of view of the relation between scaling invariance and partial regularity of weak solutions. In fact, the conditions (1.2) and (1.3) involve all components of the velocity vector field $u = (u_1, u_2, u_3)$ and are known as degree -1 growth condition, since

$$\begin{aligned} \|u(\lambda x, \lambda^2 t)\|_{L^p(0,T;L^q(\mathbb{R}^3))} &= \lambda^{-(\frac{2}{p}+\frac{3}{q})}\|u(x,t)\|_{L^p(0,\lambda^2 T;L^q(\mathbb{R}^3))} \\ &= \lambda^{-1}\|u\|_{L^p(0,\lambda^2 T;L^q(\mathbb{R}^3))}, \end{aligned}$$

$$\begin{aligned} \|\nabla u(\lambda x, \lambda^2 t)\|_{L^p(0,T;L^q(\mathbb{R}^3))} &= \lambda^{1-(\frac{2}{p}+\frac{3}{q})}\|\nabla u(x,t)\|_{L^p(0,\lambda^2 T;L^q(\mathbb{R}^3))} \\ &= \lambda^{-1}\|\nabla u\|_{L^p(0,\lambda^2 T;L^q(\mathbb{R}^3))}. \end{aligned}$$

The degree -1 growth condition is critical due to the scaling invariance property. That is, u solves (1.1) if and only if $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution of (1.1).

Regularity criteria in terms of only one component of the velocity were given in celebrated works by Zhou. It was proved in [16] (see also [13] and [17]) that regularity keeps under one of the following two conditions:

$$\begin{gathered} \nabla u_3 \in L_t^p L_x^q, \quad \text{for } \frac{2}{p} + \frac{3}{q} = \frac{3}{2},\ 2 < q \leq \infty, \\ u_3 \in L_t^p L_x^q, \quad \text{for } \frac{2}{p} + \frac{3}{q} = \frac{1}{2},\ 6 < q \leq \infty. \end{gathered}$$

Later on, some improvements and extensions were given by many authors, say [3, 4, 8, 18, 19]. Recently, Dong and Zhang [2] proved that if the horizontal derivatives of the two velocity components

$$\int_0^T \|\nabla_h \widetilde{u}(.,s)\|_{\dot{B}^0_{\infty,\infty}} ds < \infty,$$

where $\widetilde{u} = (u_1, u_2, 0)$ and $\nabla_h \widetilde{u} = (\partial_1 \widetilde{u}, \partial_2 \widetilde{u}, 0)$, then the solution keeps smoothness up to time T.

In this paper we want to prove the analogous result in the critical Morrey-Campanato space. More precisely, we show that the Leray-Hopf weak solution is regular on $(0, T]$ if the following growth condition with degree -1 is satisfied.,

$$\int_0^T \|\nabla_h \widetilde{u}(.,s)\|^{2/(2-r)}_{\dot{\mathcal{M}}_{2,3/r}} ds < \infty.$$

2. Preliminaries and main result

Now, we recall the definition and some properties of the space that will be useful in the sequel. These spaces play an important role in studying the regularity of solutions to partial differential equations; see e.g. [6] and references therein.

Definition 2.1. For $0 \leq r < 3/2$, the space $\dot{X}_r$ is defined as the space of $f(x) \in L^2_{\text{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2} < \infty.$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of the space $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{r/2}u\|_{L^2}$.

We have the homogeneity properties: for all $x_0 \in \mathbb{R}^3$,

$$\|f(.+x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r},$$
$$\|f(\lambda .)\|_{\dot{X}_r} = \frac{1}{\lambda^r}\|f\|_{\dot{X}_r}, \quad \lambda > 0.$$

The following imbedding holds

$$L^{3/r} \subset \dot{X}_r, \quad 0 \le r < \frac{3}{2}.$$

Now we recall the definition of Morrey-Campanato spaces (see e.g. [7]).

Definition 2.2. For $1 < p \le q \le +\infty$, the Morrey-Campanato space is

$$\dot{\mathcal{M}}_{p,q} = \big\{f \in L^p_{\rm loc}(\mathbb{R}^3) : \|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x\in\mathbb{R}^3}\sup_{R>0} R^{3/q-3/p}\|f\|_{L^p(B(x,R))} < \infty\big\}. \tag{2.1}$$

It is easy to check that

$$\|f(\lambda .)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{3/q}}\|f\|_{\dot{\mathcal{M}}_{p,q}}, \quad \lambda > 0.$$

We have the following comparison between Lorentz and Morrey-Campanato spaces: For $p \ge 2$,

$$L^{\frac{3}{r}}(\mathbb{R}^3) \subset \ L^{3/r,\infty}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3).$$

The relation

$$L^{\frac{3}{r},\infty}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{p,\frac{3}{r}}(\mathbb{R}^3)$$

is shown as follows. Let $f \in L^{3/r,\infty}(\mathbb{R}^3)$. Then

$$\begin{aligned}
\|f\|_{\dot{\mathcal{M}}_{p,\frac{3}{r}}} &\le \sup_E |E|^{\frac{r}{3}-\frac{1}{2}}\Big(\int_E |f(y)|^p dy\Big)^{1/p} \\
&= \Big(\sup_E |E|^{\frac{pr}{3}-1}\int_E |f(y)|^p dy\Big)^{1/p} \\
&\cong \Big(\sup_{R>0} R|\{x\in\mathbb{R}^3 : |f(y)|^p > R\}|^{\frac{pr}{3}} Big)^{1/p} \\
&= \sup_{R>0} R\big|\{x\in\mathbb{R}^p : |f(y)| > R\}\big|^{r/3} \\
&\cong \|f\|_{L^{3/r,\infty}}.
\end{aligned}$$

For $0 < r < 1$, we use the fact that

$$L^2 \cap \dot{H}^1 \subset \dot{B}^r_{2,1} \subset \dot{H}^r.$$

Thus we can replace the space $\dot{X}_r$ by the pointwise multipliers from Besov space $\dot{B}^r_{2,1}$ to L^2. Then we have the following lemma given in [10].

Lemma 2.3. *For $0 \le r < 3/2$, the space $\dot{Z}_r$ is defined as the space of $f(x) \in L^2_{\rm loc}(\mathbb{R}^3)$ such that*

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \le 1} \|fg\|_{L^2} < \infty.$$

Then $f \in \dot{\mathcal{M}}_{2,3/r}$ if and only if $f \in \dot{Z}_r$ with equivalence of norms.

To prove our main result, we need the following lemma.

Lemma 2.4. *For $0 < r < 1$, we have*

$$\|f\|_{\dot{B}^r_{2,1}} \le C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r.$$

Proof. The idea comes from [12]. According to the definition of Besov spaces,

$$\begin{aligned}
&\|f\|_{\dot{B}^r_{2,1}}\\
&= \sum_{j\in\mathbb{Z}} 2^{jr}\|\Delta_j f\|_{L^2}\\
&\le \sum_{j\le k} 2^{jr}\|\Delta_j f\|_{L^2} + \sum_{j>k} 2^{j(r-1)}2^j\|\Delta_j f\|_{L^2}\\
&\le \Big(\sum_{j\le k} 2^{2jr}\Big)^{1/2}\Big(\sum_{j\le k}\|\Delta_j f\|_{L^2}^2\Big)^{1/2} + \Big(\sum_{j>k} 2^{2j(r-1)}\Big)^{1/2}\Big(\sum_{j>k} 2^{2j}\|\Delta_j f\|_{L^2}^2\Big)^{1/2}\\
&\le C\big(2^{rk}\|f\|_{L^2} + 2^{k(r-1)}\|f\|_{\dot{H}^1}\big)\\
&= C\big(2^{rk}A^{-r} + 2^{k(r-1)}A^{1-r}\big)\|f\|_{L^2}^{1-r}\|f\|_{\dot{H}^1}^r,
\end{aligned}$$

where $A = \|f\|_{\dot{H}^1}/\|f\|_{L^2}$. Choose k such that $2^{rk}A^{-r} \le 1$; that is, $k \le [\log A^r]$, we thus obtain

$$\|f\|_{\dot{B}^r_{2,1}} \le C\big(1 + 2^{k(r-1)}A^{1-r}\big)\|f\|_{L^2}^{1-r}\|f\|_{\dot{H}^1}^r \le C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r.$$

□

Additionally, for $2 < p \le \frac{3}{r}$ and $0 \le r < \frac{3}{2}$, we have the following inclusion relations [9, 10],

$$\dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3) = \dot{Z}_r(\mathbb{R}^3).$$

The relation

$$\dot{X}_r(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$$

is shown as follows. Let $f \in \dot{X}_r(\mathbb{R}^3)$, $0 < R \le 1$, $x_0 \in \mathbb{R}^3$ and $\phi \in C_0^\infty(\mathbb{R}^3)$, $\phi \equiv 1$ on $B(\frac{x_0}{R}, 1)$. We have

$$\begin{aligned}
R^{r-\frac{3}{2}}\Big(\int_{|x-x_0|\le R}|f(x)|^2dx\Big)^{1/2} &= R^r\Big(\int_{|y-\frac{x_0}{R}|\le 1}|f(Ry)|^2dy\Big)^{1/2}\\
&\le R^r(\int_{y\in\mathbb{R}^3}|f(Ry)\phi(y)|^2dy)^{1/2}\\
&\le R^r\|f(R.)\|_{\dot{X}_r}\|\phi\|_{H^r}\\
&\le \|f\|_{\dot{X}_r}\|\phi\|_{H^r}\\
&\le C\|f\|_{\dot{X}_r}.
\end{aligned}$$

We recall the following definition of Leray-Hopf weak solution.

Definition 2.5. Let $u_0 \in L^2(\mathbb{R}^3)$ and $\nabla\cdot u_0 = 0$. A measurable vector field $u(x,t)$ is called a Leary-Hopf weak solution to the Navier-Stokes equations (1.1) on $(0,T)$, if u has the following properties:

(i) $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$;
(ii) $\partial_t u + (u\cdot\nabla)u + \nabla\pi = \Delta u$ in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$;
(iii) $\nabla\cdot u = 0$ in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$;

(iv) u satisfy the energy inequality

$$\|u(t)\|_{L^2_x}^2 + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u(x,s)|^2\, dxds \le \|u_0\|_{L^2_x}^2, \quad \text{for} \quad 0 \le t \le T. \tag{2.2}$$

By a strong solution we mean a weak solution u of the Navier-Stokes equations (1.1) that satisfies

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)). \tag{2.3}$$

It is well known that strong solutions are regular and unique in the class of weak solutions.

The following theorem is the main result of this article.

Theorem 2.6. *Suppose $u_0 \in H^1(\mathbb{R}^3)$ and $\nabla\cdot u_0 = 0$ in the sense of distributions. Assume that $u(x,t)$ is a Leray-Hopf weak solution of (1.1) on $(0,T)$. If*

$$\nabla_h \widetilde{u} \in L^{2/(2-r)}(0,T;\dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)), \quad \text{for } 0 < r < 1, \tag{2.4}$$

then u is a regular solution in $(0,T]$ in the sense that $u \in C^\infty((0,T)\times\mathbb{R}^3)$.

3. A priori estimates

Now we want to establish an a priori estimate for the smooth solution.

Theorem 3.1. *Suppose $T > 0$, $u_0 \in H^1(\mathbb{R}^3)$ and $\nabla\cdot u_0 = 0$ in the sense of distributions. Assume that u is a smooth solution of (1.1) on $\mathbb{R}^3\times(0,T)$ and satisfies any one of of the three degree -1 growth conditions (2.4). Then*

$$\begin{aligned} &\sup_{0\le t<T} \|\nabla u(.,t)\|_{L^2}^2 + \int_0^T \|\Delta u(.,t)\|_{L^2}^2\, ds \\ &\le C\|\nabla u_0\|_{L^2}^2 \exp\Big(\int_0^t \|\nabla_h\widetilde{u}(.,s)\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)} ds\Big), \end{aligned} \tag{3.1}$$

for $0 < t < T$, holds for some constant $C > 0$.

To prove this theorem, we need the following lemma.

Lemma 3.2 ([2])**.** *Let u be a smooth solution to the Navier-Stokes system (1.1) in $\mathbb{R}^3$. Furthermore, let $\widetilde{u} = (u_1,u_2,0)$ and $\nabla_h\widetilde{u} = (\partial_1\widetilde{u},\partial_2\widetilde{u},0)$. Then*

$$\Big|\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i\partial_i u_j \partial_{kk} u_j dx\Big| \le C\int_{\mathbb{R}^3} |\nabla_h\widetilde{u}||\nabla u|^2$$

for some constant $C > 0$.

The proof of this lemma is simple; see [2, Lemma 2.2]).

Proof of Theorem 3.1. Multiply the first equation of (1.1) by Δu, and integrating on $\mathbb{R}^3$, after suitable integration by parts, we obtain for $t \in (0,T)$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 \le 2\Big|\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i\partial_i u_j\partial_{kk}u_j dx\Big|. \tag{3.2}$$

Due to Hölder' s inequality and Lemma 2.4, the right-hand side (3.2) can be estimated as

$$\begin{aligned}
\Big|\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i u_j\partial_{kk}u_j dx\Big| &\leq \|\nabla_h\widetilde{u}\cdot\nabla u\|_{L^2}\|\nabla u\|_{L^2}\\
&\leq C\|\nabla_h\widetilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla u\|_{\dot{B}^{r}_{2,1}}\|\nabla u\|_{L^2}\\
&\leq C\|\nabla_h\widetilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla u\|_{L^2}^{2-r}\|\Delta u\|_{L^2}^{r}\\
&\leq C\Big(\|\nabla_h\widetilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}\|\nabla u\|_{L^2}^2\Big)^{(2-r)/2}(\|\Delta u\|_{L^2}^2)^{r/2}.
\end{aligned}$$

By Young' s inequality, we obtain

$$\Big|\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i u_j\partial_{kk}u_j dx\Big| \leq \frac{1}{2}\|\Delta u\|_{L^2}^2 + C\|\nabla_h\widetilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}\|\nabla u\|_{L^2}^2 \tag{3.3}$$

Substituting (3.3) into (3.2), it follows that

$$\frac{d}{dt}\|\nabla u(.,t)\|_{L^2}^2 + \|\Delta u(.,t)\|_{L^2}^2 \leq C\|\nabla_h\widetilde{u}\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}\|\nabla u\|_{L^2}^2.$$

Then Gronwall' s inequality yields

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \int_0^T\int_{\mathbb{R}^3}\|\Delta u(x,s)\|_{L^2}^2\,dx\,ds\\
&\leq C\|\nabla u_0\|_{L^2}^2\exp\Big(\int_0^t\|\nabla_h\widetilde{u}(.,s)\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}ds\Big).
\end{aligned}$$

This completes the proof . □

4. Proof of Theorem 2.6

After we established the key estimate in section 2, the proof of Theorem 2.6 is straightforward.

It is well known [5] that there is a unique strong solution $\overline{u}\in C([0,T^*),H^1(\mathbb{R}^3))\cap C^1([0,T^*),H^1(\mathbb{R}^3))\cap C([0,T^*),H^3(\mathbb{R}^3))$ to (1.1) for some $T^*>0$, for any given $u_0\in H^1(\mathbb{R}^3)$ with $\nabla.u_0=0$. Since u is a Leray-Hopf weak solution which satisfies the energy inequality (2.2), we have according to the Serrin's uniqueness criterion [14],

$$\bar{u}\equiv u \quad \text{on } [0,T^*).$$

By the assumption (2.4) and standard continuation argument, the local strong solution can be extended to time T. So we have proved u actually is a strong solution on $[0,T)$. This completes the proof of Theorem 2.6.

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