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EXISTENCE OF GLOBAL SOLUTIONS TO THE 2-D SUBCRITICAL DISSIPATIVE QUASI-GEOSTROPHIC EQUATION AND PERSISTENCY OF THE INITIAL REGULARITY

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Abstract. In this article, we prove that if the initial data θ_0 and its Riesz transforms ($\mathcal{R}_1(\theta_0)$ and $\mathcal{R}_2(\theta_0)$) belong to the space

$$\overline{(S(\mathbb{R}^2))}^{B^{1-2\alpha,\infty}_{\infty}}, \quad 1/2 < \alpha < 1,$$

then the 2-D Quasi-Geostrophic equation with dissipation α has a unique global in time solution θ. Moreover, we show that if in addition $\theta_0 \in X$ for some functional space X such as Lebesgue, Sobolev and Besov's spaces then the solution θ belongs to the space $C([0, +\infty[, X)$.

1. Introduction and statement of main results

In this article, we are study the initial value-problem for the two-dimensional quasi-geostrophic equation with sub-critical dissipation

$$\begin{gathered} \partial_t\theta + (-\Delta)^\alpha\theta + \nabla.(\theta u) = 0 \quad \text{on } \mathbb{R}^+_* \times \mathbb{R}^2 \\ \theta(0, x) = \theta_0(x), \quad x \in \mathbb{R}^2 \end{gathered} \tag{1.1}$$

where $\alpha \in]\frac{1}{2}, 1[$ is a fixed parameter and ∇ denotes the divergence operator with respect to the space variable $x \in \mathbb{R}^2$. The scalar function θ represents the potential temperature. The velocity $u = (u_1, u_2)$ is divergence free and determined from θ through the Riesz transforms

$$u = \mathcal{R}^\perp(\theta) \equiv (-\mathcal{R}_2(\theta), \mathcal{R}_1(\theta)).$$

The non local operator $(-\Delta)^\alpha$ is defined through the Fourier transform,

$$\mathcal{F}((-\Delta)^\alpha f)(\xi) = |\xi|^{2\alpha}\mathcal{F}(f)(\xi)$$

where $\mathcal{F}(f)$ is the Fourier transform of f defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^2} f(x)e^{-i\langle x,\xi\rangle}dx.$$

To study the existence of solutions to (1.1), we follow the Fujita-Kato method. Thus we convert (1.1) into the fixed point problem:

$$\theta(t) = e^{-t(-\Delta)^\alpha}\theta_0 + \mathcal{B}_\alpha[\theta, \theta](t). \tag{1.2}$$

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Here $(e^{-t(-\Delta)^\alpha})_{t>0}$ is the semi-group defined by

$$\mathcal{F}(e^{-t(-\Delta)^\alpha}f)(\xi) = e^{-t|\xi|^{2\alpha}}\mathcal{F}(f)(\xi)$$

and $\mathcal{B}_\alpha$ is the bi-linear operator

$$\mathcal{B}_\alpha[\theta_1, \theta_2](t) = -\mathcal{L}_\alpha(\theta_1 \mathcal{R}^\perp(\theta_2)) \tag{1.3}$$

where, for $v = (v_1, v_2)$,

$$\mathcal{L}_\alpha(v)(t) = \int_0^t \nabla . e^{-(t-s)(-\Delta)^\alpha} v ds. \tag{1.4}$$

In the sequel, by a mild solution on $]0, T[$ to (1.1) with data θ_0, we mean a function θ belonging to the space $L^2_{\rm loc}([0, T[, F_2)$ and satisfying in $\mathcal{D}'(]0, T[\times\mathbb{R}^2)$ the equation (1.2) where F_2 is the completion of $S(\mathbb{R}^2)$ with respect to the norm

$$\|f\|_{F_2} \equiv \sup_{x_0\in\mathbb{R}^2}(\|1_{B(x_0,1)}f\|_2 + \|1_{B(x_0,1)}\mathcal{R}^\perp(f)\|_2).$$

One of the main properties of (1.1) is the following scaling invariance property: If θ is a solution of (1.1) with data θ_0 then, for any $\lambda > 0$, the function $\theta_\lambda(t, x) \equiv \lambda^{2\alpha-1}\theta(\lambda^{2\alpha}t, \lambda x)$ is a solution of (1.1) with data $\theta_{0,\lambda}(x) \equiv \lambda^{2\alpha-1}\theta_0(\lambda x)$. This leads us to introduce the following notion of *super-critical space*: A Banach space X will be called *super-critical space* if $S(\mathbb{R}^2) \hookrightarrow X \hookrightarrow S(\mathbb{R}^2)$ and there exists a constant $C_X \geq 0$ such that for all $f \in X$,

$$\sup_{0<\lambda\leq 1} \lambda^{2\alpha-1}\|f(\lambda.)\|_X \leq C_X\|f\|_X.$$

For instance, the Lebesgue space $L^p(\mathbb{R}^2)$ (respectively, the Sobolev space $H^s(\mathbb{R}^2)$) is *super-critical space* if $p \geq p_c \equiv \frac{2}{2\alpha-1}$ (respectively, $s \geq s_c \equiv 2 - 2\alpha$). Moreover, one can easily prove that the Besov space $B_\infty^{1-2\alpha,\infty}(\mathbb{R}^2)$ is the greatest *super-critical space*. The first purpose of this paper, is to prove the global existence of smooth solutions of the equations (1.1) for initial data in a *super-critical space* $\tilde{\mathbf{B}}^\alpha$ closed to the space $B_\infty^{1-2\alpha,\infty}(\mathbb{R}^2)$. Our space $\tilde{\mathbf{B}}^\alpha$ is the completion of $S(\mathbb{R}^2)$ with respect to the norm

$$\|f\|_{\tilde{\mathbf{B}}^\alpha} \equiv \|f\|_{B_\infty^{1-2\alpha,\infty}} + \|\mathcal{R}^\perp(f)\|_{B_\infty^{1-2\alpha,\infty}}.$$

Before setting precisely our global existence result, let us recall some known results in this direction: in [22], Wu proved that for any initial data θ_0 in the space $L^p(\mathbb{R}^2)$ with $p > p_c = \frac{2}{2\alpha-1}$ the equations (1.1) has a unique global solution θ belonging to the space $L^\infty([0, +\infty[, L^p(\mathbb{R}^2))$. Similarly, Constantin and Wu [4] showed the global existence and uniqueness for arbitrary initial data in the Sobolev space $H^s(\mathbb{R}^2)$ where $s > s_c = 2 - 2\alpha$. However, we notice that these results don't cover the limit cases $p = p_c$ and $s = s_c$, that are critical regularity exponents.

We recall that global solutions are obtained under smallness size assumption on the initial data by several authors. For instance, one can quote the results of Wu [21] for $\theta_0 \in \dot{B}_p^{s_p,\infty}(\mathbb{R}^2)$ (critical spaces) with $s_p = \frac{2}{p} - (2\alpha - 1)$, Niche and Schonbek, [15] for $\theta_0 \in L^{p_c}(\mathbb{R}^2)$, with $p_c = \frac{2}{2\alpha-1}$, Lemarié-Rieusset and Marchand [11] for $\theta_0 \in L^{\frac{2}{2\alpha-1},\infty}(\mathbb{R}^2)$ and finally the work May and Zahrouni [12] where they considered initial data in the greatest critical homogeneous Besov space $\dot{B}_\infty^{-(2\alpha-1),\infty}(\mathbb{R}^2)$. The later one contains all the preceding critical spaces. Indeed, we have

$$\dot{L}^{p,s_p}(\mathbb{R}^2) \subset \dot{B}_p^{s_p,\infty}(\mathbb{R}^2) \subset \dot{B}_\infty^{-(2\alpha-1),\infty}(\mathbb{R}^2).$$

Our space of initial data $\tilde{\mathbf{B}}^\alpha$ introduced above contains all known critical spaces, in particular we have

$$\dot{B}_\infty^{-(2\alpha-1),\infty}(\mathbb{R}^2) \subset \tilde{\mathbf{B}}^\alpha.$$

Now we give our first result overcoming the above mentioned smallness assumption. Our global existence result reads as follows.

Theorem 1.1. *Let $\nu = 1 - (1/2\alpha)$. For any initial data $\theta_0 \in \tilde{\mathbf{B}}^\alpha$, equation* (1.1) *has a unique global solution θ belonging to the space $\cap_{T>0}\mathbf{E}_T^\nu$, where $\mathbf{E}_T^\nu$ is the completion of $C_c^\infty(]0,T] \times \mathbb{R}^2)$ with respect to the norm*

$$\|v\|_{\mathbf{E}_T^\nu} \equiv \sup_{0<t\leq T} t^\nu(\|v(t)\|_\infty + \|\mathcal{R}^\perp(v)(t)\|_\infty).$$

Moreover,

$$\theta \in C([0,+\infty[, \tilde{\mathbf{B}}^\alpha).$$

The proof of the above theorem is far from being a direct consequence of an application of a Fixed Point Theorem. We will establish a local existence result and will be able to get global existence that is essentially based on a new adapted version of the well-known maximal principle (Lemma 2.11) that we stated and proved in second section.

We can recover the results quoted above using our second main result that is a persistency Theorem stating that, the solution θ given by Theorem 1.1 keeps any further Besov or Lebesgue regularity of its initial data. Precisely, our theorem states as follows.

Theorem 1.2. *Let X be one of the following Banach spaces:*

- $X = L^p(\mathbb{R}^2)$ *with* $1 \leq p \leq \infty$;
- $X = B_p^{s,q}(\mathbb{R}^2)$ *with* $s > -1$ *and* $1 < p < \infty, 1 \leq q \leq \infty$;
- $X = \dot{B}_p^{s,q}(\mathbb{R}^2)$ *with* $s > 0$ *and* $1 \leq p, q \leq \infty$.

Assume $\theta_0 \in \tilde{\mathbf{B}}^\alpha \cap X$. Then the mild solution θ of the equation (1.1) *given by Theorem 1.1 belongs to the space $L^\infty_{\rm loc}([0,+\infty[, X)$. Moreover, if $\theta_0 \in \tilde{\mathbf{B}}^\alpha \cap \overline{S(\mathbb{R}^2)}^X$ then θ belongs to $C([0,+\infty[, \overline{S(\mathbb{R}^2)}^X)$.*

As a consequence of the previous theorems, we have the following theorem that generalizes the existence results of Wu [22] and Constantin and Wu [4] recalled above.

Theorem 1.3. *Let X be the Lebesgue space $L^p(\mathbb{R}^2)$ with $p \geq p_c = \frac{2}{2\alpha-1}$ or the Sobolev space $H^s(\mathbb{R}^2)$ with $s \geq s_c = 2 - 2\alpha$. Assume $\theta_0 \in X$. Then the equation* (1.1) *with initial data θ_0 has a unique global mild solution θ belonging to the space $C([0,+\infty[, X)$.*

We emphasize that the above stated results are new since the initial data considered here are in the nonhomogeneous space $\tilde{\mathbf{B}}^\alpha$, that is our knowledge the first time employed in this context. Moreover, we are allowed to obtain global solutions for this initial data without assuming any smallness assumption on its size. Thus we have a better results than those of Wu [21] and [4]. As a by product of our method we are able to extend the result of Wu to a large class of L^p spaces, for which we have also obtained the uniqueness issue. We focus on the fact that we have established the propagation of any further regularity of initial data belonging to $\tilde{\mathbf{B}}^\alpha$.

Our next challenge is to extend the use of our method to the critical Quasi-geostrophic equations.

The remainder of this paper is as follows: in section 2 we recall some definitions and we give some useful Lemmas that will be used in this paper. In section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2 and in section 4, we will prove Theorem 1.3.

2. Preliminaries

2.1. **Notation.** In this subsection, we introduce some notation that will be used frequently in this paper.

(1) Let X be a Banach space such that $S(\mathbb{R}^2) \hookrightarrow X \hookrightarrow S'(\mathbb{R}^2)$. We denote by $X_{\mathcal{R}}$ the space
$$X_{\mathcal{R}} = \{f \in X : \mathcal{R}^{\perp}(f) \in X^2\}$$
endowed with the norm
$$\|f\|_{X_{\mathcal{R}}} = \|f\|_X + \|\mathcal{R}^{\perp}(f)\|_X.$$
We recall that $\mathcal{R}^{\perp}(f) = (-\mathcal{R}_2 f, \mathcal{R}_1 f)$ where $\mathcal{R}_1$ and $\mathcal{R}_2$ are Riesz transforms.

(2) Let $T > 0$, $r \in [1,\infty]$ and X be a Banach space. $L^r_T X$ denotes the space $L^r([0,T[, X)$. In particular, $L^r_T L^p$ will denote the space $L^r([0,T[, L^p(\mathbb{R}^2))$.

(3) Let X be a Banach space, $T > 0$ and $\mu \in \mathbb{R}$. we denote by $L^{\infty}_{\mu}([0,T], X)$ the space of functions $f :]0,T] \to X$ such that
$$\|f\|_{L^{\infty}_{\mu}([0,T],X)} \equiv \sup_{0<t\le T} t^{\mu}\|f(t)\|_X < \infty \quad \text{and} \quad \lim_{t\to 0} t^{\mu}\|f(t)\|_X = 0.$$
The sub-space $C^0_{\mu}([0,T], X)$ of $L^{\infty}_{\mu}([0,T], X)$ is defined by
$$C^0_{\mu}([0,T], X) \equiv L^{\infty}_{\mu}([0,T], X) \cap C(]0,T], X).$$

(4) Let A and B be two reals functions. The notation $A \lesssim B$ means that there exists a constant C, independent of the effective parameters of A and B, such that $A \le CB$.

2.2. **Besov spaces.** The standard definition of Besov spaces passes through the Littlewood-Paley dyadic decomposition [1]. [7], and [10]. To this end, we take an arbitrary function $\psi \in \mathcal{S}(\mathbb{R}^2)$ whose Fourier transform $\hat{\psi}$ is such that $\mathrm{supp}(\hat{\psi}) \subset \{\xi, \frac{1}{2} \le |\xi| \le 2\}$, and for $\xi \neq 0$, $\sum_{j\in\mathbb{Z}} \hat{\psi}(\frac{\xi}{2^j}) = 1$, and define $\varphi \in \mathcal{S}(\mathbb{R}^2)$ by $\hat{\varphi}(\xi) = 1 - \sum_{j\ge 0} \hat{\psi}(\frac{\xi}{2^j})$. For $j \in \mathbb{Z}$, we write $\varphi_j(x) = 2^{2j}\varphi(2^j x)$ and $\psi_j(x) = 2^{2j}\psi(2^j x)$ and we denote the convolution operators S_j and Δ_j, respectively, the convolution operators by φ_j and ψ_j.

Definition 2.1. Let $1 \le p, q \le \infty$, $s \in \mathbb{R}$.
1. A tempered distribution f belongs to the (inhomogeneous) Besov space $B^{s,q}_p$ if and only if
$$\|f\|_{B^{s,q}_p} \equiv \|S_0 f\|_p + \Big(\sum_{j>0} 2^{jsq}\|\Delta_j f\|_p^q\Big)^{\frac{1}{q}} < \infty.$$
2. The homogeneous Besov space $\dot{B}^{s,q}_p$ is the space of $f \in \mathcal{S}'(\mathbb{R}^2)/_{\mathbb{R}[X]}$ such that
$$\|f\|_{\dot{B}^{s,q}_p} \equiv \Big(\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j f\|_p^q\Big)^{\frac{1}{q}} < \infty,$$

Where $\mathbb{R}[X]$ is the space of polynomials [17].

An equivalent definition more adapted to the Quasi-geostrophic equations involves the semigroup $(e^{-t(-\Delta)^\alpha})_{t>0}$.

Proposition 2.2. *If $s<0$ and $q=\infty$. Then*
$$f \in \dot{B}_p^{s,\infty} \Longleftrightarrow \sup_{t>0} t^{\frac{-s}{2\alpha}} \|e^{-t(-\Delta)^\alpha} f\|_p < \infty, \tag{2.1}$$
$$f \in B_p^{s,\infty} \Longleftrightarrow \forall T>0, \quad \sup_{0<t<T} t^{\frac{-s}{2\alpha}} \|e^{-t(-\Delta)^\alpha} f\|_p \le C_T. \tag{2.2}$$

The proof the above proposition can be easily done by following the same lines as in the proof in [10, Theorem 5.3] in the case of the heat Kernel. One can see also the proof in [13, Proposition 2.1].

2.3. **Intermediate results.** We shall frequently use the following estimates on the operator $e^{-t(-\Delta)^\alpha}$.

Proposition 2.3. *For $t>0$, we set $\mathcal{K}_t$ the kernel of $e^{-t(-\Delta)^\alpha}$. Then for all $r \in [1,\infty]$ we have*
$$\|\mathcal{K}_t\|_r = C_{1r} t^{\sigma_r}, \tag{2.3}$$
$$\|\nabla \mathcal{K}_t\|_r = C_{2r} t^{\sigma_r - \frac{1}{2\alpha}}, \tag{2.4}$$
$$\|\mathcal{R}_j \nabla \mathcal{K}_t\|_r = C_{3r} t^{\sigma_r - \frac{1}{2\alpha}}, \tag{2.5}$$
where $\sigma_r = \frac{1}{\alpha}(\frac{1}{r}-1)$ and C_{1r}, C_{2r} and C_{3r} are constants independent of t.

Proof. This propostion was previously proved in [22]. Equalities (2.3) and (2.4) can be found in [13]. Estimate (2.5) can be obtained by following the same argument as in [10, Proposition 11.1]. $\square$

Following the work of Lemarié-Rieusset, we introduce the notion of shift invariant functional space.

Definition 2.4. A Banach space X is called *shift invariant functional space* if
- $\mathcal{S}(\mathbb{R}^2) \hookrightarrow X \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$,
- for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and $f \in X$, $\|\varphi * f\|_X \le C_X \|\varphi\|_1 \|f\|_X$.

Remark 2.5. The Lebesgue spaces, the inhomogeneous Besov spaces $B_p^{s,q}$, with $s \in \mathbb{R}, 1 \le p,q \le \infty$, and the homogeneous Besov spaces $\dot{B}_p^{s,q}$, with $s>0, 1 \le p,q \le \infty$, are shift invariant functional spaces.

The proof of Theorem 1.1 requires the following lemmas.

Lemma 2.6. *Let X be a shift invariant functional space. If $f \in X$ then*
$$\sup_{t>0} \|e^{-t(-\Delta)^\alpha} f\|_X \le C_X \|f\|_X. \tag{2.6}$$
Moreover, if $f \in \overline{\mathcal{S}(\mathbb{R}^2)}^X$, then $e^{-t(-\Delta)^\alpha} f \in C(]0,\infty[, \overline{\mathcal{S}(\mathbb{R}^2)}^X)$ and $e^{-t(-\Delta)^\alpha} f \to f$ in X as $t \to 0^+$.

Proof. One obtain easily (2.6) from (2.3). Let us prove the last assertion. For $t>0$, we denote by $\mathcal{K}_t$ the kernel of the operator $e^{-t(-\Delta)^\alpha}$. Then $\mathcal{K}_t(.) = t^{-\frac{1}{\alpha}} \mathcal{K}(t^{-\frac{1}{2\alpha}}.)$ where $\mathcal{K} = \mathcal{K}_{t=1}$. Since $\mathcal{K} \in L^1(\mathbb{R}^2)$ and $\int \mathcal{K}(x)dx = 1$, there exists a sequence $(\mathcal{K}_{(n)})_n \in (C_c^\infty(\mathbb{R}^2))^N$ such that for all n, $\int \mathcal{K}_{(n)}(x)dx = 1$ and $(\mathcal{K}_{(n)})_n \to \mathcal{K}$ in

$L^1(\mathbb{R}^2)$. Let $(f_n)_n$ be a sequence in $C_c^\infty(\mathbb{R}^2)$ satisfying $(f_n)_n \to f$ in X. Now we consider the functions $(u_n)_n$ and u defined on $\mathbb{R}^{+*} \times \mathbb{R}^2$ by

$$u(t,x) = \mathcal{K}_t * f \quad \text{and} \quad u_n(t,x) = \mathcal{K}_{(n),t} * f_n$$

where $\mathcal{K}_{(n),t}(.) = t^{-\frac{1}{\alpha}}\mathcal{K}_{(n)}(t^{-\frac{1}{2\alpha}}.)$ and $*$ denotes the convolution in $\mathbb{R}^2$.

One can easily verify that for all n, the function $\hat{u}_n(t,\xi) = \hat{\mathcal{K}}_{(n)}(t^{\frac{1}{2\alpha}}\xi)\hat{f}_n(\xi)$ belongs to the space $C(\mathbb{R}^{+*}, S(\mathbb{R}^2))$ and satisfies $\hat{u}_n(t,.) \to \hat{f}_n$ in $S(\mathbb{R}^2)$ as t goes to 0^+. This implies that for all n, u_n can be extended to a function in $C(\mathbb{R}^+, S(\mathbb{R}^2))$ with f_n as value at $t = 0$. Consequently, to conclude the proof of the Lemma, we just need to show that the sequence $(u_n)_n$ converges to u in the space $L^\infty(\mathbb{R}^+, X)$. To do this, we notice that for any $t > 0$ and any $n \in \mathbb{N}$ we have

$$u_n(t) - u(t) = \mathcal{K}_{(n),t} * (f_n - f) + (\mathcal{K}_{(n),t} - \mathcal{K}_t) * f.$$

Hence,

$$\begin{aligned} \|u_n(t) - u(t)\|_X &\le \|\mathcal{K}_{(n),t}\|_1 \|f_n - f\|_X + \|\mathcal{K}_{(n),t} - \mathcal{K}_t\|_1 \|f\|_X \\ &\le C\|f_n - f\|_X + \|\mathcal{K}_{(n)} - \mathcal{K}\|_1 \|f\|_X, \end{aligned}$$

which leads to the desired result. □

The next lemma will be useful in the sequel.

Lemma 2.7. *Let X be a shift invariant functional space, $T > 0$ and $\mu < 1$. Then, for all $f \in L^\infty_\mu([0,T], X)$, the function $\mathcal{L}_\alpha(f)$ belongs to $L^\infty_{\mu'}([0,T], X_{\mathcal{R}})$ and satisfies*

$$\|\mathcal{L}_\alpha(f)\|_{L^\infty_{\mu'}([0,T],X_{\mathcal{R}})} \le C\|f\|_{L^\infty_\mu([0,T],X)}$$

where $\mu' = \mu - 1 + \frac{1}{2\alpha}$ and C is a constant depending only on μ, α and X. Moreover, if f belongs to $L^\infty_\mu([0,T], \overline{S(\mathbb{R}^2)}^X)$ then $\mathcal{L}_\alpha(f)$ belongs to $C^0_{\mu'}([0,T], (\overline{S(\mathbb{R}^2)}^X)_{\mathcal{R}})$.

Proof. The first assertion is a an immediate consequence of estimates (2.4)-(2.5). The last assertion can be easily proved by using the previous lemma and the Lebesgue's dominated convergence theorem, we left details to the reader. □

Lemma 2.8. *Let $T > 0$. Then the following assertions hold:*

1. *The linear operator $e^{-t(-\Delta)^\alpha}$ is continuous from $\tilde{\mathbf{B}}^\alpha$ to $\mathbb{E}^\nu_T$.*
2. *The bilinear operator $\mathcal{B}_\alpha$ is continuous from $\mathbb{E}^\nu_T \times \mathbb{E}^\nu_T \to \mathbb{E}^\nu_T$ and its norm is independent of T.*

Proof. The first assertion follows from the characterization of Besov spaces by the kernel $e^{-t(-\Delta)^\alpha}$ and the definition of $\tilde{\mathbf{B}}^\alpha$ The second assertion, is a direct consequence of the previous lemma and the fact that $\mathbb{E}^\nu_T = C^0_\nu([0,T], (C_0(\mathbb{R}^2))_{\mathcal{R}})$ □

The following lemma, which is a direct consequence of the preceding one, will be useful in the proof of Theorem 1.2.

Lemma 2.9. *Let $\theta_0 \in \tilde{B}^\alpha$. The sequence $\phi_n(\theta_0)$ defined by*

$$\begin{aligned} \phi_0(\theta_0) &= e^{-t(-\Delta)^\alpha}\theta_0, \\ \phi_{n+1}(\theta_0) &= e^{-t(-\Delta)^\alpha}\theta_0 + \mathcal{B}_\alpha[\phi_n(\theta_0), \phi_n(\theta_0)], \end{aligned}$$

belongs to $\cap_{T>0}\mathbb{E}_T^\nu$*. Moreover, there exists a constant* $\mu_0 > 0$ *(depending only on* α *) such that if for some* $T > 0$ *we have* $\|\phi_0(f)\|_{\mathbb{E}_T^\nu} \le \mu_0$ *then for all* $n \in \mathbb{N}^*$*,*

$$\|\phi_n(\theta_0)\|_{\mathbb{E}_T^\nu} \le 2\|\phi_0(\theta_0)\|_{\mathbb{E}_T^\nu}, \tag{2.7}$$

$$\|\phi_{n+1}(\theta_0) - \phi_n(\theta_0)\|_{\mathbb{E}_T^\nu} \le \frac{1}{2^n}. \tag{2.8}$$

In particular, the sequence $(\phi_n(\theta_0))_n$ *converges in the space* $\mathbb{E}_T^\nu$ *and its limit* θ *is a mild solution to the equation* (1.1) *with initial data* θ_0*.*

The following elementary lemma will play a crucial role in this paper.

Lemma 2.10 (Gronwall type Lemma)**.** *Let* $T > 0$*,* $c_1, c_2 \ge 0$*,* $\kappa \in]0,1[$ *and* $f \in L^\infty(0,T)$ *such that for all* $t \in [0,T]$*,*

$$f(t) \le c_1 + c_2 \int_0^t \frac{f(s)}{(t-s)^\kappa} ds.$$

Then for all $t \in [0,T]$*,*

$$f(t) \le 2c_1 e^{\nu t}, \tag{2.9}$$

where $\nu = \nu_{\kappa,c_2} > 0$*.*

Proof. Let $\nu > 0$ to be precise in the sequel and consider the function g defined on $[0,T]$ by

$$g(t) = \sup_{0<s<t} e^{-\nu s} f(s).$$

Clearly, we have

$$g(t) \le c_1 + c_2 \int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^\kappa} g(s)ds, \le c_1 + c_2\gamma_\kappa \nu^{\kappa-1} g(t),$$

where $\gamma_\kappa = \int_0^\infty \frac{e^{-t}}{t^\kappa}$. Thus, if we choose $\nu > 0$ such that $c_2\gamma_\kappa\nu^{\kappa-1} = \frac{1}{2}$, we obtain the estimate (2.9). □

Lemma 2.11 (Maximal Principle)**.** *Let* θ *be a mild solution of* (1.2) *belonging to the space* $C([0,T],(C_0(\mathbb{R}^2))_{\mathcal{R}})$*. Then for all* $t \in [0,T]$*, we have*

$$\|\theta(t)\|_\infty \le \|\theta_0\|_\infty, \tag{2.10}$$

$$\|\mathcal{R}^\perp(\theta)(t)\|_\infty \le 2\|\mathcal{R}^\perp(\theta_0)\|_\infty e^{\eta t}, \tag{2.11}$$

where $\eta = \eta_{\alpha,\|\theta_0\|_\infty} > 0$*.*

Proof. The inequality (2.10) is proved in [18], [5] and [22], for sufficiently smooth solution θ. To prove it in our case, we will proceed by linearization of the equations and regularization of the initial data. We consider a sequence of *linear system*

$$\begin{gathered} \partial_t v - (-\Delta)^\alpha v + \nabla.(u_n v) = 0 \\ v(0,.) = \theta_n(.). \end{gathered} \tag{2.12}$$

where $(\theta_n)_n$ is a given sequence in $C_c^\infty(\mathbb{R}^2)$ converging to $\theta(0)$ in the space $L^\infty(\mathbb{R}^2)$ and $u_n = \omega_n * \mathcal{R}^\perp(\theta)$ with $\omega_n(.) = n^2\omega(n.)$ where $\omega \in C_c^\infty(\mathbb{R}^2)$ and $\int \omega dx = 1$.

Let $n \in \mathbb{N}$. By converting the system (2.12) into the integral equation

$$v(t) = e^{-t(-\Delta)^\alpha}\theta_n - \int_0^t \nabla.e^{-(t-s)(-\Delta)^\alpha}(u_n v)ds \tag{2.13}$$

and by following a standard method, one can easily prove that the system (2.12) has a unique global solution $v_n \in \cap_{k\in\mathbb{N}} C^\infty([0,T], H^k(\mathbb{R}^2))$. Hence we are allowed to make the following computations: Let $p \in [2,\infty[$. For any $t \in [0,T]$ we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|v_n(t)\|^p &= -\int ((-\Delta)^\alpha v)v|v|^{p-2}dx - \int \nabla.(u_n v)v|v|^{p-2}dx\\ &\equiv I_1(t) + I_2(t).\end{aligned}$$

Firstly, a simple integration by parts implies that $I_2(t) = -I_2(t)$ and so

$$I_2(t) = 0.$$

Secondly, by the positivity Lemma (see [18] and [6]), we have

$$I_1(t) \le 0.$$

Therefore,

$$\sup_{t\in[0,T]} \|v_n(t)\|_p \le \|\theta_n\|_p.$$

Letting $p \to +\infty$, yields

$$\sup_{t\in[0,T]} \|v_n(t)\|_\infty \le \|\theta_n\|_\infty.$$

Consequently, to obtain (2.10), we just need to show that the sequence $(v_n)_n$ converges to the function θ in the space $L^\infty([0,T], L^\infty(\mathbb{R}^2))$. To do this, we consider the sequence $(w_n)_n = (v_n - \theta)_n$. Let $t \in [0,T]$ and $n \in \mathbb{N}$. We have

$$\begin{aligned}w_n(t) = e^{-t(-\Delta)^\alpha}(w_n(0)) &- \int_0^t \nabla.e^{-(t-s)(-\Delta)^\alpha}((u_n - \mathcal{R}^\perp(\theta))v_n)ds\\ &- \int_0^t \nabla.e^{-(t-s)(-\Delta)^\alpha}(\mathcal{R}^\perp(\theta)w_n)ds.\end{aligned}$$

Thus, by using the Young inequality and Proposition 2.3, we easily get

$$\|w_n(t)\|_\infty \le \|\theta_n - \theta(0)\|_\infty + C_\alpha T^\nu A_n B_n + C_\alpha M_\theta \int_0^t \frac{\|w_n(s)\|_\infty}{(t-s)^{1/2\alpha}}ds$$

where C_α is a constant depending only on α,

$$\begin{gathered}A_n = \sup_{0\le t\le T} \|u_n(t) - \mathcal{R}^\perp(\theta)(t)\|_\infty,\\ B_n = \sup_{0\le t\le T} \|v_n(t)\|_\infty,\\ M_\theta = \sup_{0\le t\le T} \|\mathcal{R}^\perp(\theta)(t)\|_\infty.\end{gathered}$$

Applying Lemma 2.10, we obtain

$$\sup_{0\le t\le T} \|w_n(t)\|_\infty \le C[\|\theta_n - \theta(0)\|_\infty + C_\alpha T^\nu A_n B_n]$$

where C is a constant depending on α, T and θ only.

Therefore, to obtain the desired conclusion, we just have to notice that the sequence $(B_n)_n$ is bounded and that $A_n \to 0$ as $n \to \infty$ thanks to the uniform continuity of the function $\mathcal{R}^\perp(\theta)$ on $[0,T] \times \mathbb{R}^2$, which is a consequence of the fact $\mathcal{R}^\perp(\theta) \in C([0,T], C_0(\mathbb{R}^2))$

Now, let us establish the inequality (2.11). For any $t \in [0,T]$, we have

$$\mathcal{R}^{\perp}(\theta)(t) = e^{-t(-\Delta)^{\alpha}}(\mathcal{R}^{\perp}(\theta)(0)) - \int_0^t \mathcal{R}^{\perp}\nabla . e^{-(t-s)(-\Delta)^{\alpha}}(\mathcal{R}^{\perp}(\theta)\theta)ds.$$

Applying the Young inequality and (2.5), we obtain

$$\|\mathcal{R}^{\perp}(\theta)(t)\|_{\infty} \leq \|\mathcal{R}^{\perp}(\theta)(0)\|_{\infty} + C\|\theta(0)\|_{\infty}\int_0^t \frac{\|\mathcal{R}^{\perp}(\theta)(s)\|_{\infty}}{(t-s)^{1/2\alpha}}ds$$

where the constant C depends only on α. Hence, Lemma 2.10 leads the desired inequality. $\square$

3. Proof of Theorem 1.1

According to Lemma 2.8, there exists $T > 0$ such that $\|e^{-t(-\Delta)^{\alpha}}\theta_0\|_{\mathbf{E}^{\nu}_T} \leq \mu_0$ where μ_0 is the real defined by Lemma 2.9. Therefore, the same lemma ensures that the equation (1.1) with initial data θ_0 has a mild solution θ belonging to the space $\mathbf{E}^{\nu}_T$. Following a standard arguments (see for example [10] and [2]), the uniqueness of the solution θ can be easily deduced from the continuity of the operator $\mathcal{B}_{\alpha}$ on the space $\mathbf{E}^{\nu}_T$. Hence, there exists a unique maximal solution,

$$\theta \in \cap_{0<T<T^*}\mathbf{E}^{\nu}_T.$$

where T^* is the maximal time existence. Let us show that $\theta \in C([0,T^*), \tilde{\mathbf{B}}^{\alpha})$. Thanks to the embedding,

$$(C_0(\mathbb{R}^2))_{\mathcal{R}} \subset \tilde{\mathbf{B}}^{\alpha},$$

and Lemma 2.6, we just need to prove the continuity of $N(\theta)(t) = \mathcal{B}_{\alpha}[\theta,\theta](t)$ at $t = 0^+$ in the space $\tilde{\mathbf{B}}^{\alpha}$. Furthermore, we show that

$$\lim_{t\to 0^+} N(\theta)(t) = 0, \quad \text{in} \quad \tilde{\mathbf{B}}^{\alpha}.$$

For that, we use Proposition 2.3, the Young inequality and estimates (2.4)-(2.5), to obtain

$$\|N(\theta)(t)\|_{\tilde{\mathbf{B}}^{\alpha}} \lesssim \sup_{0<t'<1} t'^{\nu}\int_0^t (t+t'-\tau)^{-\frac{1}{2\alpha}}\tau^{-2\nu}d\tau \quad \|\theta\|^2_{\mathbb{E}^{\nu}_t} \lesssim \|\theta\|^2_{\mathbb{E}^{\nu}_t}.$$

Since $\|\theta\|_{\mathbb{E}^{\nu}_t}$ goes to 0 as t goes 0^+ we obtain the desired result.

It remains to show that the solution θ is global, that is $T^* = \infty$. We argue by contradiction. If $T^* < \infty$ then, from Lemma 2.9, we must have for all $0 < t_0 < T^*$,

$$\|e^{t(-\Delta)^{\alpha}}\theta(t_0)\|_{\mathbf{E}^{\nu}_{T^*-t_0}} \geq \mu_0,$$

which yields by the Young inequality

$$\|\theta(t_0)\|_{\infty} + \|\mathcal{R}^{\perp}(\theta)(t_0)\|_{\infty} \geq \frac{c}{(T^*-t_0)^{\nu}}, \tag{3.1}$$

where $c > 0$ is a universal constant. Which contradicts the Maximal Principle (Lemma 2.10).

4. Proof of Theorem 1.2

Along this section, we consider θ_0 a given initial data belonging to the space $\tilde{\mathbf{B}}^{\alpha}$ and we denote by θ the solution to (1.1) given by Theorem 1.1. We will establish the persistency of the regularity of the initial data. That is, if moreover $\theta_0 \in X$ for a suitable Banach spaces X then the solution $\theta \in C([0,\infty), X)$.

4.1. **Propagation of the L^p regularity.** In this subsection we will prove the propagation of the initial L^p regularity. Precisely, we prove the following proposition.

Proposition 4.1. *Let $X = L^p$ with $p \in [1,\infty]$. If $\theta_0 \in X$ then θ belongs to $\bigcap_{T>0} L^\infty([0,T],X)$. Moreover, if $\theta_0 \in \overline{S(\mathbb{R}^2)}^X$ then $\theta \in C([0,\infty),\overline{S(\mathbb{R}^2)}^X)$*

Proof. Assume $\theta_0 \in X$ and let $T>0$. We consider the Banach spaces $\mathbf{Z}_1 = \mathbb{E}_T^\nu$ and $\mathbf{Z}_2 = L^\infty([0,T],X)$ endowed respectively with the norms

$$\|v\|_{\mathbf{Z}_1} = \sup_{0<t<T} e^{-\lambda t} t^\nu \|v(t)\|_\infty \quad \text{and} \quad \|v\|_{\mathbf{Z}_2} = \sup_{0<t<T} e^{-\lambda t}\|v(t)\|_p,$$

where $\lambda>0$ to be fixed later. We consider the linear integral equation,

$$v = \Psi_\theta(v) \equiv e^{t(-\Delta)^\alpha}\theta_0 + \mathcal{B}_\alpha[\theta,v]. \tag{4.1}$$

Let $k \in \{1;2\}$. According to Lemma 2.7, the affine functional $\Psi_\theta : \mathbf{Z}_k \to \mathbf{Z}_k$ is continuous. Let us estimate the norm of its linear part

$$K_\theta(v) = \mathcal{B}_\alpha[\theta,v].$$

Let $\varepsilon>0$ to be chosen later. A direct computation using (2.4) gives

$$\begin{aligned}
\|K_\theta\|_{\mathcal{L}(\mathbf{Z}_1)} &= \sup_{\|v\|_{\mathbf{Z}_1}\le 1} \|K_\theta(v)\|_{\mathbf{Z}_1}\\
&\le C_1 \sup_{0<t<T} t^\nu \int_0^t (t-\tau)^{-\frac{1}{2\alpha}} \tau^{-2\nu} e^{-\lambda(t-\tau)} \|\theta\|_{\mathbf{E}_\tau^\nu} d\tau\\
&\le C_2\Big(\|\theta\|_{\mathbf{E}_\varepsilon^\nu} \sup_{0<t<\varepsilon} t^\nu \int_0^t (t-\tau)^{-\frac{1}{2\alpha}}\tau^{-2\nu} d\tau + T^\nu \varepsilon^{-2\nu}\|\theta\|_{\mathbf{E}_T^\nu}\lambda^{-\nu}\Gamma(\nu)\Big)\\
&\le C_3\Big(\|\theta\|_{\mathbf{E}_\varepsilon^\nu} + T^\nu\varepsilon^{-2\nu}\lambda^{-\nu}\|\theta\|_{\mathbf{E}_T^\nu}\Big).
\end{aligned}$$

where the constants C_1, C_2, C_3 depend only on α. Similarly, we prove the estimate

$$\|K_\theta\|_{\mathcal{L}(\mathbf{Z}_2)} \le C(\|\theta\|_{\mathbf{E}_\varepsilon^\nu} + T^\nu\varepsilon^{-2\nu}\lambda^{-\nu}\|\theta\|_{\mathbf{E}_T^\nu}),$$

where C is a constant depending only on α. Since $\|\theta\|_{\mathbb{E}_\epsilon^\nu} \to 0$ as $\epsilon \to 0^+$, one can choose, successively, ϵ small enough and λ large enough so that Ψ_θ becomes a contraction on $\mathbf{Z}_1$ and $\mathbf{Z}_2$ and therefore on $\mathbf{Z}_1 \cap \mathbf{Z}_2$. Let v_1 and $v_{1,2}$ be the unique fixed point of Ψ_θ respectively in $\mathbf{Z}_1$ and $\mathbf{Z}_1\cap\mathbf{Z}_2$. Now, since $\mathbf{Z}_1 \cap \mathbf{Z}_2 \subset \mathbf{Z}_1$ then $v_1 = v_{1,2}$. Moreover, by construction θ is a fixed point of Ψ_θ in $\mathbf{Z}_1$ thus $\theta = v_1 = v_{1,2}$ and hence $\theta \in L^\infty([0,T],X)$.

The proof of the last statement of the proposition is identically similar, we have only to replace $\mathbf{Z}_2$ by $C([0,T],\overline{S(\mathbb{R}^2)}^X)$. □

4.2. **Propagation of $\dot{B}_p^{s,q}$ regularity for $s>0$.** In this section, we prove an abstract result, which implies in particular the persistence of the $\dot{B}_p^{s,q}$ regularity for $s>0$. Our result states as follows.

Proposition 4.2. *Let X be a shift invariant functional space such that for a constant C and all $f,g \in X \cap L^\infty(\mathbb{R}^2)$,*

$$\|fg\|_X \le C(\|f\|_\infty\|g\|_X + \|g\|_\infty\|f\|_X). \tag{4.2}$$

If the initial data θ_0 is in $X_{\mathcal{R}}$ then the solution θ belongs to $\cap_{T>0}L^\infty([0,T],X_{\mathcal{R}})$. Moreover, if θ_0 belongs to $(\overline{S(\mathbb{R}^2)}^X)_{\mathcal{R}}$ then θ belongs to $C(\mathbb{R}^+,(\overline{S(\mathbb{R}^2)}^X)_{\mathcal{R}})$).

The proof of this proposition relies essentially on the two following lemmas. The first one is an elementary compactness lemma.

Lemma 4.3. *Let $\lambda > 0$ and K a compact subset of $\tilde{\mathbf{B}}^\alpha$. Then there exists $\delta = \delta(K, \lambda) > 0$ such that for all $f \in K$,*

$$\|e^{-t(-\Delta)^\alpha} f\|_{\mathbb{E}^\nu_\delta} \le \lambda.$$

Proof. For $n \in \mathbb{N}^*$, we set

$$V_n = \{f \in \tilde{\mathbf{B}}^\alpha,\ \|e^{-t(-\Delta)^\alpha} f\|_{\mathbb{E}^\nu_{1/n}} < \lambda\}.$$

We claim that for all $n \in \mathbb{N}^*$, V_n is an open subset of $\tilde{\mathbf{B}}^\alpha$ and $\cup_n V_n = \tilde{\mathbf{B}}^\alpha$. This follows easily from the continuity of the linear operator $e^{-t(-\Delta)^\alpha}$ from $\tilde{\mathbf{B}}^\alpha$ into $\mathbb{E}^\nu_T$ for all $T > 0$ and the propriety: For all $f \in \tilde{\mathbf{B}}^\alpha$,

$$\lim_{T\to 0} \|e^{-t(-\Delta)^\alpha} f\|_{\mathbb{E}^\nu_T} = 0.$$

Thus, since K is a compact subset of $\tilde{\mathbf{B}}^\alpha$, there exists a finite subset $I \subset \mathbb{N}^*$ such that $K \subset \cup_I V_n = V_{n^*}$ where $n^* = \max(n \in I)$. Hence, we conclude that the choice $\delta = 1/n^*$ is suitable. $\square$

The second lemma establishes a local in time propagation of the X regularity.

Lemma 4.4. *Let X be as in Prop. 4.2. If θ_0 belongs to $X_{\mathcal{R}}$ (resp. $(\overline{S(\mathbb{R}^2)}^X)_{\mathcal{R}}$) then there exists $\delta = \delta(X, \alpha) > 0$ such that the solution $\theta \in L^\infty([0,\delta], X_{\mathcal{R}})$ (resp. $C([0,\delta], (\overline{S(\mathbb{R}^2)}^X)_{\mathcal{R}})$. Moreover, the time δ is bounded below by,*

$$\sup\{T > 0,\ \|e^{-t(-\Delta)^\alpha}\theta_0\|_{\mathbb{E}^\nu_T} \le \mu\},$$

where μ is a non negative constant depending on X and α only.

Proof. Let us consider the case of $\theta_0 \in X_{\mathcal{R}}$. The proof in the other case is similar. Let $\mu \in]0, \mu_0[$ to be chosen later and let $T > 0$ such that $\|e^{-t(-\Delta)^\alpha}\theta_0\|_{\mathbf{E}^\nu_T} \le \mu$. According to Lemma 2.9, the sequence $(\phi_n(\theta_0))_n$ converges in $\mathbb{E}^\nu_T$ to the solution θ and satisfies the following estimates

$$\sup_n \|\phi_n(\theta_0)\|_{\mathbf{E}^\nu_T} \le \mu \tag{4.3}$$

$$\forall n \in \mathbb{N},\quad \|\phi_{n+1}(\theta_0) - \phi_n(\theta_0)\|_{\mathbf{E}^\nu_T} \le 2^{-n}. \tag{4.4}$$

Then, to conclude we just need to show that $(\phi_n(\theta_0))_n$ is a Cauchy sequence in the Banach space $\mathbf{Z}_{\mathcal{R}} = L^\infty([0,T], X_{\mathcal{R}})$ endowed with its natural norm,

$$\|v\|_{\mathbf{Z}_{\mathcal{R}}} = \sup_{0<t<T}(\|v(t)\|_X + \|\mathcal{R}^\perp(v)(t)\|_X).$$

Firstly, using Lemma 2.7 and the fact that $(\phi_n(\theta_0))_n \in \mathbf{E}^\nu_T$, we infer inductively that the sequence $(\phi_n(\theta_0))_n$ belongs to the space $\mathbf{Z}_{\mathcal{R}}$. Secondly, once again the Lemma 2.7, implies that the sequence $(\omega_{n+1})_n \equiv (\phi_{n+1}(\theta_0) - \phi_n(\theta_0))_n$ satisfies the inequality

$$\begin{aligned}\|\omega_{n+1}\|_{\mathbf{Z}_{\mathcal{R}}} &\le C(\|\phi_n(\theta_0)\|_{\mathbf{Z}_{\mathcal{R}}} + \|\phi_{n-1}(\theta_0)\|_{\mathbf{Z}_{\mathcal{R}}})\|\omega_n\|_{\mathbf{E}^\nu_\delta}\\ &\quad + C(\|\phi_n(\theta_0)\|_{\mathbf{E}^\nu_T} + \|\phi_{n-1}(\theta_0)\|_{\mathbf{E}^\nu_\delta})\|\omega_n\|_{\mathbf{Z}_{\mathcal{R}}},\end{aligned}$$

where $C = C(X, \alpha) > 0$. This inequality combined with the estimates (4.3)-(4.4) yields

$$\|\omega_{n+1}\|_{\mathbf{Z}_{\mathcal{R}}} \leq C(\frac{1}{2})^n(\|\phi_n(\theta_0)\|_{\mathbf{Z}_{\mathcal{R}}} + \|\phi_{n-1}(\theta_0)\|_{\mathbf{Z}_{\mathcal{R}}}) + 4C\mu\|\omega_n\|_{\mathbf{Z}_{\mathcal{R}}}$$

Finally, if we choose $\mu > 0$ such that $4C\mu < 1$ one can conclude the proof by using the following lemma which is inspired from [8]. □

Lemma 4.5. *Let $(x_n)_n$ be a sequence in a normed vector space $(Z, \|.\|)$. If there exist a constant $\lambda \in [0, 1[$ and $(\sigma_n)_n \in l^1(\mathbb{N})$ such that for all $n \in \mathbb{N}^*$,*

$$\|x_{n+1} - x_n\| \leq \sigma_n(\|x_n\| + \|x_{n-1}\|) + \lambda\|x_n - x_{n-1}\|, \tag{4.5}$$

then the series $\sum_n \|x_{n+1} - x_n\|$ converges. In particular, $(x_n)_n$ is a Cauchy sequence in Z.

Proof. Let us define the sequence $M_n = \sup_{k \leq n} \|x_k\|$. It follows inductively from (4.5),

$$\|x_{n+1} - x_n\| \leq 2\sum_{k=0}^{n-1} \sigma_{n-k} M_{n-k} \lambda^k \leq \varpi_n M_n, \tag{4.6}$$

where $\varpi_n = 2\sum_{k=0}^{n-1} \sigma_{n-k}\lambda^k$. Noticing that since $(\varpi_n)_n$ is a convolution of two sequences in $l^1(\mathbb{N})$ then $(\varpi_n)_n$ belongs to $l^1(\mathbb{N})$. Therefore, we just need to show that the sequence $(M_n)_n$ is bounded. This is somehow obvious. In fact, using the triangular inequality $\|x_{n+1}\| \leq \|x_n\| + \|x_{n+1} - x_n\|$, (4.6) yields

$$M_{n+1} \leq (1 + \varpi_n)M_n.$$

Which in turn implies

$$M_n \leq \Pi_{k=0}^{n-1}(1 + \varpi_k) \leq e^{\sum_{k\geq 0} \varpi_n}.$$

The proof is complete. □

Now let us show how the two previous lemmas allow to prove Proposition 4.2.

Proof. As usual we consider only the case of $\theta_0 \in X_{\mathcal{R}}$. Let $T > 0$. By Theorem 1.1, the solution θ is continuous from $\mathbb{R}^+$ into $\tilde{\mathbf{B}}^\alpha$, then $K \equiv \theta([0, T])$ is a compact subset of $\tilde{\mathbf{B}}^\alpha$. Therefore, by Lemma 4.3, there exists $\delta > 0$ such that for all $\tau \in [0, T]$,

$$\|e^{-t(-\Delta)^\alpha}\theta(\tau)\|_{\mathbb{E}^\nu_\delta} \leq \mu_0, \tag{4.7}$$

where μ_0 is the real given by Lemma 4.4. Now, we consider a partition $0 = t_0 < \cdots < t_{N+1} = T$ of the interval $[0, T]$ such that $\sup_i t_{i+1} - t_i \leq \frac{\delta}{2}$. We will show inductively that

$$\theta \in L^\infty([t_i, t_{i+1}], X_{\mathcal{R}}), \tag{4.8}$$

which implies in turn the desired result $\theta \in L^\infty([0, T], X_{\mathcal{R}})$. First, by Lemma 4.4, the claim (4.8) is true for $i = 0$. Assume that, it is also true for $i \leq N$. Then there exists τ_0 in $]t_i, t_{i+1}[$ such that $\tilde{\theta}_0 \equiv \theta(\tau_0) \in X \cap \tilde{\mathbf{B}}^\alpha$. We notice that $\tilde{\theta} \equiv \theta(.+\tau_0)$ is the unique solution given by Theorem 1.1 of the Quasi-geostrophic equation with initial data $\tilde{\theta}_0$. Then according to Lemma 4.4 and (4.7), we obtain $\theta \in L^\infty([\tau_0, \tau_0+\delta], X_{\mathcal{R}})$. Hence, we are ready to conclude since $[t_{i+1}, t_{i+2}] \subset [\tau_0, \tau_0 + \delta]$. □

4.3. Propagation of $B_p^{s,q}$ regularity for $s<0$.

Proposition 4.6. *Let X be $B_p^{s,q}$ or $\dot B_p^{s,q}$ with $-1<s<0$ and $1\le p,\ q\le\infty$. If θ_0 belongs to $X_{\mathcal{R}}$ then the solution θ belongs to $\cap_{T>0}L^\infty([0,T],X_{\mathcal{R}})$ and satisfies*

$$t^{-\frac{s}{2\alpha}}\theta\in\cap_{T>0}L^\infty([0,T],(L^p)_{\mathcal{R}}).$$

As in the case $s>0$, by using the compactness Lemma 4.3 we just need to prove the following local persistency result.

Lemma 4.7. *If $\theta_0\in X_{\mathcal{R}}$ then there exists $\delta>0$ such that $\theta\in L^\infty([0,\delta],X_{\mathcal{R}})$ and satisfies*

$$t^{-\frac{s}{2\alpha}}\theta\in L^\infty([0,\delta],(L^p)_{\mathcal{R}}).$$

Moreover, the time δ is bounded below by

$$\sup\big\{T>0/\|e^{-t(-\Delta)^\alpha}\theta_0\|_{\mathbb{E}_T^\nu}\le\mu_0\big\},$$

where μ_0 is given by Lemma 2.9.

Proof. We consider only the case of $X=B_p^{s,q}$. The proof in the other case is similar. Let $T>0$ such that

$$\|e^{-t(-\Delta)^\alpha}\theta_0\|_{\mathbb{E}_T^\nu}\le\mu_0.$$

According to Lemma 2.9 the sequence $(\phi_n(\theta_0))_n$ satisfies

$$\|\phi_{n+1}(\theta_0)-\phi_n(\theta_0)\|_{\mathbb{E}_T^\nu}\le\frac{1}{2^n},\tag{4.9}$$

and converges to the solution θ in $\mathbb{E}_T^\nu$. Our first task is to prove that $(\phi_n(\theta_0))_n$ is a Cauchy sequence in the space

$$X_{\sigma,p}^T=\{v:(0,T]\to L^p\ \|v\|_{X_{\sigma,p}^T}\equiv\sup_{0<t<T}t^{\frac{\sigma}{2\alpha}}(\|v(t)\|_p+\|\mathcal{R}^\perp(v)(t)\|_p)<\infty\},$$

where $\sigma=-s$.

Thanks to the Besov characterization (2.2) and Lemma 2.7, we can show inductively that $(\phi_n(\theta_0))$ belongs to $X_{\sigma,p}^T$ and satisfies

$$\begin{aligned}&\|\phi_{n+1}(\theta_0)-\phi_n(\theta_0)\|_{X_{\sigma,p}^T}\\&\le C\|\phi_n(\theta_0)-\phi_{n-1}(\theta_0)\|_{\mathbb{E}_T^\nu}\max(\|\phi_n(\theta_0)\|_{X_{\sigma,p}^T},\|\phi_{n-1}(\theta_0)\|_{X_{\sigma,p}^T}).\end{aligned}\tag{4.10}$$

Thus, By (4.9) and Lemma 4.5 we deduce that $(\phi_n(\theta_0))_n$ is a Cauchy sequence in $X_{\sigma,p}^T$. Therefore its limit $\theta\in X_{\sigma,p}^T$. Now by a simple computation using the characterization (2.2) we deduce that $\theta\in L^\infty([0,T_0],(B_p^{s,\infty})_{\mathcal{R}})$. Moreover, for $\epsilon>0$ such that

$$-1<s\pm\epsilon<0,\tag{4.11}$$

one can show that the nonlinear part $N(\theta)(t)=\mathcal{B}_\alpha[\theta,\theta](t)$ satisfies

$$\|N(\theta)(t)\|_{B_p^{s\pm\epsilon,\infty}}+\|\mathcal{R}^\perp N(\theta)(t)\|_{B_p^{s\pm\epsilon,\infty}}\le C_{s,\epsilon}t^{-\pm\frac{\epsilon}{2\alpha}}\|\theta\|_{\mathbb{E}_t^\nu}\|\theta\|_{X_{\sigma,p}^T}.\tag{4.12}$$

Indeed, we have $\tau \in]0,1[$,

$$
\begin{aligned}
&\tau^{-\frac{s\pm\epsilon}{2\alpha}}\|e^{-\tau(-\Delta)^\alpha}N(\theta)(t)\|_p\\
&\le C\int_0^t (t+\tau-r)^{-\frac{1}{2\alpha}}\tau^{-\frac{s\pm\epsilon}{2\alpha}}r^{-\nu}r^{-\frac{\sigma}{2\alpha}}dr\ \|\theta\|_{\mathbb{E}_t^\nu}\|\theta\|_{X_{\sigma,p}^T},\\
&\le C\int_0^t (\frac{\tau}{t+\tau-r})^{-\frac{s\pm\epsilon}{2\alpha}}(t+\tau-r)^{\frac{-1-(s\pm\epsilon)}{2\alpha}}r^{-\nu}r^{-\frac{\sigma}{2\alpha}}dr\ \|\theta\|_{\mathbb{E}_t^\nu}\|\theta\|_{X_{\sigma,p}^T},\qquad (4.13)\\
&\le C\int_0^t (t-r)^{\frac{-1-(s\pm\epsilon)}{2\alpha}}r^{-\nu}r^{-\frac{\sigma}{2\alpha}}dr\|\theta\|_{\mathbb{E}_t^\nu}\|\theta\|_{X_{\sigma,p}^T},\\
&\le Ct^{-\frac{\pm\epsilon}{2\alpha}}\ \|\theta\|_{\mathbb{E}_t^\nu}\|\theta\|_{X_{\sigma,p}^T},
\end{aligned}
$$

Where we have used the facts that, $0\le \frac{\tau}{t+\tau-r}\le 1$, $t+\tau-r\ge t-r$ and (4.11). Similarly, we have the same estimate (4.13) for the $\mathcal{R}^\perp N(\theta)(t)$. Hence, by Proposition 2.2 we obtain (4.12). Thus, by using the interpolation inequality

$$
\|f\|_{B_p^{s,1}} \le \Big(\|f\|_{B_p^{s-\varepsilon,\infty}}\Big)^{1/2}\Big(\|f\|_{B_p^{s+\varepsilon,\infty}}\Big)^{1/2}
$$

we obtain that for all $t\in]0,T]$,

$$
\|N(\theta)(t)\|_{B_p^{s,1}} + \|\mathcal{R}^\perp N(\theta)(t)\|_{B_p^{s,1}} \le C\|\theta\|_{\mathbb{E}_t^\nu}\|\theta\|_{X_{\sigma,p}^T}. \qquad (4.14)
$$

Hence $N(\theta)\in L^\infty([0,T],(B_p^{s,1})_{\mathcal{R}})$ which implies $\theta\in L^\infty([0,T],(B_p^{s,q})_{\mathcal{R}})$. $\square$

Remark 4.8. By replacing the space $X_{\sigma,p}^T$ by $\tilde{X}_{\sigma,p}^T \equiv C^0_{\frac{\sigma}{2\alpha}}([0,T],(L^p)_{\mathcal{R}})$ in the proof of Lemma 4.7, one can show that if θ_0 is in $(\overline{S(\mathbb{R}^2)}^{B_p^{s,q}})_{\mathcal{R}}$ with $-1<s<0$ and $1\le p,q\le\infty$, then the solution θ belongs to the space $\cap_{T>0}\tilde{X}_{\sigma,p}^T$.

4.4. **The case of null regularity $s=0$.** In this subsection we aim to prove the following result.

Proposition 4.9. *Let X be $B_p^{0,q}$ or $\dot{B}_p^{0,q}$ with $1\le p,q\le\infty$. If $\theta_0\in X$ then the solution*

$$
\theta\in\cap_{T>0}L^\infty([0,T],X).
$$

Thanks to the following imbeddings

$$
\begin{gathered}
\dot{B}_p^{0,1}\subset \dot{B}_p^{0,q}\subset \dot{B}_p^{0,\infty},\\
\dot{B}_p^{0,1}\subset B_p^{0,q}\subset \dot{B}_p^{0,\infty},
\end{gathered}
$$

the proof of the above proposition is an immediate consequence of the following lemma.

Lemma 4.10. *If $\theta_0\in\dot{B}_p^{0,\infty}$ then $N(\theta)=\mathcal{B}_\alpha[\theta,\theta](t)\in\cap_{T>0}L^\infty([0,T],\dot{B}_p^{0,1})$.*

Proof. By using Young's inequality we deduce that

$$
\dot{B}_p^{0,\infty}\cap\dot{B}_\infty^{-(2\alpha-1),\infty}\subset\dot{B}_{2p}^{\frac12-\alpha,\infty}.
$$

Observe that $s^*=\frac12-\alpha<0$ and hence according to the proof of Proposition 4.6 and to the continuity of the Riesz transforms on homogeneous Besov spaces, we have $\theta\in\cap_{T>0}X_{\sigma^*,2p}^T$ where $\sigma^*=\alpha-\frac12$. Let $T>0$ and $0<\varepsilon<2\alpha-1$. The basic estimate

$$
\|\sqrt{-\Delta}^{\pm\varepsilon}\ \nabla e^{-t(-\Delta)^\alpha}f\|_p\le C_0t^{-\frac{\pm\varepsilon+1}{2\alpha}}\|f\|_p.
$$

yields immediately

$$\|(\sqrt{-\Delta})^{\pm\varepsilon}N(\theta)(t)\|_p \le Ct^{-\frac{\pm\varepsilon}{2\alpha}}\|\theta\|^2_{X^T_{\sigma^*,2p}}.$$

Now, we use the interpolation result (see [1, Theorem 6.3])

$$[(\sqrt{-\Delta})^{\varepsilon}L^p,(\sqrt{-\Delta})^{-\varepsilon}L^p]_{\frac12,1} = \dot B^{0,1}_p,$$

to deduce

$$\|N(\theta)(t)\|_{\dot B^{0,1}_p} \le C\|\theta\|^2_{X^T_{\sigma^*,2p}}, \quad \forall 0<t<T, \tag{4.15}$$

this implies

$$N(\theta)\in L^\infty([0,T],\dot B^{0,1}_p). \tag{4.16}$$

□

As in the context of the Navier-Stokes equations [3], we observe thanks to (4.16) and (4.15) that in the case $-1<s\le 0$, the fluctuation term $N(\theta)$ is more regular than the tendency $e^{-t(-\Delta)^\alpha}\theta_0$. Moreover, we have the following result.

Proposition 4.11. *Let $X = B^{s,\infty}_p$ with $s]-1,0[$ and $1\le p\le\infty$. If $\theta_0\in X_{\mathcal R}$ then $N(\theta)$ belongs to the space $C([0,\infty[;(B^{0,1}_p)_{\mathcal R})$.*

Proof. We consider the two cases:
Case $s\in]-1,0[$**:** According to Proposition 4.6, $t^{-\frac{s}{2\alpha}}\theta\in\cap_{T>0}L^\infty([0,T],(L^p)_{\mathcal R})$. Then a simple computation using that $\theta\in\cap_{T>0}\mathbf{E}^\nu_T$ gives $N(\theta)\in C(]0,\infty[;(L^p)_{\mathcal R})$ which yields $N(\theta)\in C(]0,\infty[;(B^{0,1}_p)_{\mathcal R})$ since $s<0$. On the other hand, the estimate (4.14) implies that $N(\theta)(t)\to 0$ in $(B^{s,1}_p)_{\mathcal R}$ as t goes to 0^+. Thus, we obtain the desired result.
Case $s=0$**:** By interpolation, $\theta_0\in(\overline{S(\mathbb R^2)}^{B^{s_*,\infty}_{p_*}})_{\mathcal R}$ where $s_*=\frac12-\alpha$ and $p_*=2p$. Hence, according to Remark 4.8, the solution θ belongs to $\cap_{T>0}\tilde X^T_{\sigma^*,p^*}$ where $\sigma^*=-s_*$. Let $\varepsilon\in[0,2\alpha-1[$. A simple computation gives

$$\sqrt{-\Delta}^{\,\mp\varepsilon}N(\theta)\in\cap_{T>0}C^0_{\mp\varepsilon/(2\alpha)}([0,T],(L^p)_{\mathcal R})$$

Hence, by interpolation we obtain

$$N(\theta)\in\cap_{T>0}C([0,T],(B^{0,1}_p)_{\mathcal R})\,.$$

□

Remark 4.12. Let $X = B^{s,q}_p$ with $-1<s\le 0$ and $1\le p,q\le\infty$. If $\theta_0\in(\overline{S(\mathbb R^2)}^X)_{\mathcal R}$ then Lemma 2.6 and the preceding proposition imply that the solution θ is in $C([0,\infty[;X_R)$.

5. Proof of Theorem 1.3

The existence part is a direct consequence of Theorem 1.1, Theorem 1.2 and the following embedding (consequence of Bernstein's inequality and the boundedness of the Riesz transforms on Lebesgue's and Sobolev's spaces)

$$L^p(\mathbb R^2)\subset\tilde{\mathbf B}_\alpha \quad \forall p\ge p_c,$$
$$H^s(\mathbb R^2)=B^{s,2}_2(\mathbb R^2)\subset\tilde{\mathbf B}_\alpha \quad \forall s\ge s_c.$$

Let us establish the uniqueness part. First we notice that since for $s\ge s_c$,

$$H^s(\mathbb R^2)\hookrightarrow H^{s_c}(\mathbb R^2)\hookrightarrow L^{p_c}(\mathbb R^2).$$

We just need to prove the uniqueness in the spaces $(C([0,T],L^p(\mathbb{R}^2)))_{p\geq p_c}$. This will be deduced from the following continuity result of the bilinear operator $\mathcal{B}_\alpha$.

Lemma 5.1. *Let $p\in]p_c,\infty[$, $q\in]1,\infty[$ and $T>0$. There exists a constant C independent of T such that:*

- *For any u,v in $L^\infty_T L^p$,*
$$\|\mathcal{B}_\alpha[u,v]\|_{L^\infty_T L^p}\leq CT^\sigma\|u\|_{L^\infty_T L^p}\|v\|_{L^\infty_T L^p}, \tag{5.1}$$
where $\sigma=\frac{1}{\alpha}(\frac{1}{p_c}-\frac{1}{p})$;
- *for any u,v in $L^\infty_T L^{p_c}$,*
$$\|\mathcal{B}_\alpha[u,v]\|_{L^q_T L^{p_c}}+\|\mathcal{B}_\alpha[v,u]\|_{L^q_T L^{p_c}}\leq C\|u\|_{L^\infty_T L^{p_c}}\|v\|_{L^q_T L^{p_c}.}; \tag{5.2}$$
- *for any $u\in L^\infty_T L^\infty_{\mathcal{R}}$ and $v\in L^q_T L^{p_c}$,*
$$\|\mathcal{B}_\alpha[u,v]\|_{L^q_T L^{p_c}}+\|\mathcal{B}_\alpha[v,u]\|_{L^q_T L^{p_c}}\leq C\,T^{1-\frac{1}{2\alpha}}\|u\|_{L^\infty_T L^\infty_{\mathcal{R}}}\|v\|_{L^q_T L^{p_c}.}. \tag{5.3}$$

Proof. Estimate (5.1) follows easily from the continuity of the Riesz transforms on the Lebesgue spaces $L^r(\mathbb{R}^2)$ with $1<r<\infty$, the Young and the Hölder inequality and the estimate (2.4) on the $L^r(\mathbb{R}^2)$ norm of the kernel of the operator $\nabla e^{-(t-s)(-\Delta)^\alpha}$. Estimate (5.2) is a consequence of the continuity of the Riesz transforms on the space $L^{p_c}(\mathbb{R}^2)$, the Hölder inequality, the Sobolev embedding
$$\|\frac{\nabla}{(-\Delta)^\alpha}f\|_{p_c}\lesssim\|f\|_{\frac{p_c}{2}}$$
and the maximal regularity property of the operator $(-\Delta)^\alpha$,
$$\|\int_0^t(-\Delta)^\alpha e^{-(t-s)(-\Delta)^\alpha}v\,ds\|_{L^q_T L^{p_c}}\lesssim\|v\|_{L^q_T L^{p_c}}$$
which can be proved by following [10, Theorem 7.3]. Let us now prove estimate (5.3). For any $t\in[0,T]$ we have
$$\begin{aligned}\|\mathcal{B}_\alpha[u,v](t)\|_{L^{p_c}}&\lesssim\int_0^t\frac{1}{(t-s)^{1/2\alpha}}\|\mathcal{R}^\perp(u)(s)\|_\infty\|v(s)\|_{p_c}ds\\&\lesssim\|\mathcal{R}^\perp(u)\|_{L^\infty_T L^\infty}(1_{[0,T]}s^{-\frac{1}{2\alpha}})*(1_{[0,T]}\|v(s)\|_{p_c})(t)\end{aligned}$$
where the star $*$ denotes the convolution in $\mathbb{R}$. Hence Young's inequality yields
$$\|\mathcal{B}_\alpha[u,v]\|_{L^q_T L^{p_c}}\lesssim\|\mathcal{R}^\perp(u)\|_{L^\infty_T L^\infty}T^{1-\frac{1}{2\alpha}}\|v\|_{L^q_T L^{p_c}}.$$
Similarly, we obtain
$$\begin{aligned}\|\mathcal{B}_\alpha[v,u]\|_{L^q_T L^{p_c}}&\lesssim T^{1-\frac{1}{2\alpha}}\|u\|_{L^\infty_T L^\infty}\|\mathcal{R}^\perp(v)\|_{L^q_T L^{p_c}}\\&\lesssim T^{1-\frac{1}{2\alpha}}\|u\|_{L^\infty_T L^\infty}\|v\|_{L^q_T L^{p_c}}.\end{aligned}$$
Estimate (5.3) is then proved. □

Now we are ready to finish the proof of the uniqueness. Let $p\geq p_c$ and $T>0$ be two reals number and let θ_1 and θ_2 be two mild solutions of the equation (1.1) with the same data θ_0 such that $\theta_1,\theta_2\in C([0,T],L^p(\mathbb{R}^2))$. We aim to show that $\theta_1=\theta_2$ on $[0,T]$. For this, we will argue by contradiction. Then we suppose that $t_*<T$ where
$$t_*\equiv\sup\{t\in[0,T]:\forall s\in[0,t],\ \theta_1(s)=\theta_2(s)\}.$$

To conclude, we need to prove that there exists $\delta \in]0, T - t_*]$ such that $\tilde{\theta}_1 = \tilde{\theta}_2$ on $[0, \delta]$, where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are the functions defined on $[0, T - t_*]$ by

$$\tilde{\theta}_1(t) = \theta_1(t + t_*), \quad \tilde{\theta}_2(t) = \theta_2(t + t_*).$$

We deal separately with the sub-critical case and the critical case.

Case $p > p_c$**.** Thanks to the continuity of θ_1 and θ_2 on $[0, T]$, we have $\theta_1(\tau_*) = \theta_2(t_*)$. Hence, the functions $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are two mild solutions on $[0, \delta_0 \equiv T - t_*]$ of the equation (1.1) with the same data $\theta_1(\tau_*)$. Therefore, the function $\tilde{\theta} \equiv \tilde{\theta}_1 - \tilde{\theta}_2$ satisfies the equation

$$\tilde{\theta} = \mathcal{B}_\alpha[\tilde{\theta}_1, \tilde{\theta}] - \mathcal{B}_\alpha[\tilde{\theta}, \tilde{\theta}_2]. \tag{5.4}$$

Thus, according to (5.1) we have for any $\delta \in]0, \delta_0]$,

$$\begin{aligned} \|\tilde{\theta}\|_{L^\infty_\delta L^p} &\le C\delta^\sigma (\|\tilde{\theta}_1\|_{L^\infty_\delta L^p} + \|\tilde{\theta}_2\|_{L^\infty_\delta L^p}) \|\tilde{\theta}\|_{L^\infty_\delta L^p} \\ &\le C\delta^\sigma (\|\theta_1\|_{L^\infty_T L^p} + \|\theta_2\|_{L^\infty_T L^p}) \|\tilde{\theta}\|_{L^\infty_\delta L^p}, \end{aligned}$$

where $C > 0$ is independent on δ. Consequently, for δ small enough, $\tilde{\theta} = 0$ on $[0, \delta]$ which ends the proof in the sub-critical case.

Case $p = p_c$**.** Choose a fix real $q > 1$ and let $\varepsilon > 0$ to be chosen later. By density of smooth functions in the space $C([0, T], L^{p_c}(\mathbb{R}^2))$, one can decompose $\tilde{\theta}_1$ and $\tilde{\theta}_2$ into $\tilde{\theta}_1 = u_1 + v_1$ and $\tilde{\theta}_2 = u_2 + v_2$ with

$$\|u_1\|_{L^\infty_{\delta_0} L^{p_c}} + \|u_2\|_{L^\infty_{\delta_0} L^{p_c}} \le \varepsilon, \tag{5.5}$$

$$\|v_1\|_{L^\infty_{\delta_0} L^\infty_{\mathcal{R}}} + \|v_2\|_{L^\infty_{\delta_0} L^\infty_{\mathcal{R}}} \equiv \mathcal{M} < \infty. \tag{5.6}$$

As in the previous case, the function $\tilde{\theta} \equiv \tilde{\theta}_1 - \tilde{\theta}_2$ satisfies

$$\begin{aligned} \tilde{\theta} &= \mathcal{B}_\alpha[\tilde{\theta}_1, \tilde{\theta}] + \mathcal{B}_\alpha[\tilde{\theta}, \tilde{\theta}_2] \\ &= \mathcal{B}_\alpha[u_1, \tilde{\theta}] + \mathcal{B}_\alpha[\tilde{\theta}, u_2] + \mathcal{B}_\alpha[v_1, \tilde{\theta}] + \mathcal{B}_\alpha[\tilde{\theta}, v_2]. \end{aligned}$$

Now by applying (5.2)-(5.3) and using (5.5)-(5.6) we obtain, for any $\delta \in]0, \delta_0]$, the estimate

$$\|\tilde{\theta}\|_{L^q_\delta L^p} \le C(\varepsilon + \delta^{1 - \frac{1}{2\alpha}} \mathcal{M}) \|\tilde{\theta}\|_{L^q_\delta L^p},$$

where $C > 0$ is a constant depending only on α, p and q.

Thus, by choosing ε small enough, we conclude that there exists $\delta \in]0, \delta_0]$ such that $\|\tilde{\theta}\|_{L^q_\delta L^p} = 0$, which implies that $\tilde{\theta}_1 = \tilde{\theta}_2$ on $[0, \delta]$. The proof is then achieved.

Remark 5.2. The idea of the proof of the uniqueness in the critical case is inspired from Monniaux [14].

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References

[1] J. Bergh and J. Löfstrom. *Interpolation Spaces. An introduction*. Springer Verlag, Berlin, 1976.
[2] M. Cannone. *Ondelettes, Paraproduits et Navier-Stokes*. Diderot Editeur, Arts et Sciences, Paris, (1995).
[3] M. Cannone and F. Planchon. *O*n the regularity of the bilinear term for solutions to the incompressible Navier-Stokes equations. *Rev. Mat. Iberoam. 16, No.1, 1-16 (2000)*
[4] P. Constantin and J. Wu. *B*ehavior of solutions of 2D quasi-geostrophic equations. *Siam J.Math. Anal.* , 30:937–948, (1999).

[5] P. Constantin, D. Cordoba and J. Wu. On the critical dissipative quasi-geostrophic equation. *Indiana Univ. Math. J.*, 50 Spec. Iss.: 97–107 (2001).
[6] A. Cordoba and D. Cordoba. A maximum Principale applied to Quasi-Geostrophic Equations. *Commun. Math. Phys.*, 249 511–528 (2004).
[7] M. Frazier, B. Jawerth, and G. Weiss. *Littlewood-Paley Theory and the study of function spaces.* AMS, Providence R.I., 1991.
[8] G. Furioli, P. G. Lemarié-Rieusset, A. Zhioua, and E. Zahrouni. Un théorème de persistance de la régularité en norme d'éspaces de Besov pour les solutions de Koch-Tataru des équations de Navier-Stokes dans R^3. *C.R.A.S de Paris Série I*, 330:339–342, 2000.
[9] T. Kato. Strong L^p solutions of the Navier-Stokes in R^m with applications to weak solutions. *Math. Zeit.*, 89:471–480, 1984.
[10] P. G. Lemarié-Rieusset. *Recent Developments in the Navier-Stokes Problem.* CRC Press, Boca Raton, 2002.
[11] 2181389 P. G. Lemarié-Rieusset and F. Marchand, *Solutions auto-similaires non radiales pour l'équation quasi-géostrophique dissipative critique*, C. R. Acad. Sc. Paris Série I ., **341** (2005), 535–538.
[12] R. May and E. Zahrouni. Global Existence Solutions for Subcritical Quasi-Geostrophic Equations . *Comm. in Pure and Applied Analysis,*(2007). **7** ((2008)) p. 1179–1191
[13] C. Miao, B. Yuan and B. Zhang. Well-posedness of the Cauchy problem for the fractional power dissipative equations. *Nonl. Anal : T.M.A*, 68 :461–484 (2008).
[14] S. Monniaux. Uniqueness of mild solutions of the Navier-Stokes equation and maximal Lp-regularity. *Comptes Rendus de l'Acadmie des Sciences, Paris, Srie 1, 328:663-668, (1999).*
[15] C. J. Niche and M. E. Schonbek. Decay of weak solutions to the 2D dissipative Quasi-Geostrophic equation. *Commun. Math. Phys.*, 276, 93-115 (2007).
[16] J. Pedlosky, *Geophysical Fluid Dynamics,* Springer Verlag, New York 1987.
[17] J. Peetre. *New thoughts on Besov spaces.* Duke University Mathematical Series, Durham N.C., 1976.
[18] S. Resnick. *Dynamical problems in Non-linear Advances Partial Differential Equations,* Ph.D. thesis, University of Chicago, II, 1995.
[19] H. Triebel Theory of Function Spaces . *Monograph in Mathematics, Birkhauser, 78*, (1983) .
[20] H. Triebel Theory of Function Spaces II. *Monograph in Mathematics, Birkhauser, 84*, (1992) .
[21] J. Wu Quasi-Geostrophic type equations with weak initial data. *Elec. Journal of Diff. Equa* .,Vol(1998) 16: 1–10, 1998 .
[22] J. Wu. Dissipative quasi-geostrophic equations with L^p data. *Elec. Journal of Diff. Equa* .,vol(2001) 56:1–23,2001 .

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