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EXISTENCE OF MILD SOLUTIONS FOR IMPULSIVE FRACTIONAL-ORDER SEMILINEAR EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this work we consider a class of impulsive fractional-order semilinear evolution equations with a nonlocal initial condition. By means of solution operator and application of fixed point theorems we established the existence and uniqueness of a mild solution.

1. Introduction

Recently fractional differential equations attracted many authors (see for instance [3, 8, 9, 12, 18, 19, 20, 21, 25, 27] and references in these papers). Many phenomena in engineering, physics,continuum mechanics, signal processing, electromagnetics, economics and science describes efficiently by fractional order differential equations. Impulsive differential equations have become important in recent years as mathematical models of phenomena in both physical and social sciences (see for instance [2, 7, 15, 16, 19, 26] and references in these papers). There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments.

In this article, we are concerned with the existence and uniqueness of the solution for the fractional order differential equation in a complex Banach space X,

$$\frac{d^\alpha}{dt^\alpha}x(t) + Ax(t) = f(t, x(t), x(a_1(t)), \dots, x(a_m(t))), \quad t \in J = [0, T],\ t \neq t_i, \tag{1.1}$$

$$x(0) + g(x) = x_0, \tag{1.2}$$

$$\Delta x(t_i) = I_i(x(t_i^-)), \tag{1.3}$$

where $\frac{d^\alpha}{dt^\alpha}$ is Caputo's fractional derivative of order $0 < \alpha < 1$, $i = 1, 2, \dots, p$, $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = T$. Linear operator A, defined from the domain $D(A) \subset X$ into X, is such that $-A$ generates $\alpha-$resolvent family $\{S_\alpha(t) : t \geq 0\}$ of bounded linear operators in X, the nonlinear map f is defined from $J \times X^{m+1}$ into X, for each of i the map a_i is defined on $[0, T]$ into $[0, T]$ and $\Delta x(t_i) = x(t_i^+) - x(t_i^-), x(t_i^+), x(t_i^-)$ denotes the right and the left limit of x

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at t_i, respectively. In general the derivatives $x'(t_i)$ do not exist, we assume that $x'(t_i) = x'(t_i - 0)$ at the point of discontinuity t_i of the solution $t \to x(t)$.

The nonlocal condition $g : X \to X$, defined as $g(x) = \sum_{k=1}^{p} c_k x(t_k)$, where c_k, $k = 1, \dots, p$, are given constants and $0 < t_1 < t_2 < \dots < t_p < T$. Let us recall that such nonlocal conditions were first used by Deng [13]. In this paper, Deng indicated that using the nonlocal condition $x(0) + g(x) = x_0$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy Problem $x(0) = x_0$.

The study of the impulsive fractional order semilinear functional differential problem of the type (1.1) is motivated by the paper of Byszewski and Akca [11] and Sui, Lai and Chen [26]. In [11] the authors have considered the nonlocal Cauchy problem

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t), u(a_1(t)), \dots, u(a_m(t))), \quad t \in J = [0, T], \\ u(0) + g(u) = u_0, \end{gathered} \tag{1.4}$$

where $-A$ is the generator of a compact semigroup in X, $g : C(J, X)$ into X, $u_0 \in X$ and for each $i = 1, 2, \dots, m$, $a_i : J \to J$. Further, the results obtained in [11] have been extended by Bahuguna in [4]. For more results on nonlocal conditions we refer the papers [4, 5, 6, 10, 11, 13, 14] and references therein.

In [26], the authors have investigated the existence of mild solutions of the following system

$$\begin{gathered} D^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in [0, T],\ t \neq t_k, \\ x(0) = x_0 \in X, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m, \end{gathered}$$

and corrected the errors in Mophu paper [19], and generalized some previous results.

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proofs of our main results are given in Section 3.

2. PRELIMINARIES

Throughout, in this paper X will be a complex Banach space provided with the norm $\|\cdot\|_X$ and $L(X)$ is the Banach space of bounded linear operators from X into X. In addition, $B_r(x, X)$ represents the closed ball in X with the center at x and the radius r. $-A$ is the infinitesimal generator of an analytic $\alpha-$resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ of operators on X . For the theory of resolvent operator one can see the monograph by Pazy [22]. The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an important role in the solution of non-integer order differential equations. The standard definition of the Mittag-Leffler function (see[24]) is given as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

It is also common to represent the Mittag-Leffler function in two arguments, α and β, such that

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\frac{1}{2\pi i}\int_{H_a}e^{\mu}\frac{\mu^{\alpha-\beta}}{\mu^{\alpha}-z}d\mu,\quad \alpha,\beta>0,\ z\in\mathbb{C},$$

where H_a is a Hankel path, that is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu|\le|z|^{\frac{1}{\alpha}}$ counter clockwise. It is an entire function which provides a generalization of several usual functions, for example: Exponent function: $E_{1,1}(z)=e^z$; cosine functions: $E_{2,1}(z^2)=\cosh(z)$ and $E_{2,1}(-z^2)=\cos(z)$; Sine functions: $zE_{2,2}(z^2)=\sinh(z)$ and $zE_{2,2}(-z^2)=\sin(z)$. The Laplace transform of the Mittag-Leffler function is given as:

$$L(t^{\beta-1}E_{\alpha,\beta}(-\rho^{\alpha}t^{\alpha}))=\frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}+\rho^{\alpha}},\quad \operatorname{Re}\lambda>\rho^{1/\alpha},\ \rho>0.$$

To begin with the analysis we need some basic definitions and properties from the fractional calculus theory (see [24]).

Definition 2.1. Caputo's derivative of order α for a function $f:[0,\infty)\to\mathbb{R}$ is defined as

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}}=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

for $n-1<\alpha<n,\ n\in N$. If $0<\alpha\le1$, then

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}f^{(1)}(s)ds.$$

The Laplace transform of the Caputo derivative of order $\alpha>0$ is given as

$$L\{D_t^{\alpha}f(t);\lambda\}=\lambda^{\alpha}\widehat{f}(\lambda)-\sum_{k=0}^{n-1}\lambda^{\alpha-k-1}f^{(k)}(0);\quad n-1<\alpha\le n.$$

Definition 2.2 ([3, Definition 2.3])**.** Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha>0$. Let $\rho(A)$ be the resolvent set of A. We call A the generator of an $\alpha-$resolvent family if there exists $\omega\ge0$ and a strongly continuous function $S_{\alpha}:R_+\to L(X)$ such that $\{\lambda^{\alpha}:Re\lambda>\omega\}\subset\rho(A)$ and

$$(\lambda^{\alpha}I-A)^{-1}x=\int_0^{\infty}e^{-\lambda t}S_{\alpha}(t)xdt,\quad \operatorname{Re}\lambda>\omega,\ x\in X.$$

In this case, $S_{\alpha}(t)$ is called the $\alpha-$resolvent family generated by A.

Definition 2.3 ([1, Definition 2.1])**.** Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha>0$. Let $\rho(A)$ be the resolvent set of A, then we say that A is the generator of a solution operator if there exists $\omega\ge0$ and a strongly continuous function $S_{\alpha}:R_+\to L(X)$ such that $\{\lambda^{\alpha}:Re\lambda>\omega\}\subset\rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^{\alpha}I-A)^{-1}x=\int_0^{\infty}e^{-\lambda t}S_{\alpha}(t)xdt,\quad \operatorname{Re}\lambda>\omega,\ x\in X.$$

In this case, $S_{\alpha}(t)$ is called the solution operator generated by A.

The concept of solution operator is closely related to the concept of a resolvent family (see [23, Chapter 1]). For more details on α-resolvent family and solution operators, we refer to [17, 23] and the references therein.

3. Main results

In [26], if $\alpha \in (0,1)$ and $A \in A^{\alpha}(\theta_0, \omega_0)$, then for any $x \in X$ and $t > 0$, we have

$$\|T_\alpha(t)\|_{L(X)} \le Me^{\omega t}, \quad \|S_\alpha(t)\|_{L(X)} \le Ce^{\omega t}(1 + t^{\alpha-1}), \quad t > 0,\ \omega > \omega_0.$$

Let

$$\widetilde{M_T} = \sup_{0\le t\le T} \|T_\alpha(t)\|_{L(X)}, \quad \widetilde{M_S} = \sup_{0\le t\le T} Ce^{\omega t}(1 + t^{1-\alpha}),$$

where $L(X)$ is the Banach space of bounded linear operators from X into X equipped with its natural topology. So we have

$$\|T_\alpha(t)\|_{L(X)} \le \widetilde{M_T}, \quad \|S_\alpha(t)\|_{L(X)} \le t^{\alpha-1}\widetilde{M_S}.$$

Let us consider the set of functions

$$\begin{gathered} PC(J,X) = \{x : J \to X : x \in C((t_k, t_{k+1}], X), k = 0, 1, \dots p \text{ and there exist} \\ x(t_k^-) \text{ and } x(t_k^+),\ k = 1, \dots, p \text{ with } x(t_k^-) = x(t_k)\}. \end{gathered}$$

Endowed with the norm

$$\|x\|_{PC} = \sup_{t\in J} \|x(t)\|_X,$$

the space $(PC(J,X), \|\cdot\|_{PC})$ is a Banach space.

Lemma 3.1 ([26]). *Consider the Cauchy problem*

$$\begin{gathered} D_t^\alpha x(t) + Ax(t) = f(t, x(t), x(a_1(t)), \dots, x(a_m(t))), \quad t > t_0,\ t_0 \ge 0,\ 0 < \alpha < 1, \\ x(t_0) = x_0 \in X, \end{gathered}$$

if f satisfies the uniform Holder condition with exponent $\beta \in (0,1]$ and A is a sectorial operator, then the unique solution of this Cauchy problem is

$$x(t) = T_\alpha(t - t_0)x(t_0^+) + \int_{t_0}^{t} S_\alpha(t-\theta) f(\theta, x(\theta), x(a_1(\theta)), \dots, x(a_m(\theta)))d\theta,$$

where

$$\begin{gathered} T_\alpha(t) = E_{\alpha,1}(-At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha + A}d\lambda, \\ S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-At^\alpha) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{1}{\lambda^\alpha + A}d\lambda, \end{gathered}$$

where $\widehat{B}_r$ denotes the Bromwich path. $S_\alpha(t)$ is called the $\alpha-$resolvent family and $T_\alpha(t)$ is the solution operator, generated by $-A$.

Proof. Let $t - t_0 = s$, then

$$D_s^\alpha x(s+t_0) + Ax(s+t_0) = f(s+t_0, x(s+t_0), x(a_1(s+t_0)), \dots, x(a_m(s+t_0))),$$

for $s > 0$. Now, applying the Laplace transform, we have

$$\begin{aligned} &\lambda^\alpha L\{x(s+t_0)\} - \lambda^{\alpha-1}x(t_0^+) + AL\{x(s+t_0)\} \\ &= L\{f(s+t_0, x(s+t_0), x(a_1(s+t_0)), \dots, x(a_m(s+t_0)))\}. \end{aligned} \tag{3.1}$$

Since $(\lambda^\alpha I + A)^{-1}$ exists, that is $\lambda^\alpha \in \rho(A)$, from (3.1), we obtain

$$\begin{aligned} L\{x(s+t_0)\} &= \lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}x(t_0^+) + (\lambda^\alpha I + A)^{-1} \\ &\quad \times L\{f(s+t_0, x(s+t_0), x(a_1(s+t_0)), \dots, x(a_m(s+t_0)))\}. \end{aligned}$$

Therefore, by the inverse Laplace transform, we have

$$\begin{aligned} x(s+t_0) &= E_{\alpha,1}(-As^\alpha)x(t_0^+) + \int_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A(s-\tau)^\alpha) \\ &\quad \times f(\tau+t_0, x(\tau+t_0), x(a_1(\tau+t_0)), \dots, x(a_m(\tau+t_0)))d\tau. \end{aligned}$$

Let $s+t_0=t$, we obtain

$$\begin{aligned} x(t) &= E_{\alpha,1}(-A(t-t_0)^\alpha)x(t_0^+) + \int_0^{t-t_0} (t-t_0-\tau)^{\alpha-1}E_{\alpha,\alpha}(-A(t-t_0-\tau)^\alpha) \\ &\quad \times f(\tau+t_0, x(\tau+t_0), x(a_1(\tau+t_0)), \dots, x(a_m(\tau+t_0)))d\tau. \end{aligned}$$

This is the same as

$$\begin{aligned} x(t) &= E_{\alpha,1}(-A(t-t_0)^\alpha)x(t_0^+) + \int_{t_0}^{t} (t-\theta)^{\alpha-1}E_{\alpha,\alpha}(-A(t-\theta)^\alpha) \\ &\quad \times f(\theta, x(\theta), x(a_1(\theta)), \dots, x(a_m(\theta)))d\theta. \end{aligned}$$

Let $T_\alpha(t) = E_{\alpha,1}(-At^\alpha)$ and $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(-At^\alpha)$, then we have

$$x(t) = T_\alpha(t-t_0)x(t_0^+) + \int_{t_0}^{t} S_\alpha(t-\theta)f(\theta, x(\theta), x(a_1(\theta)), \dots, x(a_m(\theta)))d\theta.$$

This completes the proof of the Lemma. □

Now, we define the definition of mild solution of (1.1).

Definition 3.2. A function $x \in PC(J,X)$ solution of the fractional integral equation

$$x(t) = \begin{cases} T_\alpha(t)(x_0 - g(x)) \\ + \int_0^t S_\alpha(t-s)f(s, x(s), x(a_1(s)), \dots, x(a_m(s)))ds, & t \in [0,t_1]; \\ T_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] \\ + \int_{t_1}^t S_\alpha(t-s)f(s, x(s), x(a_1(s)), \dots, x(a_m(s)))ds, & t \in (t_1,t_2]; \\ \dots \\ T_\alpha(t-t_p)[x(t_p^-) + I_p(x(t_p^-))] \\ + \int_{t_p}^t S_\alpha(t-s)f(s, x(s), x(a_1(s)), \dots, x(a_m(s)))ds, & t \in (t_p,T]. \end{cases}$$

will be called a mild solution of problem (1.1). From Lemma 3.1 we can verify this definition.

Now we introduce the following assumptions:

- (H1) There exists a constant $L_g > 0$ such that $\|g(x) - g(y)\|_X \le L_g\|x-y\|_X$.
- (H2) The nonlinear map $f : [0,T] \times X^{m+1} \to X$ is continuous and there exist a constant L_f such that
$$\begin{aligned} &\|f(t, x_1, x_2, \dots, x_{m+1}) - f(s, y_1, y_2, \dots, y_{m+1})\|_X \\ &\le L_f\big[|t-s| + \sum_{i=1}^{m+1}\|x_i - y_i\|_X\big] \end{aligned}$$
for all $(x_1, \dots, x_{m+1})$ and $(y_1, \dots, y_{m+1})$ in X^{m+1} and $t \in [0,T]$.
- (H3) The function $I_k : X \to X$ are continuous and there exists $L_k > 0$ such that $\|I_k(x) - I_k(y)\|_X \le L_k\|x-y\|_X$, $x, y \in X, k = 1, 2, \dots, p$, $L = \max\{L_k\} > L_g$.

Theorem 3.3. *Assume* (H1)–(H3) *are satisfied and*

$$\big[\widetilde{M}_T(1+L)+\widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha}\big]<1.$$

Then impulsive problem (1.1) *has a unique mild solution* $x\in PC(J,X)$.

Proof. Define a mapping N from $PC(J,X)$ into itself by

$$(Nx)(t)=\begin{cases} T_\alpha(t)(x_0-g(x)) & \\ +\int_0^t S_\alpha(t-s)f(s,x(s),x(a_1(s)),\dots,x(a_m(s)))ds, & t\in[0,t_1];\\ T_\alpha(t-t_1)[x(t_1^-)+I_1(x(t_1^-))] & \\ +\int_{t_1}^t S_\alpha(t-s)f(s,x(s),x(a_1(s)),\dots,x(a_m(s)))ds, & t\in(t_1,t_2];\\ \dots & \\ T_\alpha(t-t_p)[x(t_p^-)+I_p(x(t_p^-))] & \\ +\int_{t_p}^t S_\alpha(t-s)f(s,x(s),x(a_1(s)),\dots,x(a_m(s)))ds, & t\in(t_p,T]. \end{cases}$$

Now we show that N is a contraction on $PC(J,X)$. We have

$$\|Nx(t)-Ny(t)\|_X$$

$$\le\begin{cases} \|T_\alpha(t)\|_{L(X)}(\|g(x)-g(y)\|_X+\int_0^t\|S_\alpha(t-s)\|_{L(X)} & \\ \times\|f(s,x(s),x(a_1(s)),\dots,x(a_m(s))) & \\ -f(s,y(s),y(a_1(s)),\dots,y(a_m(s)))\|_X ds, & t\in[0,t_1];\\ \|T_\alpha(t-t_1)\|_{L(X)}(\|x(t_1^-)-y(t_1^-)\|_X+\|I_1(x(t_1^-))-I_1(y(t_1^-))\|_X) & \\ +\int_{t_1}^t\|S_\alpha(t-s)\|_{L(X)}\|f(s,x(s),x(a_1(s)),\dots,x(a_m(s))) & \\ -f(s,y(s),y(a_1(s)),\dots,y(a_m(s)))\|_X ds, & t\in(t_1,t_2];\\ \dots & \\ \|T_\alpha(t-t_p)\|_{L(X)}(\|x(t_p^-)-y(t_p^-)\|_X+\|I_p(x(t_p^-))-I_p(y(t_p^-))\|_X) & \\ +\int_{t_p}^t\|S_\alpha(t-s)\|_{L(X)}\|f(s,x(s),x(a_1(s)),\dots,x(a_m(s))) & \\ -f(s,y(s),y(a_1(s)),\dots,y(a_m(s)))\|_X ds, & t\in(t_p,T]; \end{cases}$$

Applying Assumptions (H1)–(H3), we obtain

$$\|Nx(t)-Ny(t)\|_X\le\begin{cases} [\widetilde{M}_T[L_g+\widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha}]\|x-y\|_{PC}, & t\in[0,t_1];\\ [\widetilde{M}_T(1+L_1)+\widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha}]\|x-y\|_{PC}, & t\in(t_1,t_2];\\ \dots & \\ [\widetilde{M}_T(1+L_p)+\widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha}]\|x-y\|_{PC}, & t\in(t_p,T]. \end{cases}$$

Which implies that for $t\in[0,T]$,

$$\|Nx-Ny\|_{PC}\le[\widetilde{M}_T(1+L)+\widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha}]\|x-y\|_{PC}.$$

Since $[\widetilde{M}_T(1+L)+\widetilde{M}_S L_f(m+1)\frac{T^\alpha}{\alpha}]<1$, N is a contraction. Therefore, N has a unique fixed point by Banach contraction principle. This completes the proof of the theorem. □

Our second result is based on the following Krasnoselkii's fixed point theorem.

Theorem 3.4. *Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that:*

(1) *$Px + Qy \in B$ whenever $x, y \in B$;*
(2) *P is compact and continuous;*
(3) *Q is a contraction mapping;*

Then there exists $z \in B$ such that $z = Pz + Qz$.

Now, we make the following assumptions:

(H4) $f \in C(J \times X^{m+1}, X)$, $g \in C(X, X)$, and $b_i \in C(J, J)$ $(i = 1, \dots, m)$. Moreover, there are $C_i > 0$ $(i = 1, 2)$ such that $\|f(s, z_0, z_1, \dots, z_m)\| \leq C_1$ for $s \in J$, $z_i \in B_r$ $(i = 0, 1, \dots, m)$ and $\|g(w)\| \leq C_2$ for $w \in X$.

(H5) The function $I_k : X \to X$ are continuous and there exists $\rho > C_2$ such that

$$\rho = \max_{1 \leq k \leq m, x \in B_r} \{\|I_k(x)\|_X\}.$$

Theorem 3.5. *Assume* (H2), (H4), (H5) *are satisfied and*

$$[\widetilde{M}_S L_f (m+1) \frac{T^\alpha}{\alpha}] < 1.$$

Then the impulsive problem (1.1) *has at least one mild solution on J.*

Proof. Choose $r \geq [\widetilde{M}_T(r + \rho) + \widetilde{M}_S C_1 \frac{T^\alpha}{\alpha}]$ and consider $B_r = \{x \in PC(J, X) : \|x\|_{PC} \leq r, \}$ then B_r is a bounded, closed convex subset in $PC(J, X)$. Define on B_r the operators P and Q by:

$$(Px)(t) = \begin{cases} T_\alpha(t)(x_0 - g(x)), & t \in [0, t_1]; \\ T_\alpha(t - t_1)[x(t_1^-) + I_1(x(t_1^-))], & t \in (t_1, t_2]; \\ \dots \\ T_\alpha(t - t_p)[x(t_p^-) + I_p(x(t_p^-))], & t \in (t_p, T], \end{cases}$$

$$(Qx)(t) = \begin{cases} \int_0^t S_\alpha(t-s) f(s, x(s), x(a_1(s)), \dots, x(a_m(s))) ds, & t \in [0, t_1]; \\ \int_{t_1}^t S_\alpha(t-s) f(s, x(s), x(a_1(s)), \dots, x(a_m(s))) ds, & t \in (t_1, t_2]; \\ \dots \\ \int_{t_p}^t S_\alpha(t-s) f(s, x(s), x(a_1(s)), \dots, x(a_m(s))) ds, & t \in (t_p, T]. \end{cases}$$

Now we present the proof in five steps:

Step 1. We show that $Px + Qy \in B_r$ whenever $x, y \in B_r$. Let $x, y \in B_r$, then

$$\begin{aligned} &\|Px + Qy\|_{PC} \\ &\leq \begin{cases} \|T_\alpha(t)\|_{L(X)}(\|x_0\|_X + \|g(x)\|_X) \\ + \int_0^t \|S_\alpha(t-s)\|_{L(X)} \|f(s, y(s), y(a_1(s)), \dots, y(a_m(s)))\|_X ds, & t \in [0, t_1]; \\ \|T_\alpha(t - t_1)\|_{L(X)}[\|x(t_1^-)\|_X + \|I_1(x(t_1^-))\|_X] \\ + \int_{t_1}^t \|S_\alpha(t-s)\|_{L(X)} \|f(s, y(s), y(a_1(s)), \dots, y(a_m(s)))\|_X ds, & t \in (t_1, t_2]; \\ \dots \\ \|T_\alpha(t - t_p)\|_{L(X)}[\|x(t_p^-)\|_X + \|I_p(x(t_p^-))\|_X] \\ + \int_{t_p}^t \|S_\alpha(t-s)\|_{L(X)} \|f(s, y(s), y(a_1(s)), \dots, y(a_m(s)))\|_X ds, & t \in (t_p, T]. \end{cases} \\ &\leq \begin{cases} \widetilde{M}_T(r + C_2) + \widetilde{M}_S C_1 \frac{T^\alpha}{\alpha}, & t \in [0, t_1]; \\ \widetilde{M}_T(r + \rho) + \widetilde{M}_S C_1 \frac{T^\alpha}{\alpha}, & t \in (t_1, t_2]; \\ \dots \\ \widetilde{M}_T(r + \rho) + \widetilde{M}_S C_1 \frac{T^\alpha}{\alpha}, & t \in (t_p, T]. \end{cases} \end{aligned}$$

Which implies

$$\|Px + Qy\|_{PC} \le [\widetilde{M}_T(r+\rho) + \widetilde{M}_S C_1 \frac{T^\alpha}{\alpha}] \le r.$$

Step 2. Continuity of P. For this purpose, let $\{x^n\}_{n=0}^{\infty}$ be a sequence in B_r with $\lim x^n \to x$ in B_r. Then for every $t \in J$, we have

$$\|(Px^n)(t) - (Px)(t)\|_X \le \begin{cases} \|T_\alpha(t)\|_{L(X)} \|g(x^n) - g(x)\|_X, & t \in [0, t_1]; \\ \|T_\alpha(t-t_1)\|_{L(X)} [\|x^n(t_1^-) - x(t_1^-)\|_X & \\ + \|I_1(x^n(t_1^-)) - I_1 x(t_1^-)\|_X], & t \in (t_1, t_2]; \\ \dots & \\ \|T_\alpha(t-t_p)\|_{L(X)} [\|x^n(t_p^-) - x(t_p^-)\|_X & \\ + \|I_p(x^n(t_p^-)) - I_p x(t_p^-)\|_X], & t \in (t_p, T]. \end{cases}$$

Since the functions g and I_k, $k = 1, \dots, p$ are continuous, $\lim_{n\to\infty} \|Px^n - Px\|_{PC} = 0$ in B_r. This implies that the mapping P is continuous on B_r.

Step 3. P maps bounded sets into bounded sets in $PC(J, X)$. So, let us prove that for any $r > 0$ there exists a $\gamma > 0$ such that for each $x \in B_r = \{x \in PC(J, X) : \|x\|_{PC} \le r\}$, we have $\|Px\|_{PC} \le \gamma$. Indeed, we have for any $x \in B_r$,

$$\begin{aligned} \|Px(t)\|_X &\le \begin{cases} \|T_\alpha(t)\|_{L(X)} (\|x_0\|_X + \|g(x)\|_X), & t \in [0, t_1]; \\ \|T_\alpha(t-t_1)\|_{L(X)} [\|x(t_1^-)\|_X + \|I_1(x(t_1^-))\|_X], & t \in (t_1, t_2]; \\ \dots & \\ \|T_\alpha(t-t_p)\|_{L(X)} [\|x(t_p^-)\|_X + \|I_p(x(t_p^-))\|_X], & t \in (t_p, T]. \end{cases} \\ &\le \begin{cases} \widetilde{M}_T(r + C_2), & t \in [0, t_1]; \\ \widetilde{M}_T(r+\rho), & t \in (t_1, t_2]; \\ \dots & \\ \widetilde{M}_T(r+\rho), & t \in (t_p, T]. \end{cases} \end{aligned}$$

Which implies that $\|Px\|_{PC} \le \widetilde{M}_T(r+\rho) = \gamma$.

Step 4. We prove that $P(B_r)$ is equicontinuous with B_r. For $0 \le u < v \le T$, we have

$$\begin{aligned} &\|(Px)(v) - (Px)(u)\|_X \\ &\le \begin{cases} \|T_\alpha(v) - T_\alpha(u)\|_{L(X)} [\|x_0\|_X + \|g(x)\|_X], & 0 \le u < v \le t_1; \\ \|T_\alpha(v-t_1) - T_\alpha(u-t_1)\|_{L(X)} & \\ \times [\|x(t_1^-)\|_X + \|I_1(x(t_1^-))\|_X], & t_1 < u < v \le t_2; \\ \dots & \\ \|T_\alpha(v-t_p) - T_\alpha(u-t_p)\|_{L(X)} & \\ \times [\|x(t_p^-)\|_X + \|I_p(x(t_p^-))\|_X, & t_p < u < v \le T. \end{cases} \\ &\le \begin{cases} (r + C_2)\|T_\alpha(v) - T_\alpha(u)\|_{L(X)}, & 0 \le u < v \le t_1; \\ (r+\rho)\|T_\alpha(v-t_1) - T_\alpha(u-t_1)\|_{L(X)}, & t_1 < u < v \le t_2 \\ \dots & \\ (r+\rho)\|T_\alpha(v-t_p) - T_\alpha(u-t_p)\|_{L(X)}, & t_p < u < v \le T. \end{cases} \end{aligned}$$

Therefore, the continuity of the function $t \mapsto \|T(t)\|$ allows us to conclude that $\lim_{u\to v} \|T_\alpha(v - t_i) - T_\alpha(u - t_i)\|_{L(X)} = 0$, $i = 1, \dots, p$ and $\lim_{u\to v} \|T_\alpha(v) -$

$T_\alpha(u)\|_{L(X)} = 0$. Finally, combining Step 2 to Step 4 with the Ascoli's Theorem, we deduce that the operator P is a compact.

Step 5. We show that Q is a contraction mapping. Let $x, y \in B_r$ and we have

$$
\begin{aligned}
&\|(Qx)(t) - (Qy(t))(t)\|_X \\
&\le \begin{cases}
\int_0^t \|S_\alpha(t-s)\|_{L(X)} \|f(s, x(s), x(a_1(s)), \dots, x(a_m(s))) \\
-f(s, y(s), y(a_1(s)), \dots, y(a_m(s)))\|_X ds, & t \in [0, t_1]; \\
\int_{t_1}^t \|S_\alpha(t-s)\|_{L(X)} \|f(s, x(s), x(a_1(s)), \dots, x(a_m(s))) \\
-f(s, y(s), y(a_1(s)), \dots, y(a_m(s)))\|_X ds, & t \in (t_1, t_2]; \\
\dots \\
\int_{t_p}^t \|S_\alpha(t-s)\|_{L(X)} \|f(s, x(s), x(a_1(s)), \dots, x(a_m(s))) \\
-f(s, y(s), y(a_1(s)), \dots, y(a_m(s)))\|_X ds, & t \in (t_p, T].
\end{cases} \\
&\le \begin{cases}
\widetilde{M}_S L_f (m+1) \frac{T^\alpha}{\alpha} \|x - y\|_{PC}, & t \in [0, t_1]; \\
\widetilde{M}_S L_f (m+1) \frac{T^\alpha}{\alpha} \|x - y\|_{PC}, & t \in (t_1, t_2]; \\
\dots \\
\widetilde{M}_S L_f (m+1) \frac{T^\alpha}{\alpha} \|x - y\|_{PC}, & t \in (t_p, T].
\end{cases}
\end{aligned}
$$

Since $(\widetilde{M}_S L_f (m+1) \frac{T^\alpha}{\alpha}) < 1$ then Q is a contraction mapping. Hence, by the Krasnoselkii theorem, we can conclude that (1.1) has at least one solution on $[0, T]$. This completes the proof of the theorem. $\square$

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