Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 111, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GORDON TYPE THEOREM FOR MEASURE PERTURBATION

CHRISTIAN SEIFERT

Abstract. Generalizing the concept of Gordon potentials to measures we prove a version of Gordon's theorem for measures as potentials and show absence of eigenvalues for these one-dimensional Schrödinger operators.

1. Introduction

According to [2], the one-dimensional Schrödinger operator $H = -\Delta + V$ has no eigenvalues if the potential $V \in L_{1,\mathrm{loc}}(\mathbb{R})$ can be approximated by periodic potentials (in a suitable sense). The aim of this paper is to generalize this result to measures μ instead of potential functions V; i.e., to more singular potentials.

Although all statements remain valid for complex measures we only focus on real (but signed) measures μ, since we are interested in self-adjoint operators.

In the remaining part of this section we explain the situation and define the operator in question. We also describe the class of measures we are concerned with. Section 2 provides all the tools we need to prove the main theorem: $H = -\Delta+\mu$ has no eigenvalues for suitable μ. In section 3 we show some examples for Schrödinger operators with measures as potentials.

We consider a Schrödinger operator of the form $H = -\Delta + \mu$ on $L_2(\mathbb{R})$. Here, $\mu = \mu_+ - \mu_-$ is a signed Borel measure on $\mathbb{R}$ with locally finite total variation $|\mu|$.

We define H via form methods. To this end, we need to establish form boundedness of μ_-. Therefore, we restrict the class of measures we want to consider.

Definition 1.1. A signed Borel measure μ on $\mathbb{R}$ is called *uniformly locally bounded*, if

$$\|\mu\|_{\mathrm{loc}} := \sup_{x\in\mathbb{R}} |\mu|([x, x + 1]) < \infty.$$

We call μ a *Gordon measure* if μ is uniformly locally bounded and if there exists a sequence $(\mu^m)_{m\in\mathbb{N}}$ of uniformly locally bounded periodic Borel measures with period sequence (p_m) such that $p_m \to \infty$ and for all $C \in \mathbb{R}$ we have

$$\lim_{m\to\infty} e^{Cp_m} |\mu - \mu^m|([-p_m, 2p_m]) = 0;$$

i.e., (μ^m) approximates μ on increasing intervals. Here, a Borel measure is p-periodic, if $\mu = \mu(\cdot + p)$.

2000 *Mathematics Subject Classification.* 34L05, 34L40, 81Q10.
Key words and phrases. Schrödinger operators; eigenvalue problem;
quasiperiodic measure potentials.

Submitted May 31, 2011. Published August 29, 2011.

Clearly, every generalized Gordon potential $V \in L_{1,\mathrm{loc}}$ as defined in [2] induces a Gordon measure $\mu = V\lambda$, where λ is the Lebegue measure on $\mathbb{R}$. Therefore, also every Gordon potential (see the original work [4]) induces a Gordon measure.

Lemma 1.2. *Let μ be a uniformly locally bounded measure. Then $|\mu|$ is $-\Delta$-form bounded, and for all $0 < c < 1$ there is $\gamma \geq 0$ such that*

$$\int_{\mathbb{R}} |u|^2 \, d|\mu| \leq c\|u'\|_2^2 + \gamma\|u\|_2^2 \quad (u \in W_2^1(\mathbb{R})).$$

Proof. For $\delta \in (0,1)$ and $n \in \mathbb{Z}$ we have

$$\|u\|^2_{\infty,[n\delta,(n+1)\delta]} \leq 4\delta\|u'\|^2_{L_2(n\delta,(n+1)\delta)} + \frac{4}{\delta}\|u\|^2_{L_2(n\delta,(n+1)\delta)}$$

by Sobolev's inequality.

Now, we estimate

$$\begin{aligned}
\int_{\mathbb{R}} |u|^2 \, d|\mu| &= \sum_{n\in\mathbb{Z}} \int_{n\delta}^{(n+1)\delta} |u|^2 \, d|\mu| \\
&\leq \sum_{n\in\mathbb{Z}} \|u\|^2_{\infty,[n\delta,(n+1)\delta]} \|\mu\|_{\mathrm{loc}} \\
&\leq \|\mu\|_{\mathrm{loc}} \sum_{n\in\mathbb{Z}} \Big(4\delta\|u'\|^2_{L_2(n\delta,(n+1)\delta)} + \frac{4}{\delta}\|u\|^2_{L_2(n\delta,(n+1)\delta)}\Big) \\
&= 4\delta\|\mu\|_{\mathrm{loc}}\|u'\|_2^2 + \frac{4\|\mu\|_{\mathrm{loc}}}{\delta}\|u\|_2^2.
\end{aligned}$$

□

Let μ be a Gordon measure and define

$$D(\tau) := W_2^1(\mathbb{R}), \quad \tau(u,v) := \int u'\overline{v}' + \int u\overline{v}\, d\mu.$$

Then τ is a closed symmetric semibounded form. Let H be the associated self-adjoint operator.

In [1], Ben Amor and Remling introduced a direct approach for defining the Schrödinger operator $H = -\Delta + \mu$. Since we will use some of their results we sum up the main ideas: For $u \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ define $Au \in L_{1,\mathrm{loc}}(\mathbb{R})$ by

$$Au(x) := u'(x) - \int_0^x u(t)\, d\mu(t),$$

where

$$\int_0^x u(t)\, d\mu(t) := \begin{cases} \int_{[0,x]} u(t)\, d\mu(t) & \text{if } x \geq 0, \\ -\int_{(x,0)} u(t)\, d\mu(t) & \text{if } x < 0. \end{cases}$$

Clearly, Au is only defined as an $L_{1,\mathrm{loc}}(\mathbb{R})$-element. We define the operator T in $L_2(\mathbb{R})$ by

$$D(T) := \big\{u \in L_2(\mathbb{R});\ u, Au \in W^1_{1,\mathrm{loc}}(\mathbb{R}),\, (Au)' \in L_2(\mathbb{R})\big\}, \quad Tu := -(Au)'.$$

Lemma 1.3. *$H \subseteq T$.*

Proof. Let $u \in D(H)$. Then $u \in W_2^1(\mathbb{R}) \subseteq W_{1,\mathrm{loc}}^1(\mathbb{R})$ and $Au \in L_{1,\mathrm{loc}}(\mathbb{R})$. Let $\varphi \in C_c^\infty(\mathbb{R}) \subseteq D(\tau)$. Using Fubini's Theorem, we compute

$$\begin{aligned}
\int_{\mathbb{R}} (Au)(x)\varphi'(x)\,dx &= \int_{\mathbb{R}} \Big(u'(x) - \int_0^x u(t)\,d\mu(t)\Big)\varphi'(x)\,dx \\
&= \int_{\mathbb{R}} u'(x)\varphi'(x)\,dx - \int_{\mathbb{R}}\int_0^x u(t)\,d\mu(t)\varphi'(x)\,dx \\
&= \int_{\mathbb{R}} u'(x)\varphi'(x)\,dx + \int_{-\infty}^0 \int_{(-\infty,t)} \varphi'(x)\,dx u(t)\,d\mu(t) \\
&\quad - \int_0^\infty \int_{[t,\infty)} \varphi'(x)\,dx u(t)\,d\mu(t) \\
&= \int_{\mathbb{R}} u'(x)\varphi'(x)\,dx + \int_{-\infty}^0 u(t)\varphi(t)\,d\mu(t) + \int_0^\infty u(t)\varphi(t)\,d\mu(t) \\
&= \int_{\mathbb{R}} u'\varphi' + \int_{\mathbb{R}} u(x)\varphi(x)\,dx \\
&= \tau(u,\overline{\varphi}) = (Hu \mid \overline{\varphi}) = \int_{\mathbb{R}} Hu(x)\varphi(x)\,dx.
\end{aligned}$$

Hence, $(Au)' = -Hu \in L_2(\mathbb{R})$. We conclude that $Au \in W_{1,\mathrm{loc}}^1(\mathbb{R})$ and therefore $u \in D(T)$, $Tu = -(Au)' = Hu$. $\square$

Remark 1.4. For $u \in D(H)$ we obtain

$$u'(x) = Au(x) + \int_0^x u(t)\,d\mu(t)$$

for a.a. $x \in \mathbb{R}$. Since $Au \in W_{1,\mathrm{loc}}^1(\mathbb{R})$ and $x \mapsto \int_0^x u(t)\,d\mu(t)$ is continuous at all $x \in \mathbb{R}$ with $\mu(\{x\}) = 0$, u' is continuous at x for all $x \in \mathbb{R} \setminus \mathrm{spt}\mu_p$, where μ_p is the point measure part of μ.

2. Absence of eigenvalues

We show that H has no eigenvalues. The proof is based on two observations. The first one is a stability result and will be achieved in Lemma 2.6, the second one is an estimate of the solution for periodic measure perturbations, see Lemma 2.8.

As in [2] we start with a Gronwall Lemma, but in a more general version for locally finite measures. For the proof, see [3].

Lemma 2.1 (Gronwall)**.** *Let μ be a locally finite Borel measure on $[0,\infty)$, $u \in \mathcal{L}_{1,\mathrm{loc}}([0,\infty),\mu)$ and $\alpha : [0,\infty) \to [0,\infty)$ measurable. Suppose, that*

$$u(x) \le \alpha(x) + \int_{[0,x]} u(s)\,d\mu(s) \quad (x \ge 0).$$

Then

$$u(x) \le \alpha(x) + \int_{[0,x]} \alpha(s)\exp\big(\mu([s,x])\big)\,d\mu(s) \quad (x \ge 0).$$

For $x \in \mathbb{R}$ we abbreviate

$$I_x := [x \wedge 0, x \vee 0], \quad I_x(t) := I_x \cap ([t,x] \cup [x,t]) \quad (t \in \mathbb{R}).$$

Let μ be uniformly locally bounded. Then

$$|\mu|(I_x) \le (|x|+1)\|\mu\|_{\mathrm{loc}} \quad (x \in \mathbb{R}).$$

Furthermore, if μ is periodic and locally bounded, μ is uniformly locally bounded.

Let $H := -\Delta + \mu$ and $E \in \mathbb{R}$. Then $u \in W^1_{1,\mathrm{loc}}(\mathbb{R})$ $(= D(A))$ is a *solution* of $Hu = Eu$, if $-(Au)' = Eu$ in the sense of distributions (i.e., u satisfies the eigenvalue equation but without being an L_2-function).

Lemma 2.2. *Let μ_1, μ_2 be two uniformly locally bounded measures, $E \in \mathbb{R}$ and u_1 and u_2 solutions of*

$$H_1 u_1 = E u_1, \quad H_2 u_2 = E u_2$$

subject to

$$u_1(0) = u_2(0), \quad u_1'(0+) = u_2'(0+), \quad |u_1(0)|^2 + |u_1'(0+)|^2 = 1.$$

Then there are $C_0, C \geq 0$ such that for all $x \in \mathbb{R}$

$$\begin{aligned} &\Big\| \begin{pmatrix} u_1(x) \\ u_1'(x) \end{pmatrix} - \begin{pmatrix} u_2(x) \\ u_2'(x) \end{pmatrix} \Big\| \\ &\leq C_0 + \int_{I_x} |u_2(t)|\, d|\mu_1 - \mu_2|(t) \\ &\quad + C \int_{I_x} \Big(C_0 + \int_{I_t} |u_2| d|\mu_1 - \mu_2| \Big) e^{C(\lambda + |\mu_1 - E\lambda|)(I_x(t))}\, d(\lambda + |\mu_1 - E\lambda|)(t). \end{aligned}$$

Proof. Write

$$u_1(x) - u_2(x) = \int_0^x (u_1'(t) - u_2'(t))\, dt$$

and

$$\begin{aligned} u_1'(x) - u_2'(x) &= u_1'(0+) - u_2'(0+) - (u_1(0)\mu_1(\{0\}) - u_2(0)\mu_2(\{0\})) \\ &\quad + \int_0^x u_1(t)\, d\mu_1(t) - \int_0^x u_2(t)\, d\mu_2(t) - \int_0^x E(u_1(t) - u_2(t))\, dt \\ &= u_2(0)\, (\mu_2(\{0\}) - \mu_1(\{0\})) \\ &\quad + \int_0^x u_2(t)\, d(\mu_1 - \mu_2)(t) + \int_0^x (u_1(t) - u_2(t))\, d(\mu_1 - E\lambda)(t). \end{aligned}$$

Hence,

$$\begin{aligned} \begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} &= \begin{pmatrix} 0 \\ u_2(0)\, (\mu_2(\{0\}) - \mu_1(\{0\})) \end{pmatrix} + \int_0^x \begin{pmatrix} 0 \\ u_2(t) \end{pmatrix} d(\mu_1 - \mu_2)(t) \\ &\quad + \int_0^x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1(t) - u_2(t) \\ u_1'(t) - u_2'(t) \end{pmatrix} d \begin{pmatrix} \lambda \\ \mu_1 - E\lambda \end{pmatrix} (t). \end{aligned}$$

We conclude, that

$$\begin{aligned} \Big\| \begin{pmatrix} u_1(x) \\ u_1'(x) \end{pmatrix} - \begin{pmatrix} u_2(x) \\ u_2'(x) \end{pmatrix} \Big\| &\leq C_0 + \int_{I_x} |u_2(t)|\, d|\mu_1 - \mu_2|(t) \\ &\quad + C \int_{I_x} \Big\| \begin{pmatrix} u_1(t) \\ u_1'(t) \end{pmatrix} - \begin{pmatrix} u_2(t) \\ u_2'(t) \end{pmatrix} \Big\|\, d(\lambda + |\mu_1 - E\lambda|)(t). \end{aligned}$$

An application of Lemma 2.1 with $\alpha(x) = C_0 + \int_{I_x} |u_2(t)|\, d|\mu_1 - \mu_2|(t)$ and $\mu = C(\lambda + |\mu_1 - E\lambda|)$ yields the assertion. $\square$

Remark 2.3. Regarding the proof of Lemma 2.2 we can further estimate $C_0 \leq |u_2(0)||\mu_1 - \mu_2|(I_x)$ $(x \in \mathbb{R})$.

Lemma 2.4. *Let $E \in \mathbb{R}$ and u_0 be a solution of $-\Delta u_0 = Eu_0$. Then there is $C \geq 0$ such that $|u_0(x)| \leq Ce^{C|x|}$ for all $x \in \mathbb{R}$.*

In the following lemmas and proofs the constant C may change from line to line, but we will always state the dependence on the important quantities.

Lemma 2.5. *Let μ_1 be a locally bounded p-periodic measure, $E \in \mathbb{R}$, u_1 a solution of $H_1 u_1 = Eu_1$. Then there is $C \geq 0$ such that*

$$|u_1(x)| \leq Ce^{C|x|} \quad (x \in \mathbb{R}).$$

Proof. Let u_0 be a solution of $-\Delta u_0 = Eu_0$ subject to the same boundary conditions at 0 as u_1. By Lemma 2.2 we have

$$\begin{aligned}
&|u_1(x) - u_0(x)| \\
&\leq C + \int_{I_x} |u_0(t)|\, d|\mu_1|(t) \\
&\quad + C \int_{I_x} \Big(C + \int_{I_t} |u_0(s)|\, d|\mu_1|(s) \Big) e^{C(\lambda + |\mu_1 - E\lambda|)(I_x(t))}\, d(\lambda + |\mu_1 - E\lambda|)(t) \\
&\leq C + |\mu_1|(I_x) Ce^{C|x|} \\
&\quad + \int_{I_x} \Big(C + C|\mu_1|(I_t) e^{C|t|} \Big)\, e^{(\lambda + |\mu_1 - E\lambda|)(I_x(t))}\, d(\lambda + |\mu_1 - E\lambda|)(t) \\
&\leq \Big(C + C|\mu_1|(I_x) e^{C|x|} \Big) \Big(1 + e^{(\lambda + |\mu_1 - E\lambda|)(I_x)} (\lambda + |\mu_1 - E\lambda|)(I_x) \Big).
\end{aligned}$$

Since μ_1 is periodic and locally bounded it is uniformly locally bounded and we have $|\mu_1|(I_x) \leq (|x| + 1)\|\mu_1\|_{\rm loc}$. Furthermore, $\mu_1 - E\lambda$ is periodic and uniformly locally bounded, so $|\mu_1 - E\lambda|)(I_x) \leq (|x| + 1)\|\mu_1 - E\lambda\|_{\rm loc}$. We conclude that

$$\begin{aligned}
|u_1(x) - u_0(x)| &\leq \Big(C + C(|x| + 1)\|\mu_1\|_{\rm loc} e^{C|x|} \Big) \\
&\quad \times \Big(1 + e^{(|x|+1)(1 + \|\mu_1 - E\lambda\|_{\rm loc})} (|x| + 1)(1 + \|\mu_1 - E\lambda\|_{\rm loc}) \Big) \\
&\leq Ce^{C|x|},
\end{aligned}$$

where C is depending on E, $\|\mu_1\|_{\rm loc}$ and $\|\mu_1 - E\lambda\|_{\rm loc}$. Hence,

$$|u_1(x)| \leq |u_1(x) - u_0(x)| + |u_0(x)| \leq Ce^{C|x|}.$$

□

Lemma 2.6. *Let μ be a Gordon measure and (μ^m) the p_m-periodic approximants, $E \in \mathbb{R}$. Let u be a solution of $Hu = Eu$, u_m a solution of $H_m u_m = Eu_m$ for $m \in \mathbb{N}$ (obeying the same boundary conditions at 0). Then there is $C \geq 0$ such that*

$$\Big\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u_m'(x) \end{pmatrix} \Big\| \leq Ce^{C|x|} |\mu - \mu^m|(I_x) \quad (x \in \mathbb{R}).$$

Proof. By Lemma 2.2 and Remark 2.3 we know that

$$\begin{aligned}
\Big\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u_m'(x) \end{pmatrix} \Big\| &\leq |u_m(0)||\mu - \mu^m|(I_x) + \int_{I_x} |u_m(t)|\, d|\mu - \mu^m|(t) \\
&\quad + C \int_{I_x} \Big(|u_m(0)||\mu - \mu^m|(I_t) + \int_{I_t} |u_m| d|\mu - \mu^m| \Big) \\
&\quad \times e^{C(\lambda + |\mu - E\lambda|)(I_x(t))}\, d(\lambda + |\mu - E\lambda|)(t).
\end{aligned}$$

We have

$$M := \sup_{m\in\mathbb{N}} \|\mu^m\|_{\mathrm{loc}} < \infty,$$

since (μ^m) approximates μ. Hence, also

$$\sup_{m\in\mathbb{N}} \|\mu^m - E\lambda\|_{\mathrm{loc}} < \infty$$

and Lemma 2.5 yields

$$|u_m(x)| \le Ce^{C|x|},$$

where C can be chosen independently of m. Therefore

$$\begin{aligned}\Big\| \begin{pmatrix} u(x) \\ u'(x)\end{pmatrix} - \begin{pmatrix} u_m(x) \\ u_m'(x)\end{pmatrix} \Big\| &\le \Big(Ce^{C|x|}|\mu-\mu^m|(I_x) + Ce^{C|x|}|\mu-\mu^m|(I_x)\Big) \\ &\quad \times \Big(1 + e^{C(\lambda+|\mu-E\lambda|)(I_x)}(\lambda+|\mu-E\lambda|)(I_x)\Big).\end{aligned}$$

Since

$$|\mu - E\lambda|(I_x) \le (|x|+1)\|\mu-E\lambda\|_{\mathrm{loc}},$$

we further estimate

$$\Big\| \begin{pmatrix} u(x) \\ u'(x)\end{pmatrix} - \begin{pmatrix} u_m(x) \\ u_m'(x)\end{pmatrix} \Big\| \le Ce^{C|x|}|\mu-\mu^m|(I_x)$$

where C is depending on $\|\mu - E\lambda\|_{\mathrm{loc}}$ (and of course on M, $\|\mu\|_{\mathrm{loc}}$ and E). □

Lemma 2.6 can be regarded as a stability (or continuity) result: if the measures converge in total variation, the corresponding solutions converge as well.

Now, we focus on periodic measures and estimate the solutions. This will then be applied to the periodic approximations of our Gordon measure μ.

Remark 2.7. (a) Let f, g be two solutions of the equation $Hu = Eu$. Define their Wronskian by $W(f,g)(x) := f(x)g'(x+) - f'(x+)g(x)$. By [1], Proposition 2.5, $W(f,g)$ is constant.

(b) Let u be a solution of the equation $Hu = Eu$. Define the transfer matrix $T_E(x)$ mapping $(u(0), u'(0+))^\top$ to $(u(x), u'(x+))^\top$. Consider now the two solutions u_N, u_D subject to

$$\begin{pmatrix} u_N(0) \\ u_N'(0+)\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix}, \quad \begin{pmatrix} u_D(0) \\ u_D'(0+)\end{pmatrix} = \begin{pmatrix} 0 \\ 1\end{pmatrix}.$$

Then

$$T_E(x) = \begin{pmatrix} u_N(x) & u_D(x) \\ u_N'(x+) & u_D'(x+)\end{pmatrix}.$$

We obtain $\det T_E(x) = W(u_N, u_D)(x)$ and $\det T_E$ is constant, hence equals 1 for all $x \in \mathbb{R}$.

Lemma 2.8. *Let μ be p-periodic and $E \in \mathbb{R}$. Let u be a solution of $Hu = Eu$ subject to*

$$|u(0)|^2 + |u'(0+)|^2 = 1.$$

Then

$$\max\Big\{\Big\| \begin{pmatrix} u(-p) \\ u'(-p+)\end{pmatrix} \Big\|, \Big\| \begin{pmatrix} u(p) \\ u'(p+)\end{pmatrix} \Big\|, \Big\| \begin{pmatrix} u(2p) \\ u'(2p+)\end{pmatrix} \Big\|\Big\} \ge \frac{1}{2}.$$

The proof of this lemma is completely analoguous to the proof of [2, Lemma 2.2].

Lemma 2.9. *Let $v \in L_2 \cap BV_{\rm loc}(\mathbb{R})$ and assume that for all $r > 0$ we have*

$$|v(x) - v(x+r)| \to 0 \quad (|x| \to \infty).$$

Then $|v(x)| \to 0$ as $|x| \to \infty$.

Proof. Without restriction, we can assume that $v \geq 0$. We prove this lemma by contradiction. Assume that $v(x) \to 0$ does not hold for $x \to \infty$. Then we can find $\delta > 0$ and (q_k) in $\mathbb{R}$ with $q_k \to \infty$ such that $v(q_k) \geq \delta$ for all $k \in \mathbb{N}$. By square integrability of v we have $\|v\mathbf{1}_{[q_k,q_k+1]}\|_2 \to 0$. Therefore, we can find a subsequence (r_n) of (q_k) satisfying

$$\|v\mathbf{1}_{[r_n,r_n+1]}\|_2 \leq 2^{-3n/2} \quad (n \in \mathbb{N}).$$

Now, Chebyshev's inequality implies

$$\lambda(\{x \in [r_n, r_n+1];\ v(x) \geq 2^{-n}\}) \leq 2^{2n}\|v\mathbf{1}_{[r_n,r_n+1]}\|_2^2 \leq 2^{-n} \quad (n \in \mathbb{N}).$$

Denote $A_n := \{x \in [r_n, r_n+1];\ v(x) \geq 2^{-n}\} - r_n \subseteq [0,1]$. Then $\lambda(A_n) \leq 2^{-n}$ and

$$\lambda\big(\cup_{n\geq 3} A_n\big) \leq \sum_{n\geq 3} \lambda(A_n) \leq 2^{-2} < 1.$$

Hence, $G := [0,1] \setminus (\cup_{n\geq 3} A_n)$ has positive measure. For $r \in G$, $r > 0$ it follows $v(r_n + r) \leq 2^{-n}$ $(n \geq 3)$. Therefore,

$$\liminf_{n\to\infty} |v(r_n) - v(r_n + r)| \geq \delta > 0,$$

a contradiction. □

Lemma 2.10. *Let μ be a Gordon measure, $E \in \mathbb{R}$, $u \in D(H)$ a solution of $Hu = Eu$. Then $u(x) \to 0$ as $x \to \infty$ and $u'(x) \to 0$ as $x \to \infty$.*

Proof. Since $u \in D(H) \subseteq D(\tau) \subseteq W_2^1(\mathbb{R})$ we have $u(x) \to 0$ as $|x| \to \infty$. Lemma 1.3 yields $u \in D(T)$ and $-(Au)' = Hu = Eu$. Let $r > 0$. Then, for almost all $x \in \mathbb{R}$,

$$\begin{aligned} u'(x+r) - u'(x) &= Au(x+r) - Au(x) + \int_{(x,x+r]} u(t)\, d\mu(t) \\ &= \int_x^{x+r} (Au)'(y)\, dy + \int_{(x,x+r]} u(t)\, d\mu(t). \end{aligned}$$

Hence,

$$\begin{aligned} |u'(x+r) - u'(x)| &\leq |E| \int_x^{x+r} |u(y)|\, dy + \int_{(x,x+r]} |u(t)|\, d|\mu|(t) \\ &\leq |E| r \|u\|_{\infty,[x,x+r]} + \|u\|_{\infty,[x,x+r]} |\mu|([x,x+r]) \\ &\leq \|u\|_{\infty,[x,x+r]} \left(|E|r + (r+1)\|\mu\|_{\rm loc}\right). \end{aligned}$$

By Sobolev's inequality, there is $C \in \mathbb{R}$ (depending on r, but r is fixed anyway) such that

$$\|u\|_{\infty,[x,x+r]} \leq C\|u\|_{W_2^1(x,x+r)} \to 0 \quad (|x| \to \infty).$$

Thus,

$$|u'(x+r) - u'(x)| \to 0 \quad (|x| \to \infty).$$

An application of Lemma 2.9 with $v := u'$ yields $u'(x) \to 0$ as $|x| \to \infty$. □

Now, we can state the main result of this paper.

Theorem 2.11. *Let μ be a Gordon measure. Then H has no eigenvalues.*

Proof. Let (μ^m) be the periodic approximants of μ. Let $E \in \mathbb{R}$ and u be a solution of $Hu = Eu$. Let (u_m) be the solutions for the measures (μ^m). By Lemma 2.6 we find $m_0 \in \mathbb{N}$ such that

$$\Big\| \begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u_m'(x) \end{pmatrix} \Big\| \leq \frac{1}{4}$$

for $m \geq m_0$ and almost all $x \in [-p_m, 2p_m]$. By Lemma 2.8 we have

$$\limsup_{|x|\to\infty} \left(|u(x)|^2 + |u'(x)|^2\right) \geq \frac{1}{4} > 0.$$

Hence, u cannot be in $D(H)$ by Lemma 2.10. □

3. Examples

Remark 3.1 (periodic measures)**.** Every locally bounded periodic measure on $\mathbb{R}$ is a Gordon measure. Thus, for $\mu := \sum_{n\in\mathbb{Z}} \delta_{n+\frac{1}{2}}$ the operator $H := -\Delta + \mu$ has no eigenvalues.

Some examples of quasi-periodic $L_{1,\mathrm{loc}}$-potentials can be found in [2].

For a measure μ and $x \in \mathbb{R}$ let $T_x\mu := \mu(\cdot - x)$. If μ is periodic with period p, then $T_p\mu = \mu$.

Example 3.2. Let $\alpha \in (0,1) \setminus \mathbb{Q}$. There is a unique continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots}}}}$$

with $a_n \in \mathbb{N}$. For $m \in \mathbb{N}$ we set $\alpha_m = \frac{p_m}{q_m}$, where

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_m = a_m p_{m-1} + p_{m-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_m = a_m p_{m-1} + q_{m-2}. \end{aligned}$$

The number α is called *Liouville number*, if there is $B \geq 0$ such that

$$|\alpha - \alpha_m| \leq Bm^{-q_m}.$$

The set of Liouville numbers is a dense G_δ.

Let $\nu, \tilde{\nu}$ be 1-periodic measures and assume that there is $\gamma > 0$ such that

$$|\nu(\cdot - x) - \nu|([0,1]) \leq |x|^\gamma \quad (x \in \mathbb{R}).$$

Define $\mu := \tilde{\nu} + \nu\circ\alpha$ and $\mu^m := \tilde{\nu} + \nu\circ\alpha_m$ for $m \in \mathbb{N}$. Then μ^m is q_m-periodic and

$$\begin{aligned} |\mu - \mu^m|([-q_m, 2q_m]) &= |\nu\circ\alpha - \nu\circ\alpha_m|([-q_m, 2q_m]) \\ &= \Big|\nu\circ\frac{\alpha}{\alpha_m} - \nu\Big|([-p_m, 2p_m]) \\ &\leq \sum_{n=-p_m}^{2p_m-1} \Big|\nu\circ\frac{\alpha}{\alpha_m} - \nu\Big|([n, n+1]). \end{aligned}$$

Now, we have

$$\begin{aligned}\big|\nu\circ\frac{\alpha}{\alpha_m}-\nu\big|([n,n+1]) &= T_{-n}\big|\nu\circ\frac{\alpha}{\alpha_m}-\nu\big|([0,1])\\ &= \big|T_{-n}(\nu\circ\frac{\alpha}{\alpha_m})-T_{-n}\nu\big|([0,1])\\ &= \big|T_{-n}(\nu\circ\frac{\alpha}{\alpha_m})-\nu\big|([0,1]).\end{aligned}$$

With $g_{m,n}(y) := y + (\frac{\alpha}{\alpha_m} - 1)(y+n)$ and using periodicity of ν we obtain

$$T_{-n}(\nu\circ\frac{\alpha}{\alpha_m}) = \nu\circ g_{m,n}.$$

Hence,

$$|\nu\circ\frac{\alpha}{\alpha_m}-\nu|([n,n+1]) = |\nu\circ g_{m,n}-\nu|([0,1]).$$

For $y\in[0,1]$ and $n\in\{-p_m,\dots,2p_m-1\}$ we have

$$|\big(\frac{\alpha}{\alpha_m}-1\big)(y+n)| \le |\frac{\alpha}{\alpha_m}-1| \le 2q_m B m^{-q_m}.$$

Thus,

$$|\nu\circ g_{m,n}-\nu|([0,1]) \le \big(2q_m B m^{-q_m}\big)^{\gamma} = (2q_mB)^{\gamma} m^{-q_m\gamma}.$$

We conclude that

$$|\mu-\mu^m|([-q_m,2q_m]) \le 3p_m(2q_mB)^{\gamma}m^{-q_m\gamma}$$

and therefore for arbitrary $C\ge 0$

$$e^{Cq_m}|\mu-\mu^m|([-q_m,2q_m]) \to 0 \quad (m\to\infty).$$

Hence μ is a Gordon potential and $H := -\Delta+\mu$ does not have any eigenvalues.

Acknowledgements. We want to thank Peter Stollmann for advices and hints and especially for providing a proof of Lemma 2.9. Many thanks to the anonymous referee for the useful comments.

References

[1] A. Ben Amor and C. Remling; *Direct and Inverse Spectral Theory of One-dimensional Schrödinger Operators with Measures.* Integr. Equ. Oper. Theory 52 (2005) 395–417.
[2] D. Damanik and G. Stolz; *A Generalization of Gordon's Theorem and Applications to quasiperiodic Schrödinger Operators.* El. J. Diff. Equ. 55 (2000) 1–8.
[3] S. N. Ethier and T. G. Kurtz; *Markov processes, characterization and convergence.* Wiley Series in Probability and Statistics, 2005.
[4] A. Gordon; *On the point spectrum of the one-dimensional Schrödinger operator.* Usp. Math. Nauk 31 (1976) 257–258.
[5] S. Klassert, D. Lenz and P. Stollmann; *Delone measures of finite local complexity and applications to spectral theory of one-dimensional continuum models of quasicrystals.* DCDS-A 29(4) (2011) 1553–1571.
[6] B. Simon; *Schrödinger Semigroups.* Bull. Amer. Math. Soc. 7(3) (1982) 447–526.
[7] P. Stollmann and J. Voigt; *Perturbation of Dirichlet forms by measures.* Pot. An. 5 (1996) 109–138.

Christian Seifert
Fakultät Mathematik, Technische Universität Chemnitz, 09107 Chemnitz, Germany
E-mail address: christian.seifert@mathematik.tu-chemnitz.de