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NONLINEAR QUARTER-PLANE PROBLEM FOR THE KORTEWEG-DE VRIES EQUATION

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Abstract. This article concerns an initial-boundary value problem in a quarter-plane for the Korteweg-de Vries (KdV) equation. For general nonlinear boundary conditions we prove the existence and uniqueness of a global regular solution.

1. Introduction

This work concerns the existence and uniqueness of global solutions for the KdV equation posed on the first quarter-plane with a general nonlinear boundary condition. Such initial-boundary value problems may serve as models for waves generated by wavemakers in a channel, or for shallow water waves of the shore, [1, 2, 3]. There is a number of papers where initial value problems and initial-boundary value problems in a quarter-plane and in a bounded domain for dispersive equations were studied (see [2, 3, 4, 6, 7, 18, 10, 11, 12, 16, 19]). As a rule, simple boundary conditions at $x = 0$ such as $u = 0$ for the KdV equation or $u = u_x = 0$ for the Kawahara equation were imposed. On the other hand, general initial-boundary value problems for odd-order evolution equations attracted little attention. We must mention a classical paper of Volevich and Gindikin [14], where general mixed problems for linear $(2b + 1)$-hyperbolic equations were studied by means of functional analysis methods. It is difficult to apply their method directly to nonlinear dispersive equations due to complexity of this theory.

In [4, 5], Bubnov considered general mixed problems for the KdV equation posed on a bounded interval and proved solvability results. In [18] also were considered general mixed problems with linear boundary conditions for the KdV equation on a bounded interval and for small initial data the existence and uniqueness of global solutions as well as the exponential decay of L^2-norms of solutions while $t \to +\infty$ were proved.

Here we study a mixed problem for the KdV equation in a quarter plane with a general nonlinear nonhomogeneous condition:

$$x = 0, \quad \partial_x^2 u(0,t) = \varphi(t, u(0,t), \partial_x u(0,t))$$

and prove the existence and uniqueness of global in t solutions as well as smoothing effect for the initial data. The presence of a dissipative nonlinear term in the

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boundary condition guarantees the existence of global solutions without smallness conditions for the initial data, whereas posing a general linear boundary condition we did not succeed to prove a global existence result because general linear conditions did not imply the first estimate for the KdV equation which is crucial for global solvability of a corresponding mixed problem as was pointed out in [3].

To prove our results, we use a linearization technique, semi-discretization in t to solve the linear problem, the Banach fixed point theorem for local in t existence and uniqueness results and, finally, a priori estimates, independent of t, for the nonlinear problem. To prove solvability of a linearized problem with a nonhomogeneous boundary condition, we exploit the method of semi-discretization which is transparent and proved its universality, see [6, 7, 16, 17], instead of very popular in the theory of the KdV equation with homogeneous boundary conditions semigroups technique, because it is difficult to adapt this technique for mixed problems with nonhomogeneous and nonlinear boundary conditions.

This article has the following structure: Section 1 is Introduction. In Section 2 we formulate the principal problem and some useful known facts. In Section 3 the related stationary problem is studied. Section 4 is devoted to regular solutions to the linear evolution problem which are obtained by the method of semi-discretization with respect to t. In Section 5, using the contraction mapping arguments, we obtain a local in time regular solution to the nonlinear problem. Finally, in Section 6, necessary a priori estimates are proved which allow us to extend the local solution to the whole interval $t \in (0,T)$ with arbitrary finite $T > 0$.

2. Formulation of the problem and main results

Denote $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and for a positive number T, $\mathbb{Q}_T = \{(x,t) \in \mathbb{R}^2 : x \in \mathbb{R}^+,\ t \in (0,T)\}$. In $\mathbb{Q}_T$ we consider the KdV equation

$$u_t + Du + D^3u + uDu = 0 \tag{2.1}$$

subject to initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^+, \tag{2.2}$$

$$D^2u(0,t) + \alpha Du(0,t) + \beta u(0,t) + |u(0,t)|u(0,t) + g(t) = 0, \quad t \in (0,T); \tag{2.3}$$

where $g(t)$ is a given function, α and β are real coefficients such that

$$2\beta - |\alpha| - 1 = 2a_1 > 0, \quad 1 - |\alpha| = a_2 > 0. \tag{2.4}$$

Remark 2.1. From the technical reasons, we chose the simple nonlinearity in (2.3). Of course, more general dissipative functions may be used.

Boundary conditions (2.3) follow from more general conditions

$$\gamma D^2u(0,t) + \alpha Du(0,t) + \beta u(0,t) + |u(0,t)|u(0,t) + g(t) = 0, \quad t \in (0,T) \tag{2.5}$$

when $\gamma \neq 0$. Explicitly, the simple boundary condition $u(0,t) = 0$ does not follow from (2.3), but it is a singular case of (2.5): when $\gamma = 0$, in order to get the first $L^2(\mathbb{R}^+)$ estimate, which is crucial for solvability of (2.1)-(2.3), see [3], we must put $\alpha = 0$ and $u(0,t) = 0$ that gives exactly the simple boundary condition.

Here $u : \mathbb{R}^+ \times (0,T) \to \mathbb{R}$, or $u : (0,L) \times (0,T) \to \mathbb{R}$ $D^j = \partial^j/\partial x^j$; $D = D^1$. In this article, we adopt the usual notation

$$(u,v)(t) = \int_0^{+\infty} u(x,t)v(x,t)dx, \quad \|u\|^2(t) = (u,u)(t),$$

$$(u,v)=\int_0^{+\infty} u(x)v(x)dx, \quad \|u\|^2=(u,u),$$
$$(u,v)_L=\int_0^{L} u(x)v(x)dx, \quad \|u\|_L^2=(u,u)_L.$$

Symbols C, C_0, C_i, for $i\in\mathbb{N}$, mean positive constants appearing during the text. The main result of this article is the following theorem.

Theorem 2.2. *Let $u_0\in H^3(\mathbb{R}^+)$, $g\in H^1(0,T)$, α and β satisfy (2.4) and for a real $k=\min\{a_2/2,\ (-1+\sqrt{1+2a_1})/2\}$ the following inequality holds:*

$$\Big(e^{kx},\Big[\sum_{i=0}^{3}|D^iu_0|^2+|u_0Du_0|^2\Big]\Big)<\infty.$$

Then for all finite $T>0$, problem (2.1)-(2.3) has a unique regular solution:

$$u\in L^\infty(0,T;H^3(\mathbb{R}^+))\cap L^2(0,T;H^4(\mathbb{R}^+)),$$
$$u_t\in L^\infty(0,T;L^2(\mathbb{R}^+))\cap L^2(0,T;H^1(\mathbb{R}^+))$$

and the following estimate holds:

$$\begin{aligned}
&\sup_{t\in(0,T)}\big\{\Big(e^{kx},\sum_{i=0}^{3}|D^iu|^2\Big)(t)+(e^{kx},u_t^2)(t)\big\}\\
&+\int_0^T\big[(e^{kx},\sum_{i=0}^{4}|D^iu|^2)(t)+(e^{kx},\sum_{i=0}^{1}|D^iu_t|^2)(t)\big]\,dt\\
&+\int_0^T(u^2(0,t)+|Du(0,t)|^2)\,dt+\int_0^T(u_t^2(0,t)+|Du_t(0,t)|^2)\,dt\\
&\le C(T,k)\big[\sum_{i=0}^{3}(e^{kx},|D^iu_0|^2)+(e^{kx},|u_0Du_0|^2)+\int_0^T(g^2+g_t^2)(t)\,dt\big].
\end{aligned}$$

3. Stationary problem

Our purpose in this section is to solve the stationary boundary-value problem

$$D^3u(x)+du(x)=f(x), \quad x\in\mathbb{R}^+, \tag{3.1}$$
$$D^2u(0)+\alpha Du(0)+\beta u(0)+q_1=0, \tag{3.2}$$

where $d>0$ and q_1 are real constants, α and β satisfy (2.4) and f is such that

$$e^{kx/2}f\in L^2(\mathbb{R}^+), \quad k>0. \tag{3.3}$$

Theorem 3.1. *Let $d>2k^3$ and f satisfy (3.3). Then (3.1)-(3.2) admits a unique solution $u\in H^3(\mathbb{R}^+)$ such that*

$$\sum_{i=0}^{3}(e^{kx},|D^iu|^2)\le C[(e^{kx},f^2)+q_1^2].$$

Proof. Consider on an interval $(0,L)$ the problem

$$D^3u(x)+du(x)=f(x), \quad x\in(0,L), \tag{3.4}$$
$$D^2u(0)+\alpha Du(0)+\beta u(0)+q_1=0, \tag{3.5}$$
$$u(L)=Du(L)=0, \tag{3.6}$$

where L is an arbitrary finite positive number, $f(x)$ is a restriction on $(0, L)$ of $f(x) : e^{kx/2} f \in L^2(\mathbb{R}^+) \cap C(\mathbb{R}^+)$. □

It is known (see [8]) that (3.4)-(3.6) has a unique classical solution if the corresponding homogeneous problem has only trivial solution.

Proposition 3.2. *Let $f(x) \equiv 0$, $q_1 = 0$ and α and β satisfy* (2.4)*. Then* (3.4)-(3.6) *has only the trivial solution.*

Proof. Multiplying (3.4) by $2u$ and using (3.5), (3.6), we come to the inequality

$$2d\|u\|_L^2 + (2\beta - |\alpha|)u^2(0) + (1 - |\alpha|)|Du(0)|^2 \leq 0.$$

Taking into account (2.4), we obtain $\|u\|_L^2 = 0$ which completes the proof. □

Corollary 3.3. *For all finite $L > 0$ there exists a unique classical solution of* (3.4)-(3.6).

To prove Theorem 3.1, we must extend an interval $(0, L)$ to $\mathbb{R}^+$. To do this, we need a priori estimates of solutions to the problem (3.4)-(3.6) independent of $L > 0$. These estimates provides the following result.

Lemma 3.4. *Let $d > 2k^3$ and $f(x) : e^{kx/2} f \in L^2(\mathbb{R}^+) \cap C(\mathbb{R}^+)$. Then for all finite $L > 0$ solutions of* (3.4)-(3.6) *satisfy the inequality*

$$\sum_{i=0}^{3} (e^{kx}, ||D^i u||^2)_L \leq C_R[(e^{kx}, f^2) + q_1^2],$$

where the constant C_R does not depend on L.

Proof. Multiplying (3.4) by u and integrating over $(0, L)$, we obtain

$$(D^3 u, u)_L + d\|u\|_L^2 = (f, u)_L, \tag{3.7}$$

and

$$\begin{aligned} I_1 = (D^3 u, u)_L &\geq \frac{1}{2}(1 - |\alpha|)|Du(0)|^2 + (\beta - \frac{|\alpha|}{2} - \frac{1}{2})u^2(0) - \frac{q_1^2}{2} \\ &\geq C_4(|Du(0)|^2 + u^2(0)) - \frac{q_1^2}{2}, \end{aligned}$$

where $C_4 = \min\{\frac{1-|\alpha|}{2}, \beta - \frac{|\alpha|}{2} - \frac{1}{2}\}$. Since $(D^3 u, u)_L + \frac{q_1^2}{2} \geq 0$, then

$$d\|u\|_L^2 \leq (D^3 u, u)_L + \frac{q_1^2}{2} + d\|u\|_L^2 \leq \frac{1}{2d}\|f\|_L^2 + \frac{d}{2}\|u\|_L^2 + \frac{q_1^2}{2}$$

and

$$\|u\|_L^2 \leq C(d)\big(\|f\|^2 + q_1^2\big). \tag{3.8}$$

Returning to (3.7), we obtain

$$\frac{d}{2}\|u\|_L^2 + C_4(|Du(0)|^2 + u^2(0)) \leq C(d)\|f\|^2 + \frac{q_1^2}{2}$$

which implies

$$|Du(0)|^2 + u^2(0) \leq C(d)(\|f\|^2 + q_1^2). \tag{3.9}$$

Multiplying (3.4) by $e^{kx}u$ and integrating over $(0, L)$, we obtain

$$d(e^{kx}, u^2)_L + (D^3 u, e^{kx} u)_L = (f, e^{kx} u)_L, \tag{3.10}$$

and

$$(e^{kx}D^3u,u)_L \geq K_1u^2(0)+K_2|Du(0)|^2+\frac{3k}{2}(e^{kx},|Du|^2)_L-\frac{k^3}{2}(e^{kx},u^2)_L-\frac{1}{2}q_1^2,$$

where

$$K_1=\beta-\frac{1}{2}-\frac{k^2}{2}-\frac{|\alpha+k|}{2},\quad K_2=\frac{1}{2}(1-|\alpha+k|).$$

With this, (3.10) becomes

$$\begin{aligned}&\frac{d}{2}(e^{kx},u^2)_L-\frac{k^3}{2}(e^{kx},u^2)_L+\frac{3k}{2}(e^{kx},|Du|^2)_L\\&\leq C_5\big(|Du(0)|^2+u^2(0)\big)+C(d)(e^{kx},f^2)+\frac{1}{2}q_1^2,\end{aligned}$$

where $C_5=\max\{|K_1|,|K_2|,1\}$. Using (3.9), we have

$$(e^{kx},u^2)_L+(e^{kx},|Du|^2)_L\leq C(d,k)\left((e^{kx},f^2)+q_1^2\right). \tag{3.11}$$

Now, multiplying (3.4) by $e^{kx}D^3u$ and integrating over $(0,L)$, we obtain

$$(e^{kx},|D^3u|^2)_L\leq C(d,k)\left((e^{kx},f^2)+q_1^2\right) \tag{3.12}$$

and multiplying (3.4) by $-e^{kx}Du$, we obtain

$$\begin{aligned}&D^2u(0)Du(0)+(e^{kx},|D^2u|^2)_L+k(e^{kx}D^2u,Du)_L-d(e^{kx}Du,u)_L\\&=-(e^{kx},fDu)_L.\end{aligned}$$

Taking into account (3.2), (3.9), (3.11), we find that

$$(e^{kx},|D^2u|^2)_L\leq C(d,k)\left((e^{kx},f^2)+q_1^2\right).$$

Adding to this inequality (3.8), (3.12), we complete the proof. □

Since estimates of this Lemma do not depend on L, it allows us to extend an interval $(0,L)$ to $\mathbb{R}^+$ and by compactness arguments we can eliminate condition $f\in C(\mathbb{R}^+)$. Uniqueness of a solution follows from (3.8). This completes the proof of Theorem 3.1.

4. Linear evolution problem

Consider the linear initial-boundary value problem

$$u_t+D^3u=f(x,t),\quad (x,t)\in\mathbb{Q}_T; \tag{4.1}$$

$$D^2u(0,t)+\alpha Du(0,t)+\beta u(0,t)+q(t)=0,\quad t\in(0,T); \tag{4.2}$$

$$u(x,0)=u_0(x),\quad x\in\mathbb{R}^+; \tag{4.3}$$

where

$$u_0\in H^3(\mathbb{R}^+),\quad f,f_t\in C(0,T;L^2(\mathbb{R}^+)),\quad q\in H^1(0,T); \tag{4.4}$$

$$\Big(e^{kx},\sum_{i=0}^{3}|D^iu_0|^2\Big)+\int_0^T[(e^{kx},f^2)(t)+(e^{kx},f_t^2)(t)]dt+\int_0^T(q^2(t)+q_t^2(t))dt<\infty. \tag{4.5}$$

Henceforth we will use the following Lemma (see [9]).

Lemma 4.1 (Discrete Gronwall Lemma). *Let k_n be a sequence of non-negative real numbers. Consider a sequence $\phi_n \geq 0$ such that*

$$\phi_0 \leq g_0, \quad \phi_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s \phi_s, \quad n \geq 1$$

with $g_0 \geq 0$ and $p_s \geq 0$. Then for all $n \geq 1$ it holds

$$\phi_n \leq \Big(g_0 + \sum_{s=0}^{n-1} p_s\Big) \exp\big\{\sum_{s=0}^{n-1} k_s\big\}.$$

To study (4.1)-(4.3), we use the method of semi-discretization with respect to t, [6, 15]. Define

$$\begin{gathered}
h = \frac{T}{N} > 0, \quad N \in \mathbb{N}, \\
u^n(x) = u(x, nh), \quad q^n = q(nh), \quad f^n(x) = f(x, nh), \quad n = 1, \dots, N; \\
u^0(x) = u(x, 0) = u_0(x); \\
u_h^n(x) = \frac{u^n(x) - u^{n-1}(x)}{h}, \quad q_h^n = \frac{q^n - q^{n-1}}{h}, \quad n = 1, \dots, N; \\
u_h^0 \equiv u_t(x, 0) = f(x, 0) - D^3 u(x, 0).
\end{gathered}$$

We approximate (4.1)-(4.3) with the system

$$Lu^n \equiv \frac{u^n}{h} + D^3 u^n = \frac{u^{n-1}}{h} + f^{n-1}, \quad x \in \mathbb{R}^+; \tag{4.6}$$

$$D^2 u^n(0) + \alpha D u^n(0) + \beta u^n(0) + q^{n-1} = 0, \quad n = 1, \dots, N; \tag{4.7}$$

$$u^0(x) = u_0(x) \in H^3(\mathbb{R}^+), \quad x \in \mathbb{R}^+. \tag{4.8}$$

By Theorem 3.1, given f^{n-1}, q^{n-1} and u^{n-1} satisfying

$$(e^{kx}, |f^{n-1}(x)|^2 + |u^{n-1}(x)|^2) + |q^{n-1}|^2 \leq C,$$

there exists a unique solution $u^n(x) \in H^3(\mathbb{R}^+)$ of (4.6)-(4.8) such that

$$(e^{kx}, |u^n(x)|^2) \leq C. \tag{4.9}$$

Proposition 4.2. *Let u_0 and $f(x,t)$ be such that for all $t \in (0,T)$*

$$(e^{kx}, |u_0(x)|^2 + |f(x,t)|^2) \leq C.$$

Then for all $n = 1, \dots, N$ and $N > 2k^3 T$ problem (4.6)-(4.8) admits a unique solution $u^n \in H^3(\mathbb{R}^+)$ such that

$$(e^{kx}, |u^n(x)|^2) \leq C.$$

Proof. For $n = 1$ we have $f^0(x) = f(x, 0)$, $u^0(x) = u_0(x)$, $q^0 = q(0)$ and (4.6)-(4.8) becomes

$$\begin{gathered}
\frac{u^1(x)}{h} + D^3 u^1(x) = f(x, 0) + \frac{u^0(x)}{h} \equiv F^1(x), \\
D^2 u^1(0) + \alpha D u^1(0) + \beta u^1(0) + q^0 = 0.
\end{gathered}$$

Due to (4.4)-(4.5),

$$(e^{kx}, |F^1(x)|^2) \leq C.$$

Taking $1/h > 2k^3$, by Theorem 3.1, there exists a unique solution $u^1 \in H^3(\mathbb{R}^+)$ of the above problem satisfying $(e^{kx}, |u^1(x)|^2) \le C$. Repeating this procedure, the result follows. □

To prove solvability of (4.1)-(4.3), it is sufficient to pass to the limit in (4.6)-(4.8) as $h \to 0$. For this purpose we need the following lemma.

Lemma 4.3. *Assume condition* (4.5)*. Then for all $h > 0$ sufficiently small and $l = 1, \dots, N$ the solutions $u^n(x)$ of* (4.6)-(4.8) *satisfy*

$$\begin{aligned} &\sup_{1\le l\le N} \big\{(e^{kx}, |u^l|^2) + (e^{kx}, |u^l_h|^2)\big\} + \sum_{n=1}^{l} \big\{(e^{kx}, |Du^n|^2)h + (e^{kx}, |Du^n_h|^2)h\big\} \\ &\le C\big\{\sum_{i=0}^{3}(e^{kx}, |D^i u_0|^2) + \int_0^T (e^{kx}, f^2 + f_t^2)(t)\,dt + \int_0^T (q^2(t) + q_t^2(t))\,dt\big\}, \end{aligned} \tag{4.10}$$

where the constant $C > 0$ does not depend on $h > 0$.

Proof. First we prove a priori estimates independent of $h > 0$ for u^n and u^n_h.

Estimate I. Taking $1/h > 2k^3$, multiplying (4.6) by $e^{kx}u^n$ and integrating over $\mathbb{R}^+$, we obtain

$$\frac{1}{h}(u^n - u^{n-1}, e^{kx}u^n) + (D^3u^n, e^{kx}u^n) = (f^{n-1}, e^{kx}u^n). \tag{4.11}$$

We estimate

$$I_1 = \frac{1}{h}(u^n - u^{n-1}, e^{kx}u^n) \ge \frac{(e^{kx}, |u^n|^2)}{2h} - \frac{(e^{kx}, |u^{n-1}|^2)}{2h};$$

$$\begin{aligned} I_2 &= (D^3u^n, e^{kx}u^n) \\ &\ge K_1|u^n(0)|^2 + K_2|Du^n(0)|^2 + \frac{3k}{2}(e^{kx}, |Du^n|^2) - \frac{k^3}{2}(e^{kx}, |u^n|^2) - \frac{1}{2}|q^{n-1}|^2, \end{aligned}$$

where $K_1 = \beta - \frac{1}{2} - \frac{k^2}{2} - \frac{|\alpha+k|}{2}$ and $K_2 = \frac{1}{2}(1 - |\alpha + k|)$.

$$I_3 = (f^{n-1}, e^{kx}u^n) \le \frac{1}{2}(e^{kx}, |u^n|^2) + \frac{1}{2}(e^{kx}, |f^{n-1}|^2).$$

Substituting I_1, I_2, I_3 in (4.11), multiplying the result by $2h$ and summing from $n = 1$ to $n = l \le N$, we obtain

$$\begin{aligned} &3k\sum_{n=1}^{l}(e^{kx}, |Du^n|^2)h + (e^{kx}, |u^l|^2) - (e^{kx}, u_0^2) \\ &\le C_5\big[\sum_{n=1}^{l}\big(|Du^n(0)|^2 + |u^n(0)|^2\big)h\big] + (k^3 + 1)\sum_{n=1}^{l}(e^{kx}, |u^n|^2)h \\ &\quad + \sum_{n=1}^{l}(e^{kx}, |f^{n-1}|^2)h + \sum_{n=1}^{l}|q^{n-1}|^2h. \end{aligned} \tag{4.12}$$

Making $k = 0$ in (4.11), using $I_1 - I_3$, multiplying the result by $2h$ and summing from $n = 1$ till $n = l \leq N$, we obtain

$$\begin{aligned} &2hC_4 \sum_{n=1}^{l} \big(|Du^n(0)|^2 + |u^n(0)|^2\big) + \|u^l\|^2 - \|u_0\|^2 \\ &\leq h \sum_{n=1}^{l} \|u^n\|^2 + h \sum_{n=0}^{l-1} \|f^n\|^2 + h \sum_{n=0}^{l-1} |q^n|^2, \end{aligned} \tag{4.13}$$

where

$$C_4 = \min\{\frac{1-|\alpha|}{2},\ \beta - \frac{|\alpha|}{2} - \frac{1}{2}\}.$$

Considering $0 < h < 1/2$, we have

$$\|u^l\|^2 \leq 2\|u_0\|^2 + 2h \sum_{n=0}^{l-1} \|u^n\|^2 + 2h \sum_{n=0}^{l-1} \|f^n\|^2 + 2h \sum_{n=0}^{l-1} |q^n|^2.$$

Taking into account (4.4),

$$2h \sum_{n=0}^{N} \|f^n\|^2 = 2 \sum_{n=0}^{N} h \int_0^\infty |f^n(x)|^2\, dx \leq M \int_0^T \int_0^\infty f^2(x,t)\, dx\, dt$$

and

$$2h \sum_{n=0}^{l-1} |q^n|^2 \leq M_0 \int_0^T q^2(t)\, dt,$$

where the constants M and M_0 do not depend on h.
Using the Discrete Gronwall Lemma, we find

$$\begin{aligned} \|u^l\|^2 &\leq \Big(2\|u_0\|^2 + 2h \sum_{n=0}^{l-1} \|f^n\|^2 + 2h \sum_{n=0}^{l-1} |q^n|^2\Big) \exp(2T) \\ &\leq \Big(2\|u_0\|^2 + M \int_0^T \int_0^\infty f^2(x,t)\, dx\, dt + M_0 \int_0^T q^2(t)\, dt\Big) \exp(2T) \\ &\leq M_1\Big(\|u_0\|^2 + \int_0^T \int_0^\infty f^2(x,t)\, dx\, dt + \int_0^T q^2(t)\, dt\Big), \end{aligned}$$

with $M_1 = \max\{2e^{2T}, Me^{2T}, M_0e^{2T}\}$. Now, from (4.13),

$$\begin{aligned} &\sum_{n=1}^{l} \Big(|Du^n(0)|^2 + |u^n(0)|^2\Big)h \\ &\leq M_2\Big[\|u_0\|^2 + \int_0^T \int_0^\infty f^2(x,t)\, dx\, dt + \int_0^T q^2(t)\, dt\Big] \end{aligned}$$

and $M_2 = (1 + M_1T + M + M_0)/(2C_4)$. Considering $0 < h < \frac{1}{2}$ and using (4.12),

$$\begin{aligned} &3k \sum_{n=1}^{l} (e^{kx}, |Du^n|^2)h + (e^{kx}, |u^l|^2) \\ &\leq (e^{kx}, u_0^2) + \ C(T,k)\Big[\|u_0\|^2 + \int_0^T \int_0^\infty f^2(x,t)\, dx\, dt + \int_0^T q^2(t)\, dt\Big] \end{aligned}$$

$$+ (k^3+1)\sum_{n=1}^{l}(e^{kx},|u^n|^2)h + \sum_{n=0}^{l-1}(e^{kx},|f^n|^2)h + h\sum_{n=0}^{l-1}|q^n|^2.$$

Taking h such that $0 < (1+k^3)h < 1/2$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{l}(e^{kx},|Du^n|^2)h + (e^{kx},|u^l|^2)\\
&\le C(T,k)\Big[(e^{kx},u_0^2) + \int_0^T\int_0^\infty e^{kx}f^2(x,t)\,dx\,dt + \int_0^T q^2(t)\,dt\Big]\\
&\quad + 2(k^3+1)\sum_{n=0}^{l-1}(e^{kx},|u^n|^2)h + 2h\sum_{n=0}^{l-1}|q^n|^2 \quad \forall l \le N.
\end{aligned}$$

Applying Discrete Gronwall Lemma, we obtain

$$(e^{kx},|u^l|^2) \le C(k,T)\Big[(e^{kx},u_0^2) + \int_0^T\int_0^\infty e^{kx}f^2(x,t)\,dx\,dt + \int_0^T q^2(t)\,dt\Big]. \quad (4.14)$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{l}(e^{kx},|Du^n|^2)h\\
&\le C(T,k)\Big[(e^{kx},u_0^2) + \int_0^T\int_0^\infty e^{kx}f^2(x,t)\,dx\,dt + \int_0^T q^2(t)\,dt\Big].
\end{aligned} \quad (4.15)$$

Estimate II. Writing (4.6) as $(Lu^n - Lu^{n-1})/h$, we have

$$\begin{gathered}
L_h u_h^n = \frac{u_h^n - u_h^{n-1}}{h} + D^3u_h^n = f_h^{n-1}, \quad x \in \mathbb{R}^+; \quad (4.16)\\
D^2u_h^n(0) + \alpha Du_h^n(0) + \beta u_h^n(0) + q_h^{n-1} = 0, \quad n = 1,\dots,N;\\
u_h^0 \equiv u_t(x,0) = f(x,0) - D^3u_0(x);\\
f_h^0(x) \equiv f_t(x,0).
\end{gathered}$$

Multiplying (4.16) by $e^{kx}u_h^n$, integrating over $\mathbb{R}^+$ and acting as by proving the estimate I, we obtain

$$\begin{aligned}
&(e^{kx},|u_h^l|^2) + \sum_{n=1}^{l}(e^{kx},|Du_h^n|^2)h\\
&\le C(T,k)\Big[(e^{kx},u_t^2(x,0)) + \int_0^T(e^{kx},f_t^2)(t)\,dt + \int_0^T q_t^2(t)\,dt\Big]\\
&\le C(T,k)\Big[\sum_{i=0}^{3}(e^{kx},|D^iu_0|^2) + \int_0^T(e^{kx},f_t^2+f^2)(t)\,dt + \int_0^T q_t^2(t)\,dt\Big].
\end{aligned} \quad (4.17)$$

This and (4.14), (4.15) completes the proof. □

Theorem 4.4. *Let $u_0(x)$, $q(t)$ and $f(x,t)$ satisfy* (4.4), (4.5)*. Then there exists a unique solution of* (4.1)-(4.3)*, such that*

$$u \in L^\infty(0,T;H^3(\mathbb{R}^+)), \quad u_t \in L^\infty(0,T;L^2(\mathbb{R}^+)) \cap L^2(0,T;H^1(\mathbb{R}^+)).$$

Proof. Rewriting (4.6)-(4.8) as

$$D^3u^n(x)+2k^3u^n(x)=2k^3u^n(x)-u^n_h(x)+f^{n-1}(x)\equiv F(x),\quad x\in\mathbb{R}^+;$$
$$D^2u^n(0)+\alpha Du^n(0)+\beta u^n(0)+q^{n-1}=0,\quad n=1,\dots,N;$$
$$u^0(x)=u_0(x),\quad x\in\mathbb{R}^+;$$

and taking into account Theorem 3.1, we find a solution $u^n\in H^3(\mathbb{R}^+)$ such that

$$\sum_{i=0}^{3}(e^{kx},|D^iu^n|^2)\le C\left((e^{kx},F^2)+|q^{n-1}|^2\right).$$

Hence

$$\begin{aligned}\|u^n\|^2_{H^3(\mathbb{R}^+)}&\le\sum_{i=0}^{3}(e^{kx},|D^iu^n|^2)\\&\le C\left((e^{kx},|u^n_h|^2+|u^n|^2+|f^{n-1}|^2)+|q^{n-1}|^2\right)\\&\le C\Big\{\sum_{i=0}^{3}(e^{kx},|D^iu_0|^2)+\int_0^T[(e^{kx},f^2+f_t^2)(t)+q^2(t)+q_t^2(t)]\,dt\Big\},\end{aligned}$$

where the constant C for $h>0$ sufficiently small does not depend on h. Because the estimates (4.14), (4.15), (4.17) and the inequality above are uniform in $h>0$, the standard arguments (see [15]) imply that there exists a function $u(x,t)$ such that

$$\overline{u^n}\to u\quad\text{weakly-* in } L^\infty(0,T;H^3(\mathbb{R}^+)),$$
$$\overline{u^n_h}\to u_t\quad\text{weakly-* in } L^\infty(0,T;L^2(\mathbb{R}^+))\cap L^2(0,T;H^1(\mathbb{R}^+)).$$

Here $\overline{u^n}$ and $\overline{u^n_h}$ are interpolations of u^n and u^n_h, respectively, and $u(x,t)$ is a solution of (4.1)-(4.3). For more details, see [15]. □

5. Nonlinear problem. Local solutions

In this section we prove the existence of local regular solutions to the nonlinear problem

$$u_t+D^3u=-uDu-Du,\quad (x,t)\in\mathbb{Q}_T; \tag{5.1}$$
$$D^2u(0,t)+\alpha Du(0,t)+\beta u(0,t)+|u(0,t)|u(0,t)+g(t)=0,\quad t\in(0,T); \tag{5.2}$$
$$u(x,0)=u_0(x),\quad x\in\mathbb{R}^+; \tag{5.3}$$

where $g(t)$ is a given function, α and β satisfy (2.4). The main result here is as follows.

Theorem 5.1. *Let α and β satisfy* (2.4)*, $u_0(x)\in H^3(\mathbb{R}^+)$, $g\in H^1(0,T)$ and for some $k>0$*

$$\sum_{i=0}^{3}(e^{kx},|D^iu_0|^2)+(e^{kx},|u_0Du_0|^2)<\infty.$$

Then there exists a positive number T_0 such that (5.1)-(5.3) *possesses a unique regular solution in $\mathbb{Q}_{T_0}$ such that*

$$u\in L^\infty(0,T_0;H^3(\mathbb{R}^+)),\quad u_t\in L^\infty(0,T_0;L^2(\mathbb{R}^+))\cap L^2(0,T_0;H^1(\mathbb{R}^+))$$

and the following inequality holds:

$$\sup_{t\in(0,T_0)}\{(e^{kx},u^2)(t)+(e^{kx},u_t^2)(t)\}+\int_0^{T_0}[(e^{kx},|Du|^2)(t)+(e^{kx},|Du_t|^2)(t)]dt$$
$$+\int_0^{T_0}\big(|u(0,t)|^2+|u_t(0,t)|^2\big)dt$$
$$\le C(T_0,k)\Big[\sum_{i=0}^{3}(e^{kx},|D^iu_0|^2)+(e^{kx},|u_0Du_0|^2)+\int_0^{T_0}(g^2(t)+g_t^2(t))\,dt\Big].$$

Proof. We prove this theorem using the Banach fixed point theorem. Consider $X=L^\infty(0,T;H^3(\mathbb{R}^+))$, $Y=L^\infty(0,T;L^2(\mathbb{R}^+))\cap L^2(0,T;H^1(\mathbb{R}^+))$. Let V be the space

$$V=\{v:\mathbb{R}^+\times[0,T]\to\mathbb{R}: v\in X, v_t\in Y, v(x,0)=u_0(x)\}$$

with the norm

$$\begin{aligned}\|v\|_V^2=&\sup_{t\in(0,T)}\{(e^{kx},v^2)(t)+(e^{kx},v_t^2)(t)\}\\&+\int_0^T[(e^{kx},|Dv|^2)(t)+(e^{kx},|Dv_t|^2)(t)]dt\\&+\int_0^T\big(|v(0,t)|^2+|v_t(0,t)|^2\big)dt.\end{aligned}\tag{5.4}$$

Obviously, $(V,\|\cdot\|)$ is a Banach space. Define

$$B_R=\{v\in V:\|v\|_V\le R\sqrt{12C^*}\},\quad C^*=\max\{1+2\frac{C_5}{C_4},\ 1+3\frac{C_5\delta}{C_4},\ \delta\},$$

with $\delta=1/\min\{1,k,C_5\}$, and $R>1$ is such that

$$\sum_{i=0}^{3}(e^{kx},|D^iu_0|^2+|u_0Du_0|^2)+2\int_0^T(g^2(t)+g_t^2(t))\,dt\le R^2.\tag{5.5}$$

For any $v(x,t)\in B_R$ consider the linear problem

$$u_t+D^3u=-vDv-Dv,\quad (x,t)\in Q_T;\tag{5.6}$$
$$D^2u(0,t)+\alpha Du(0,t)+\beta u(0,t)+|v(0,t)|v(0,t)+g(t)=0,\quad t\in(0,T);\tag{5.7}$$
$$u(x,0)=u_0(x),\quad x\in\mathbb{R}^+;\tag{5.8}$$

where $g(t)\in H^1(0,T)$, α and β satisfy (2.4).

It is easy to verify that $|v(0,t)|v(0,t)\in H^1(0,T)$; $f(x,t)=-vDv-Dv$ satisfies conditions (4.4) and (4.5). Therefore, by Theorem 4.4, there exists a unique function $u(x,t): u\in L^\infty(0,T;H^3(\mathbb{R}^+))$, $u_t\in L^\infty(0,T;L^2(\mathbb{R}^+))\cap L^2(0,T;H^1(\mathbb{R}^+))$ which solves (5.6)-(5.8). Hence, one can define an operator P related to (5.6)-(5.8) such that $u=Pv$.

Lemma 5.2. *There is a real $T_0=T_0(R)>0$ such that an operator $P:u=Pv$ maps B_R into itself.*

Proof. To prove this lemma it suffices to show the necessary a priori estimates:

Estimate I. Multiplying (5.6) by $2u$, integrating over $\mathbb{R}^+$ and repeating the calculations from Lemma 3.4, we find

$$\begin{aligned} &\frac{d}{dt}\|u\|^2(t) + C_4\big(|u(0,t)|^2 + |Du(0,t)|^2\big) \\ &\le \|u\|^2(t) + 2\|vDv\|^2(t) + 2\|Dv\|^2(t) + C|v(0,t)|^2 v(0,t)^2 + 2g^2(t). \end{aligned} \tag{5.9}$$

We estimate

$$\begin{aligned} \int_{\mathbb{R}^+} e^{kx}|Dv(x,t)|^2\,dx &= \int_{\mathbb{R}^+} e^{kx}|Dv(x,0)|^2\,dx + \int_0^t \frac{\partial}{\partial\tau}(e^{kx},|Dv|^2)(\tau)\,d\tau \\ &\le (e^{kx},|Du_0|^2) + \int_0^t\int_{\mathbb{R}^+} e^{kx}(|Dv|^2 + |Dv_\tau|^2)\,dxd\tau \end{aligned}$$

and

$$\begin{aligned} v^2(0,t) &\le \sup_{x\in\mathbb{R}^+} v^2(x,t) \le 2\|v\|(t)\|Dv\|(t) \\ &\le 2\Big(\int_{\mathbb{R}^+} e^{kx}v^2(x,t)\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^+} e^{kx}|Dv(x,t)|^2\,dx\Big)^{1/2} \\ &\le 2\big(R\sqrt{12C^*}\big)(R^2 + 12R^2C^*)^{1/2}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|vDv\|^2(t) &\le \sup_{x\in\mathbb{R}^+} v^2(x,t)\int_{\mathbb{R}^+}|Dv(x,t)|^2\,dx \\ &\le 2\big(R\sqrt{12C^*}\big)(R^2 + 12R^2C^*)^{1/2}(R^2 + 12R^2C^*). \end{aligned}$$

Hence (5.9) may be rewritten as

$$\frac{d}{dt}\|u\|^2(t) + C_4\big(|u(0,t)|^2 + |Du(0,t)|^2\big) \le \|u\|^2(t) + C(R,C^*) + 2g^2(t). \tag{5.10}$$

Ignoring the second term on the left side of this inequality and applying Gronwall's lemma, we obtain

$$\|u\|^2(t) \le e^{T_0}\big(\|u_0\|^2 + C(R,C^*)T_0 + 2\int_0^t g^2(\tau)\,d\tau\big).$$

Taking $T_0 > 0$ such that $e^{T_0} \le 2$ e $C(R,C^*)T_0 \le R^2$, we have

$$\|u\|^2(t) \le 6R^2, \quad t \in (0,T_0).$$

Using this inequality and integrating (5.10) over $(0,t)$, we obtain

$$\begin{aligned} \int_0^t [|u(0,\tau)|^2 + |Du(0,\tau)|^2]d\tau &\le \frac{1}{C_4}\big[C(R,C^*)T_0 + \|u_0\|^2 + 2\int_0^t g^2(\tau)\,d\tau\big] \\ &\le \frac{1}{C_4}[C(R,C^*)T_0 + \|u_0\|^2 + R^2]. \end{aligned} \tag{5.11}$$

Multiplying (5.6) by $2e^{kx}u$ and integrating over $\mathbb{R}^+$, we find

$$\begin{aligned} &\frac{d}{dt}(e^{kx},u^2)(t) + 3k(e^{kx},|Du|^2)(t) \\ &\le 2C_5\big(|u(0,t)|^2 + |Du(0,t)|^2\big) + C_6(e^{kx},u^2)(t) \\ &\quad + 2(e^{kx},|vDv|^2 + |Dv|^2)(t) + C\big(|v(0,t)|^2 v(0,t)^2\big) + 2g^2(t) \\ &\le 2C_5\big(|u(0,t)|^2 + |Du(0,t)|^2\big) + C_6(e^{kx},u^2)(t) + C(R,C^*) + 2g^2(t), \end{aligned} \tag{5.12}$$

where $C_6 = 1 + k^3$. Ignoring the second term on the left side of (5.12), using (5.11) and applying the Gronwall lemma, we obtain

$$
\begin{aligned}
(e^{kx}, u^2)(t) &\le e^{C_6T_0}(e^{kx}, u_0^2) + 2C_5 e^{C_6T_0}\int_0^t [|u(0,\tau)|^2 + |Du(0,\tau)|^2]d\tau \\
&\quad + e^{C_6T_0}C(R,C^*)T_0 + e^{C_6T_0}2\int_0^t g^2(\tau)\,d\tau \\
&\le e^{C_6T_0}(e^{kx}, u_0^2)\big(1+\frac{2C_5}{C_4}\big) + e^{C_6T_0}[\frac{2C_5}{C_4}C(R,C^*) + C(R,C^*)]T_0 \\
&\quad + e^{C_6T_0}2\int_0^t g^2(\tau)\,d\tau + e^{C_6T_0}\frac{2C_5}{C_4}R^2.
\end{aligned}
$$

Choosing $T_0 > 0$ such that $e^{C_6T_0} \le 2$ and $[\frac{2C_5}{C_4}C(R,C^*) + C(R,C^*)]T_0 \le R^2C^*$, we obtain

$$
(e^{kx}, u^2)(t) \le 6R^2C^* + 2R^2; \quad t \in (0,T_0).
$$

Returning to (5.12), we rewrite it as

$$
\begin{aligned}
&\frac{d}{dt}(e^{kx}, u^2)(t) + 3k(e^{kx}, |Du|^2)(t) + C_5|u(0,t)|^2 \\
&\le 3C_5|u(0,t)|^2 + 3C_5|Du(0,t)|^2 + C(R,C^*,k) + C(R,C^*) + 2g^2(t).
\end{aligned} \tag{5.13}
$$

Integrating (5.13) over $(0,t)$ and using (5.11), we find

$$
\begin{aligned}
&(e^{kx}, u^2)(t) + \int_0^t (e^{kx}, |Du|^2)(\tau)\,d\tau + \int_0^t |u(0,\tau)|^2\,d\tau \\
&\le (e^{kx}, u_0^2) + \delta\frac{3C_5}{C_4}[C(R,C^*)T_0 + (e^{kx}, u_0^2) + R^2] + \delta T_0[C(R,C^*,k) + C(R,C^*)] \\
&\quad + 2\delta\int_0^t g^2(\tau)\,d\tau \\
&\le (e^{kx}, u_0^2)[1 + \frac{3\delta C_5}{C_4}] + \delta T_0[\frac{3C_5}{C_4}C(R,C^*) + C(R,C^*,k) + C(R,C^*)] + 2R^2C^*,
\end{aligned}
$$

where $\delta = 1/\min\{1,k,C_5\}$. Choosing $T_0 > 0$ such that

$$
\delta T_0[\frac{3C_5}{C_4}C(R,C^*) + C(R,C^*,k) + C(R,C^*)] \le 3R^2C^*,
$$

we obtain

$$
(e^{kx}, u^2)(t) + \int_0^t (e^{kx}, |Du|^2)(\tau)\,d\tau + \int_0^t |u(0,\tau)|^2\,d\tau \le 6R^2C^*; \quad t \in (0,T_0). \tag{5.14}
$$

Estimate II. Differentiating (5.6) with respect to t, multiplying by $2u_t$, integrating over $\mathbb{R}^+$ and acting as by proving (5.14), we have

$$
\begin{aligned}
&\frac{d}{dt}\|u_t\|^2(t) + C_4\big(|u_t(0,t)|^2 + |Du_t(0,t)|^2\big) \\
&\le C(\epsilon_0)\|u_t\|^2(t) + 2\epsilon_0\|v_tDv\|^2(t) + 2\epsilon_0\|vDv_t\|^2(t) + 2\epsilon_0\|Dv_t\|^2(t) \\
&\quad + C(R,C^*)\|Dv_t\|(t) + 2g_t^2(t),
\end{aligned} \tag{5.15}
$$

where $\epsilon_0 > 0$ will be chosen later. We estimate

$$
2\|v_tDv\|^2(t) \le 2\sup_{x\in\mathbb{R}^+}|v_t(x,t)|^2\|Dv\|^2(t)
$$

$$\leq 4\|v_t\|(t)\|Dv_t\|(t)\|Dv\|^2(t) \leq C(R,C^*)\|Dv_t\|(t)$$

and

$$\|vDv_t\|^2(t) \leq C(R,C^*)\|Dv_t\|^2(t).$$

Then (5.15) becomes

$$\begin{aligned}
&\frac{d}{dt}\|u_t\|^2(t) + C_4\big(|u_t(0,t)|^2 + |Du_t(0,t)|^2\big) \\
&\leq C(\epsilon_0)\|u_t\|^2(t) + \epsilon_0 C(R,C^*)\|Dv_t\|(t) + \epsilon_0 C(R,C^*)\|Dv_t\|^2(t) \\
&\quad + C(R,C^*)\|Dv_t\|(t) + 2g_t^2(t).
\end{aligned} \tag{5.16}$$

By the Gronwall lemma,

$$\begin{aligned}
&\|u_t\|^2(t) \\
&\leq e^{C(\epsilon_0)T_0}\Big(\|u_t(x,0)\|^2 + \epsilon_0 C(R,C^*)\int_0^t \|Dv_\tau\|(\tau)\,d\tau + C(R,C^*)\int_0^t \|Dv_\tau\|(\tau)d\tau\Big) \\
&\quad + e^{C(\epsilon_0)T_0}\Big(\epsilon_0 C(R,C^*)\int_0^t \|Dv_\tau\|^2(\tau)\,d\tau + 2\int_0^t g_\tau^2(\tau)\,d\tau\Big).
\end{aligned}$$

Due to (5.6),

$$\|u_t(x,0)\|^2 \leq 3\big(\|u_0Du_0\|^2 + \|D^3u_0\|^2 + \|Du_0\|^2\big) \leq 3R^2. \tag{5.17}$$

Since

$$\int_0^t \|Dv_\tau\|(\tau)\,d\tau \leq \Big(\int_0^t d\tau\Big)^{1/2}\Big(\int_0^t \|Dv_\tau\|^2(\tau)\,d\tau\Big)^{1/2} \leq T_0^{1/2}R\sqrt{12C^*},$$

we can take $T_0 > 0$ and $\epsilon_0 > 0$ such that $e^{C(\epsilon_0)T_0} \leq 2$ and $\epsilon_0 C(R,C^*)T_0^{1/2}R\sqrt{12C^*} + \epsilon_0 C(R,C^*) + C(R,C^*)T_0^{1/2}R\sqrt{12C^*} \leq C(R,C^*)$ in order to obtain

$$\|u_t\|^2(t) \leq C(R,C^*), \quad t \in (0,T_0). \tag{5.18}$$

Substituting (5.18) into (5.16) and integrating over $(0,t)$, we obtain

$$\begin{aligned}
&\|u_t\|^2(t) + C_4\int_0^t \big(|u_\tau(0,\tau)|^2 + |Du_\tau(0,\tau)|^2\big)d\tau \\
&\leq \|u_t\|^2(0) + C(R,C^*,\epsilon_0)T_0 + C(R,C^*,\epsilon_0)\int_0^t \|Dv_\tau\|(\tau)\,d\tau \\
&\quad + \epsilon_0 C(R,C^*)\int_0^t \|Dv_\tau\|^2(\tau)\,d\tau + 2\int_0^t g_\tau^2(\tau)\,d\tau \\
&\leq \|u_t\|^2(0) + C(R,C^*,\epsilon_0)T_0 + C(R,C^*,\epsilon_0)T_0^{1/2} + \epsilon_0 C(R,C^*) + R^2
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^t \big(|u_\tau(0,\tau)|^2 + |Du_\tau(0,\tau)|^2\big)d\tau \\
&\leq \frac{1}{C_4}[\|u_t\|^2(0) + C(R,C^*,\epsilon_0)T_0 + C(R,C^*,\epsilon_0)T_0^{1/2}] \\
&\quad + \frac{1}{C_4}[\epsilon_0 C(R,C^*) + R^2].
\end{aligned} \tag{5.19}$$

Differentiating (5.6) with respect to t, multiplying by $2e^{kx}u_t$, integrating over $\mathbb{R}^+$ and acting as earlier, we find

$$\begin{aligned} &\frac{d}{dt}(e^{kx},u_t^2)(t)+3k(e^{kx},|Du_t|^2)(t)\\ &\le 2C_5\big(|u_t(0,t)|^2+|Du_t(0,t)|^2\big)+k^3(e^{kx},u_t^2)(t)-2(e^{kx},(vDv)_tu_t)(t)\\ &\quad -2(e^{kx},Dv_tu_t)(t)+C(R,C^*)\|Dv_t\|(t)+2g_t^2(t). \end{aligned}\tag{5.20}$$

We estimate

$$-2(e^{kx},Dv_tu_t)(t)\le \epsilon_1(e^{kx},|Dv_t|^2)(t)+C(\epsilon_1)(e^{kx},u_t^2)(t),$$

where $\epsilon_1>0$ will be chosen later, and

$$\begin{aligned} -2(e^{kx},(vDv)_tu_t)(t) &= -2(e^{kx},(vv_t)_xu_t)(t)\\ &= 2v(0,t)v_t(0,t)u_t(0,t)+2(vv_t,e^{kx}[Du_t+ku_t])(t)\\ &\le 2v(0,t)v_t(0,t)u_t(0,t)+k(e^{kx},|vv_t|^2+|u_t|^2)(t)\\ &\quad +k(e^{kx},|Du_t|^2)(t)+\frac{1}{k}(e^{kx},|vv_t|^2)(t). \end{aligned}$$

Since

$$|v(0,t)v_t(0,t)u_t(0,t)|\le C(R,C^*)\|Dv_t\|^{1/2}(t)\|Du_t\|^{1/2}(t),$$

by the Young inequality,

$$|v(0,t)v_t(0,t)u_t(0,t)|\le C(R,C^*,k)\|Dv_t\|^{2/3}(t)+k\|Du_t\|^2(t).$$

Then (5.20) becomes

$$\begin{aligned} &\frac{d}{dt}(e^{kx},u_t^2)(t)+k(e^{kx},|Du_t|^2)(t)\\ &\le 2C_5\big(|u_t(0,t)|^2+|Du_t(0,t)|^2\big)+C(k,\epsilon_1)(e^{kx},u_t^2)(t)\\ &\quad +C(k)(e^{kx},|vv_t|^2)(t)+\epsilon_1(e^{kx},|Dv_t|^2+C(R,C^*)\|Dv_t\|^{2/3}(t)\\ &\quad +C(R,C^*)\|Dv_t\|(t)+2g_t^2(t). \end{aligned}\tag{5.21}$$

Ignoring the second term on the left-hand side and applying the Gronwall lemma, we obtain

$$\begin{aligned} (e^{kx},u_t^2)(t) &\le e^{C(k,\epsilon_1)T_0}\big[(e^{kx},u_t^2)(0)+2C_5\int_0^t\big(|u_\tau(0,\tau)|^2+|Du_\tau(0,\tau)|^2\big)d\tau\big]\\ &\quad +e^{C(k,\epsilon_1)T_0}\big[C(k)\int_0^t(e^{kx},|vv_\tau|^2)(\tau)\,d\tau+\epsilon_1\int_0^t(e^{kx},|Dv_\tau|^2)(\tau)\,d\tau\big]\\ &\quad +e^{C(k,\epsilon_1)T_0}\Big[C(R,C^*)\Big[T_0^{2/3}\Big(\int_0^t\|Dv_\tau\|^2(\tau)\,d\tau\Big)^{1/3}\\ &\quad +T_0^{1/2}\Big(\int_0^t\|Dv_\tau\|^2(\tau)\,d\tau\Big)^{1/2}\Big]\Big]+e^{C(k,\epsilon_1)T_0}\big[2\int_0^t g_\tau^2(\tau)\,d\tau\big]. \end{aligned}$$

Using (5.17), (5.19), (5.5) and (5.4) together with the choice of B_R, we have

$$\begin{aligned} &(e^{kx},u_t^2)(t)\\ &\le e^{C(k,\epsilon_1)T_0}\big[3R^2+\frac{2C_5}{C_4}[C(R,C^*,\epsilon_0)T_0+C(R,C^*,\epsilon_0)T_0^{1/2}+\epsilon_0C(R,C^*)+4R^2]\big]\\ &\quad +e^{C(k,\epsilon_1)T_0}[C(k,R)T_0+\epsilon_1 12R^2C^*+C(R,C^*)(T_0^{2/3}+T_0^{1/2})+R^2]. \end{aligned}$$

Choose $T_0 > 0$, $\epsilon_0 > 0$ and $\epsilon_1 > 0$ sufficiently small to obtain

$$(e^{kx}, u_t^2)(t) \leq C(R, C^*). \tag{5.22}$$

Returning to (5.21), it gives

$$\begin{aligned}
&\frac{d}{dt}(e^{kx}, u_t^2)(t) + k(e^{kx}, |Du_t|^2)(t) + C_5|u_t(0,t)|^2 \\
&\leq 3C_5\big(|u_t(0,t)|^2 + |Du_t(0,t)|^2\big) + C(k, R, C^*, \epsilon_1) + C(k)(e^{kx}, |vv_t|^2)(t) \\
&\quad + \epsilon_1(e^{kx}, |Dv_t|^2)(t) + C(R, C^*)\|Dv_t\|^{2/3}(t) + C(R, C^*)\|Dv_t\|(t) + 2g_t^2(t).
\end{aligned}$$

Integration over $(0, t)$ yields

$$\begin{aligned}
&(e^{kx}, u_t^2)(t) + \int_0^t (e^{kx}, |Du_\tau|^2)(\tau)\,d\tau + \int_0^t |u_\tau(0,\tau)|^2\,d\tau \\
&\leq (e^{kx}, u_t^2)(0) + \delta\Big[3C_5\int_0^t \big(|u_\tau(0,\tau)|^2 + |Du_\tau(0,\tau)|^2\big)\,d\tau + C(k, R, C^*, \epsilon_1)T_0\Big] \\
&\quad + \delta\Big[C(k)\int_0^t (e^{kx}, |vv_\tau|^2)(\tau)\,d\tau + \epsilon_1\int_0^t (e^{kx}, |Dv_\tau|^2)(\tau)\,d\tau \\
&\quad + C(R, C^*)\int_0^t \|Dv_\tau\|^{2/3}(\tau)\,d\tau\Big] + \delta\Big[C(R, C^*)\int_0^t \|Dv_\tau\|(\tau)\,d\tau + 2\int_0^t g_\tau^2(\tau)\,d\tau\Big],
\end{aligned}$$

where $\delta = 1/\min\{1, k, C_5\}$. Taking into account (5.17) and (5.19), we find

$$\begin{aligned}
&(e^{kx}, u_t^2)(t) + \int_0^t (e^{kx}, |Du_\tau|^2)(\tau)\,d\tau + \int_0^t |u_\tau(0,\tau)|^2\,d\tau \\
&\leq (e^{kx}, u_t^2)(0)\big(1 + 3\delta\frac{C_5}{C_4}\big) + 3\delta\frac{C_5}{C_4}\big[C(R, C^*, \epsilon_0)T_0 + C(R, C^*, \epsilon_0)T_0^{1/2} \\
&\quad + \epsilon_0 C(R, C^*) + R^2\big] + \delta C(k, R, C^*, \epsilon_1)T_0 + \delta\big[C(R, C^*, k)T_0 + \epsilon_1 12R^2C^* \\
&\quad + C(R, C^*)T_0^{2/3} + C(R, C^*)T_0^{1/2} + R^2\big].
\end{aligned}$$

Choose $T_0 > 0$, $\epsilon_0 > 0$ and $\epsilon_1 > 0$ sufficiently small in order to obtain

$$(e^{kx}, u_t^2)(t) + \int_0^t (e^{kx}, |Du_\tau|^2)(\tau)\,d\tau + \int_0^t |u_\tau(0,\tau)|^2\,d\tau \leq 6R^2C^*, \quad t \in (0, T_0).$$

This inequality and (5.14) completes the proof. □

Lemma 5.3. *For $T_0 > 0$ sufficiently small, the operator P is a contraction mapping.*

Proof. For arbitrary $v_1, v_2 \in B_R$ we denote $u_i = Pv_i$, $i = 1, 2$, $s = v_1 - v_2$ and $z = u_1 - u_2$. Then $z(x,t)$ satisfies the initial boundary value problem

$$z_t(x,t) + D^3z(x,t) = -\frac{1}{2}D(v_1^2 - v_2^2) - Ds, \quad (x,t) \in Q_T; \tag{5.23}$$

$$D^2z(0,t) + \alpha Dz(0,t) + \beta z(0,t) + |v_1(0,t)|v_1(0,t) - |v_2(0,t)|v_2(0,t) = 0, \tag{5.24}$$

$$z(x,0) = 0, \quad x \in \mathbb{R}^+. \tag{5.25}$$

Define

$$\rho^2(v_1, v_2) = \rho^2(s) = \sup_{t>0}(e^{kx}, s^2)(t) + \int_0^{T_0} (e^{kx}, |Ds|^2)(t)\,dt. \tag{5.26}$$

Estimate III. Multiplying (5.23) by $2z$ and integrating over $\mathbb{R}^+$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|z\|^2(t) + 2C_4\big(|z(0,t)|^2 + |Dz(0,t)|^2\big) \\
&- 2\Big||v_1(0,t)|v_1(0,t) - |v_2(0,t)|v_2(0,t)\Big||z(0,t)| \\
&\le -(D(v_1^2 - v_2^2), z)(t) - 2(Ds, z)(t).
\end{aligned} \tag{5.27}$$

We estimate

$$\begin{aligned}
I_1 &= -(D(v_1^2 - v_2^2), z)(t) \\
&\le \|D(v_1^2 - v_2^2)\|(t)\|z\|(t) \le C(R, C^*)\big(\|s\|(t) + \|Ds\|(t)\big)\|z\|(t) \\
&\le C(R, C^*)\|s\|^2(t) + \epsilon_4\|Ds\|^2(t) + C(R, C^*, \epsilon_4)\|z\|^2(t),
\end{aligned}$$

and

$$I_2 = -2(Ds, z)(t) \le \|Ds\|(t)\|z\|(t) \le \epsilon_4\|Ds\|^2(t) + C(\epsilon_4)\|z\|^2(t),$$

where $\epsilon_4 > 0$ will be chosen later;

$$\begin{aligned}
&- 2\Big||v_1(0,t)|v_1(0,t) - |v_2(0,t)|v_2(0,t)\Big||z(0,t)| \\
&= -2\Big||v_1(0,t)|s(0,t) + v_2(0,t)\Big(|v_1(0,t)| - |v_2(0,t)|\Big)\Big||z(0,t)| \\
&\ge -2\Big(|v_1(0,t)||s(0,t)| + |v_2(0,t)||v_1(0,t) - v_2(0,t)|\Big)|z(0,t)| \\
&= -2\Big(|v_1(0,t)||s(0,t)| + |v_2(0,t)||s(0,t)|\Big)|z(0,t)| \\
&\ge -\Big(C(R, C^*)\|s\|^{1/2}(t)\|Ds\|^{1/2}(t)\Big)|z(0,t)| \\
&\ge -C(R, C^*, \epsilon)\|s\|(t)\|Ds\|(t) - \epsilon|z(0,t)|^2 \\
&\ge -C(R, C^*, \epsilon, \epsilon_4)\|s\|^2(t) - \epsilon_4\|Ds\|^2(t) - \epsilon|z(0,t)|^2,
\end{aligned}$$

where $\epsilon = \frac{1}{2}\big(\beta - \frac{|\alpha|}{2} - \frac{1}{2}\big)$. Then (5.27) becomes

$$\begin{aligned}
&\frac{d}{dt}\|z\|^2(t) + C_4\big(|z(0,t)|^2 + |Dz(0,t)|^2\big) \\
&\le 3\epsilon_4\|Ds\|^2(t) + C(R, C^*, \epsilon, \epsilon_4)\|s\|^2(t) + C(R, C^*, \epsilon_4)\|z\|^2(t).
\end{aligned} \tag{5.28}$$

Ignoring the second term on the left-hand side of (5.28) and applying the Gronwall lemma, we find

$$\|z\|^2(t) \le e^{C(R,C^*,\epsilon_4)T_0}\Big[3\epsilon_4\int_0^t \|Ds\|^2(\tau)\,d\tau + C(R, C^*, \epsilon, \epsilon_4)\int_0^t \|s\|^2(\tau)\,d\tau\Big].$$

Taking $T_0 > 0$ for a fixed $\epsilon_4 > 0$ such that $e^{C(R,C^*,\epsilon_4)T_0} \le 2$, we have

$$\|z\|^2(t) \le \Big(6\epsilon_4 + C(R, C^*, \epsilon, \epsilon_4)T_0\Big)\rho^2(s).$$

The choice of $T_0 > 0$ such that $C(R, C^*, \epsilon, \epsilon_4)T_0 \le 4\epsilon_4$ yields

$$\|z\|^2(t) \le 10\epsilon_4\rho^2(s). \tag{5.29}$$

Integrating (5.28) over $(0, t)$ and using (5.29), we obtain

$$\int_0^t \big(|z(0,\tau)|^2 + |Dz(0,\tau)|^2\big)d\tau \le \frac{(3\epsilon_4 + C(R, C^*, \epsilon, \epsilon_4)T_0)}{C_4}\rho^2(s). \tag{5.30}$$

Multiplying (5.23) by $2e^{kx}z$, we find

$$\begin{aligned} &\frac{d}{dt}(e^{kx},z^2)(t)+3k(e^{kx},|Dz|^2)(t)\\ &\le 2C_5\big(|z(0,t)|^2+|Dz(0,t)|^2\big)+C(R,C^*,k,\epsilon_4)(e^{kx},z^2)(t)\\ &\quad +C(R,C^*,\epsilon_4,\epsilon)(e^{kx},s^2)(t)+3\epsilon_4(e^{kx},|Ds|^2)(t). \end{aligned} \tag{5.31}$$

Due to the Gronwall lemma and (5.30),

$$\begin{aligned} (e^{kx},z^2)(t)&\le e^{C(R,C^*,k,\epsilon_4)T_0}\big[2C_5\frac{(3\epsilon_4+C(R,C^*,\epsilon,\epsilon_4)T_0)}{C_4}\big]\rho^2(s)\\ &\quad+\ e^{C(R,C^*,k,\epsilon_4)T_0}[C(R,C^*,\epsilon_4)T_0+3\epsilon_4]\rho^2(s). \end{aligned}$$

Choose $T_0>0$ and $\epsilon_4>0$ sufficiently small to obtain

$$(e^{kx},z^2)(t)\le\frac{1}{4}\rho^2(s). \tag{5.32}$$

Integrating (5.31) over $(0,t)$ and using (5.32) and (5.30), we obtain

$$\begin{aligned} &(e^{kx},z^2)(t)+\int_0^t(e^{kx},|Dz|^2)(\tau)\,d\tau\\ &\le 2C_5\delta\frac{(3\epsilon_4+C(R,C^*,\epsilon,\epsilon_4)T_0)}{C_4}\rho^2(s)+\delta C(R,C^*,k,\epsilon_4)T_0\rho^2(s)\\ &\quad+\delta C(R,C^*,\epsilon_4,\epsilon)T_0\sup_{t>0}(e^{kx},s^2)(t)+3\delta\epsilon_4\int_0^t(e^{kx},|Ds|^2)(\tau)\,d\tau. \end{aligned}$$

Finally, we choose $T_0>0$ and $\epsilon_4>0$ sufficiently small to obtain

$$\rho^2(z)\le\gamma\rho^2(s),\quad 0<\gamma<1,$$

with $\rho=\rho(s)\ge 0$ defined in (5.26). This competes the proof. □

Lemmas 5.2 and 5.3 and the contraction mapping arguments imply the existence and uniqueness of a generalized solution $u\in B_R$ of (5.1)-(5.3):

$$u,u_t\in L^\infty(0,T_0;L^2(\mathbb{R}^+))\cap L^2(0,T_0;H^1(\mathbb{R}^+)).$$

It is easy to verify that $uDu+Du\in H^1(0,T_0;L^2(\mathbb{R}^+))$. Hence, by Theorem 3.1, $u\in L^\infty(0,T_0;H^3(\mathbb{R}^+))$. This completes the proof of Theorem 5.1. □

Remark 5.4. One can observe that to prove the existence of local solutions, we do not pose any restrictions on k while in conditions of Theorem 2.2 the value of k depends on coefficients a_1,a_2.

6. Global solutions

To prove Theorem 2.2, we must extend the obtained local solution to the whole $(0,T)$ with an arbitrary fixed $T>0$. For this purpose, we need a priori estimates of local solutions uniform in $t\in(0,T)$.

Estimate I. Multiplying (5.1) by $2u$, we obtain

$$\frac{d}{dt}\|u\|^2(t)+2(D^3u,u)(t)+2(Du,u^2)(t)+2(Du,u)(t)=0. \tag{6.1}$$

We calculate

$$I_1=2(D^3u,u)(t)$$

$$\geq 2\big(\beta - \frac{|\alpha|}{2} - \frac{\epsilon}{2}\big)|u(0,t)|^2 + (1-|\alpha|)|Du(0,t)|^2 + 2|u(0,t)|^3 - \frac{1}{\epsilon}|g(t)|^2,$$

where ϵ is an arbitrary positive number;

$$I_2 = 2(Du,u)(t) = -u^2(0,t), \quad I_3 = 2(Du,u^2)(t) = -\frac{2}{3}u^3(0,t) \geq -\frac{2}{3}|u(0,t)|^3.$$

Since $\beta - \frac{|\alpha|}{2} - \frac{1}{2} = a_1 > 0$ and $1 - |\alpha| = a_2 > 0$, choosing $\epsilon = a_1$ and substituting $I_1 - I_3$ into (6.1), we obtain

$$\frac{d}{dt}\|u\|^2(t) + a_1|u(0,t)|^2 + a_2|Du(0,t)|^2 + \frac{4}{3}|u(0,t)|^3 \leq \frac{1}{a_1}|g(t)|^2. \tag{6.2}$$

This implies

$$\begin{aligned}&\|u\|^2(t) + \int_0^t \Big(a_1|u(0,\tau)|^2 + a_2|Du(0,\tau)|^2 + \frac{4}{3}|u(0,\tau)|^3\Big)d\tau\\ &\leq \|u\|^2(0) + \frac{1}{a_1}\int_0^t |g(\tau)|^2\,d\tau.\end{aligned} \tag{6.3}$$

Estimate II. Multiplying (5.1) by $2e^{kx}u$, we obtain

$$\begin{aligned}&\frac{d}{dt}(e^{kx},u^2)(t) + 3k(e^{kx},|Du|^2)(t) + 2|u(0,t)|^3\\ &+ 2[\beta - \frac{1}{2} - \frac{|\alpha|}{2} - \frac{k^2+k}{2} - \frac{a_1}{4}]|u(0,t)|^2 + (1-|\alpha|-k)|Du(0,t)|^2\\ &\leq (k+k^3)(e^{kx},u^2)(t) - 2(e^{kx}u^2,Du)(t) + \frac{2}{a_1}|g(t)|^2.\end{aligned} \tag{6.4}$$

The conditions of Theorem 2.2 imply

$$\frac{k+k^2}{2} \leq \frac{a_1}{4}, \quad k \leq \frac{a_2}{2}$$

which transforms (6.4) into

$$\begin{aligned}&\frac{d}{dt}(e^{kx},u^2)(t) + 3k(e^{kx},|Du|^2)(t) + 2|u(0,t)|^3 + a_1|u(0,t)|^2 + \frac{a_2}{2}|Du(0,t)|^2\\ &\leq (k+k^3)(e^{kx},u^2)(t) - 2(e^{kx}u^2,Du)(t) + \frac{2}{a_1}|g(t)|^2.\end{aligned} \tag{6.5}$$

We estimate

$$\begin{aligned}I_1 &= -2|(e^{kx}u^2,Du)(t)| \leq k(e^{kx},|Du|^2)(t) + \frac{1}{k}(e^{kx},u^4)(t)\\ &\leq k(e^{kx},|Du|^2)(t) + \frac{1}{k}\sup_{x\in\mathbb{R}^+}|e^{kx}u^2(x,t)|\|u\|^2(t).\end{aligned}$$

Taking into account (6.3), we have

$$\begin{aligned}I_1 &\leq k(e^{kx},|Du|^2)(t) + \frac{2}{k}\Big(\|u\|^2(0) + \frac{1}{a_1}\int_0^t |g(\tau)|^2\,d\tau\Big)\|e^{kx/2}u\|(t)\|D(e^{kx/2}u)\|(t)\\ &\leq k(e^{kx},|Du|^2)(t) + \frac{2}{k}\Big(\|u\|^2(0) + \frac{1}{a_1}\int_0^t |g(\tau)|^2\,d\tau\Big)\big\{\frac{k}{2}\|e^{kx/2}u\|^2(t)\big\}\\ &\quad + \frac{2}{k}\Big(\|u\|^2(0) + \frac{1}{a_1}\int_0^t |g(\tau)|^2\,d\tau\Big)\big\{\|e^{kx/2}u\|(t)\|e^{kx/2}Du\|(t)\big\}.\end{aligned}$$

Using the Young inequality, we obtain

$$I_1 \le 2k(e^{kx}, |Du|^2)(t) + C(\|u_0\|, \|g\|^2_{L^2(0,T)}, k)(e^{kx}, u^2)(t). \tag{6.6}$$

Substituting I_1 into (6.5), we obtain

$$\frac{d}{dt}(e^{kx}, u^2)(t) + \frac{k}{2}(e^{kx}, |Du|^2)(t) \le C(\|u_0\|, \|g\|^2_{L^2(0,T)}, k)[(e^{kx}, u^2)(t) + g^2(t)].$$

Due to the Gronwall lemma and (6.3),

$$(e^{kx}, u^2)(t) \le C(\|u_0\|, \|g\|^2_{L^2(0,T)}, k, T)\Big[(e^{kx}, u_0^2) + \int_0^t g^2(\tau)\, d\tau\Big]. \tag{6.7}$$

Hence

$$(e^{kx}, u^2)(t) + \int_0^t (e^{kx}, |Du|^2)(\tau)\, d\tau \le C\Big[(e^{kx}, u_0^2) + \int_0^t g^2(\tau)\, d\tau\Big], \quad t \in (0,T); \tag{6.8}$$

where C depends on $k > 0$, $T > 0$, $\|g\|_{L^2(0,T)}$ and $\|u_0\|$.

Estimate III. Differentiate (5.1) with respect to t, multiply by $2e^{kx}u_t$ and acting as by proving Estimate II, we obtain

$$\begin{aligned} &\frac{d}{dt}(e^{kx}, u_t^2)(t) + 3k(e^{kx}, |Du_t|^2)(t) + (e^{kx}(u^2)_{xt}, u_t)(t) \\ &+ 4|u(0,t)|u_t^2(0,t) + a_1|u_t(0,t)|^2 + \frac{a_2}{2}|Du_t(0,t)|^2 \\ &\le (k + k^3)(e^{kx}, u_t^2)(t) + \frac{2}{a_1}|g_t(t)|^2. \end{aligned} \tag{6.9}$$

We define

$$I_1 = (e^{kx}(u^2)_{xt}, u_t)(t) = -u(0,t)u_t^2(0,t) - k(e^{kx}u, u_t^2)(t) + (e^{kx}Du, u_t^2)(t)$$

which we estimate as follows:

$$\begin{aligned} I_{11} = k(e^{kx}u, u_t^2)(t) &\le k \sup_{x\in\mathbb{R}^+} |u(x,t)|(e^{kx}, u_t^2)(t) \\ &\le k\big(1 + \|u\|(t) + (e^{kx}, |Du|^2)(t)\big)(e^{kx}, u_t^2)(t), \end{aligned}$$

and

$$\begin{aligned} I_{12} = (e^{kx}Du, u_t^2)(t) &\le \sup_{x\in\mathbb{R}^+} |u_t(x,t)|(e^{kx}|Du|, |u_t|)(t) \\ &\le \sqrt{2}\|u_t\|^{1/2}(t)\|Du_t\|^{1/2}(t)\|e^{kx/2}Du\|(t)\|e^{kx/2}u_t\|(t) \\ &\le k(e^{kx}, |Du_t|^2)(t) + \frac{1}{k}[1 + (e^{kx}, |Du|^2)(t)](e^{kx}, u_t^2)(t). \end{aligned}$$

Substituting I_1 into (6.9) and using (6.3), we come to the inequality

$$\begin{aligned} &\frac{d}{dt}(e^{kx}, u_t^2)(t) + 2k(e^{kx}, |Du_t|^2)(t) + a_1|u_t(0,t)|^2 \\ &+ \frac{a_0}{2}|Du_t(0,t)|^2 + 3|u(0,t)|u_t^2(0,t) \\ &\le C(k)[1 + \|u\|^2(t) + \|Du\|^2(t)](e^{kx}, u_t^2)(t) + \frac{2}{a_1}|g_t(t)|^2. \end{aligned}$$

Taking into account (6.8), it is easy to see that

$$1 + \|u\|^2(t) + \|Du\|^2(t) \in L^1(0,T).$$

Hence, by the Gronwall lemma, we obtain

$$\begin{aligned}&(e^{kx}, u_t^2)(t) + \int_0^t (e^{kx}, |Du_\tau|^2)(\tau)\,d\tau\\&\le C\Big[\sum_{i=0}^3 (e^{kx}, |D^i u_0|^2) + (e^{kx}, |u_0 Du_0|^2)\Big] + C\|g\|^2_{H^1(0,T)}, \quad t\in(0,T);\end{aligned}$$

where C depends on $k>0$, $T>0$, $\|g\|_{L^2(0,T)}$ and $\|u_0\|$. Returning to (5.1), we can write it as

$$D^3u + u = u - u_t - uDu - Du. \tag{6.10}$$

By Theorem 3.1,

$$\sum_{i=0}^3 (e^{kx}, |D^i u|^2)(t) \le C\Big(1 + \big[\sum_{i=0}^3 (e^{kx}, |D^i u_0|^2) + (e^{kx}, |u_0 Du_0|^2) + \|g\|^2_{H^1(0,T)}\big]\Big). \tag{6.11}$$

On the other hand,

$$D^4u = -Du_t - D^2u - |Du|^2 - uD^2u \in L^2(0,T;L^2(\mathbb{R}^+))$$

and

$$\begin{aligned}\int_0^t (e^{kx}, |D^4u|^2)(\tau)\,d\tau &\le \frac{2}{3}\int_0^t (e^{kx}, |D^4u|^2)(\tau)\,d\tau + \frac{3}{2}\int_0^t (e^{kx}, |Du_\tau|^2)(\tau)\,d\tau\\&\quad + \frac{3}{2}\int_0^t (e^{kx}, |D^2u|^2)(\tau)\,d\tau + \frac{3}{2}\int_0^t (e^{kx}, |Du|^4)(\tau)\,d\tau\\&\quad + \frac{3}{2}\int_0^t (e^{kx}, |uD^2u|^2)(\tau)\,d\tau.\end{aligned}$$

This gives

$$\begin{aligned}&\int_0^t (e^{kx}, |D^4u|^2)(\tau)\,d\tau\\&\le \frac{2}{3}\int_0^t (e^{kx}, |D^4u|^2)(\tau)\,d\tau + \frac{3}{2}\int_0^t (e^{kx}, |Du_\tau|^2)(\tau)\,d\tau\\&\quad + \frac{3}{2}\int_0^t (e^{kx}, |D^2u|^2)(\tau)\,d\tau + 3\|Du\|(t)\|D^2u\|(t)\int_0^t (e^{kx}, |Du|^2)(\tau)\,d\tau\\&\quad + 3\|u\|(t)\|Du\|(t)\int_0^t (e^{kx}, |D^2u|^2)(\tau)\,d\tau.\end{aligned}$$

This and (6.11) imply

$$\begin{aligned}&\int_0^t (e^{kx}, |D^4u|^2)(\tau)\,d\tau\\&\le C\Big(1 + \big[\sum_{i=0}^3 (e^{kx}, |D^i u_0|^2) + (e^{kx}, |u_0 Du_0|^2) + \|g\|^2_{H^1(0,T)}\big]\Big).\end{aligned}$$

Hence, $u \in L^2(0,T;H^4(\mathbb{R}^+))$.

Estimates I, II, III complete Section 6. Uniqueness of the obtained regular solution follows from the contraction mapping principle. The proof of Theorem 2.2 is complete.

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