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p-HARMONIOUS FUNCTIONS WITH DRIFT ON GRAPHS VIA GAMES

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Abstract. In a connected finite graph E with set of vertices $\mathfrak{X}$, choose a nonempty subset, not equal to the whole set, $Y \subset \mathfrak{X}$, and call it the boundary $Y = \partial\mathfrak{X}$. Given a real-valued function $F : Y \to \mathbb{R}$, our objective is to find a function u, such that $u = F$ on Y, and for all $x \in \mathfrak{X} \setminus Y$,

$$u(x) = \alpha \max_{y \in S(x)} u(y) + \beta \min_{y \in S(x)} u(y) + \gamma\Big(\frac{\sum_{y \in S(x)} u(y)}{\#(S(x))}\Big).$$

Here α, β, γ are non-negative constants such that $\alpha+\beta+\gamma = 1$, the set $S(x)$ is the collection of vertices connected to x by an edge, and $\#(S(x))$ denotes its cardinality. We prove the existence and uniqueness of a solution of the above Dirichlet problem and study the qualitative properties of the solution.

1. Introduction

The goal of this paper is to study functions that satisfy

$$u(x) = \alpha \max_{y \in S(x)} u(y) + \beta \min_{y \in S(x)} u(y) + \gamma\Big(\frac{\sum_{y \in S(x)} u(y)}{\#(S(x))}\Big). \tag{1.1}$$

We denote a graph by E and the collection of vertices by $\mathfrak{X}$. We choose Y to be a proper nonempty subset of $\mathfrak{X}$ and call it the boundary. In equation (1.1) the set $S(x)$ is the collection of vertices connected to the given vertex x by a single edge, and α, β and γ are predetermined non-negative constants such that $\alpha + \beta + \gamma = 1$. The cardinality of $S(x)$ is denoted by $\#S(x)$. A function satisfying (1.1) is called p-harmonious with drift, by analogy with continuous case studied in [5]. Functions of this type arise as approximations of p-harmonic functions. In particular, an approximating sequence could be generated by running zero-sum stochastic games on a graph of decreasing step-size. The value of the game function satisfies a nonlinear equation, which is directly linked to the existence and uniqueness of the solution of the p-Laplacian as demonstrated in [9, 8, 4]. We present the connections between equation (1.1) and game theory in Theorem 5.1.

We formally pose the Dirichlet problem: For a given $F : Y \to \mathbb{R}$ find a function u defined on $\mathfrak{X}$, such that $u = F$ on Y and u satisfies (1.1). We address questions of existence and uniqueness of the solution of this Dirichlet problem in Theorems

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3.1 and 4.1. We state the strong comparison principle in Theorem 6.1. We also study the question of unique continuation for p-harmonious functions with drift. In particular we present an example of p-harmonious function which does not have the unique continuation property. The current manuscript is based on the results obtained in [10].

The equation (1.1) can be restated in a more traditional notation with the help of the following definitions, which we borrowed from [1].

Definition 1.1. The Laplace operator on the graph is given by

$$\Delta u(x) = \fint_{S(x)} u - u(x).$$

Definition 1.2. The infinity Laplacian on the graph is given by

$$\Delta_\infty u(x) = \frac{1}{2}(\max_{S(x)} u + \min_{S(x)} u) - u(x).$$

Definition 1.3. For $X = (x, y, z) \in \mathbb{R}^3$ we define the analog of the maximal directional derivative

$$\langle X \rangle_\infty = \max\{x, y, z\}.$$

With the above definitions we can restate (1.1) as

$$(\alpha - \beta)\langle \nabla u \rangle_\infty + 2\beta \Delta_\infty u + \gamma \Delta u = 0. \tag{1.2}$$

2. Game setup and definitions

Most of our results are proved using the following game. We consider a connected graph E with vertex set $\mathfrak{X}$. The set $\mathfrak{X}$ is finite unless stated otherwise. We equip $\mathfrak{X}$ with the σ-algebra $\mathcal{F}$ of all subsets of $\mathfrak{X}$. For an arbitrary vertex x we define $S(x)$ the collection of vertices, which are connected to the vertex x by a single edge. In case $\mathfrak{X}$ is infinite, we require that $\mathfrak{X}$ is at least locally finite; i.e. the cardinality of $S(x)$ is finite. At the beginning of the game a token is placed at some point $x_0 \in \mathfrak{X}$. Then we toss a three-sided virtual coin. The side of a coin labelled 1 comes out with probability α and in this case player I chooses where to move the token among all vertices in $S(x)$. The side of a coin labelled 2 comes out with probability β and in this case player II chooses where to move the token among all vertices in $S(x)$. Finally, the side of a coin labelled 3 comes out with probability γ and in this case we choose the next point randomly (uniformly) among all vertices in $S(x)$. This setup has been described in [9] and in [7] and is known as "biased tug-of-war with noise". The game stops once we hit the boundary set Y. The set Y is simply predetermined non-empty set of vertices at which game terminates. In the game literature the set Y is called set of absorbing states. Let $F : Y \to \mathbb{R}$ be the payoff function defined on Y. If game ends at some vertex $y \in Y$, then player I receives from player II the sum of $F(y)$ dollars.

Let us define the value of the game for player I. Firstly, we formalize the notion of a pure strategy. We define a strategy S_I for player I as a collection of maps $\{\sigma_I^k\}_{k\in\mathbb{N}}$, such that for each k,

$$\sigma_I^k : \mathfrak{X}^k \to \mathfrak{X},$$
$$\sigma_I^k(x_0, \dots, x_{k-1}) = x_k,$$

where

$$\mathfrak{X}^k = \underbrace{\mathfrak{X} \times \mathfrak{X} \times \cdots \times \mathfrak{X}}_{k\,\text{times}}.$$

Hence, σ_I^k tells player I where to move given $(x_0, \dots, x_{k-1})$ - the history of the game up to the step k, if he wins the toss. We call a strategy *stationary* if it depends only on the current position of the token. Given two strategies for player I and II the transition probabilities for $k \geq 1$ are given by

$$\pi_k(x_0, \dots, x_{k-1}; y) = \alpha\delta_{\sigma_I^k(x_0,\dots,x_{k-1})}(y) + \beta\delta_{\sigma_{II}^k(x_0,\dots,x_{k-1})}(y) + \gamma U_{S(x_{k-1})}(y),$$

where we have set

$$U_{S(x_{k-1})} \text{ is a uniform distribution on } S(x_{k-1}) \text{ and } \pi_0(y) = \delta_{x_0}(y).$$

We equip $\mathfrak{X}^k$ with product σ-algebra $\mathcal{F}^k$,

$$\mathcal{F}^k = \underbrace{\mathcal{F} \otimes \mathcal{F} \otimes \cdots \otimes \mathcal{F}}_{k\,\text{times}}$$

and then we define a probability measure on $(\mathfrak{X}^k, \mathcal{F}^k)$ in the following way:

$$\mu_0 = \pi_0 = \delta_{x_0},$$
$$\mu_k(A^k \times A) = \int_{A^k} \pi_k(x_0, \dots, x_{k-1}; A)\, d\mu_{k-1},$$

where $A^{k-1} \times A$ is a rectangle in $(\mathfrak{X}^k, \mathcal{F}^k)$. The space of infinite sequences with elements from $\mathfrak{X}$ is $\mathfrak{X}^\infty$. Let $X_k : \mathfrak{X}^\infty \to \mathfrak{X}$ be the coordinate process defined by

$$X_k(h) = x_k, \quad \text{for } h = (x_0, x_1, x_2, x_3, \dots) \in \mathfrak{X}^\infty.$$

We equip $\mathfrak{X}^\infty$ with product σ-algebra $\mathcal{F}^\infty$. For precise definition of $\mathcal{F}^\infty$ see [2].

The family of $\{\mu_k\}_{k\geq 0}$ satisfies the conditions of Kolmogorov extension theorem [11], therefore, we can conclude that there exists a unique measure $\mathbb{P}^{x_0}$ on $(\mathfrak{X}^\infty, \mathcal{F}^\infty)$ with the following property:

$$\mathbb{P}^{x_0}(B_k \times \mathfrak{X} \times \mathfrak{X} \times \times \mathfrak{X} \dots) = \mu_k(B_k), \quad \text{for } B_k \in \mathcal{F}^k \tag{2.1}$$

and

$$\mathbb{P}^{x_0}[X_k \in A | X_0 = x_0, X_1 = x_1, \dots, X_{k-1} = x_{k-1}] = \pi_k(x_0, \dots, x_{k-1}; A). \tag{2.2}$$

We are now ready to define the value of the game for player I. The boundary hitting time is given by

$$\tau = \inf_k \{X_k \in Y\}.$$

Consider strategies S_I and S_{II} for player I and player II respectively. We define

$$F_-^x(S_I, S_{II}) = \begin{cases} \mathbb{E}_{S_I,S_{II}}^x[F(X_\tau)] & \text{if } \mathbb{P}_{S_I,S_{II}}^x(\tau < \infty) = 1 \\ -\infty & \text{otherwise} \end{cases} \tag{2.3}$$

$$F_+^x(S_I, S_{II}) = \begin{cases} \mathbb{E}_{S_I,S_{II}}^x[F(X_\tau)] & \text{if } \mathbb{P}_{S_I,S_{II}}^x(\tau < \infty) = 1 \\ +\infty & \text{otherwise} \end{cases} \tag{2.4}$$

The value of the game for player I is

$$u_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{F}_-^x(S_I, S_{II})$$

and the value of the game for player II is

$$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{F}_+^x(S_I, S_{II})$$

These definitions penalize players severely for not being able to force the game to end. Whenever player I has a strategy to finish the game almost surely, then we simplify notation by setting

$$u_I(x) = \sup_{S_I} \inf_{S_{II}} E^x_{S_I,S_{II}}[F(X_\tau)].$$

Similarly, for player II we set

$$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} E^x_{S_I,S_{II}}[F(X_\tau)].$$

The following lemma states rigorously whether player I has a strategy to finish the game almost surely:

Lemma 2.1. *If $\mathfrak{X}$ is a finite set, then player I (player II) has strategies to finish the game almost surely.*

Proof. When $\gamma = 0$, this result was already proven by Peres, Schramm, Sheffield, and Wilson in [9, Theorem 2.2]. When $\gamma \neq 0$, the statement follows from the fact that random walk on a finite graph is recurrent. □

We always have $u_I(x) \le u_{II}(x)$. Whenever $u_I(x) = u_{II}(x)$ for all $x \in \mathfrak{X}$ we say that game has a value.

3. Existence

Here is the first existence result for equation (1.1).

Theorem 3.1 (Dynamic Programming Principle equals Mean Value Property)**.** *The value functions u_I and u_{II} satisfy the Dynamic Programming Principle (DPP) or the Mean Value Property (MVP):*

$$u_I(x) = \alpha \max_{y\in S(x)} u_I(y) + \beta \min_{y\in S(x)} u_I(y) + \gamma \fint_{S(x)} u_I(y)dy, \tag{3.1}$$

$$u_{II}(x) = \alpha \max_{y\in S(x)} u_{II}(y) + \beta \min_{y\in S(x)} u_{II}(y) + \gamma \fint_{S(x)} u_{II}(y)dy. \tag{3.2}$$

The above result is true in the general setting of discrete stochastic games (see Maitra and Sudderth, [3, chapter 7]). Here we provide a simpler proof in Markovian case. It turns out that optimal strategies are Markovian (see [3, chapter 5]).

Proposition 3.2 (The stationary case)**.** *In a game with stationary strategies the value functions u_I and u_{II} satisfy the Dynamic Programming Principle (DPP) or the Mean Value Property (MVP):*

$$u_I(x) = \alpha \max_{y\in S(x)} u_I(y) + \beta \min_{y\in S(x)} u_I(y) + \gamma \fint_{S(x)} u_I(y)dy, \tag{3.3}$$

$$u_{II}(x) = \alpha \max_{y\in S(x)} u_{II}(y) + \beta \min_{y\in S(x)} u_{II}(y) + \gamma \fint_{S(x)} u_{II}(y)dy. \tag{3.4}$$

Proof. We will provide a proof only for u_I; the proof for u_{II} follows by symmetry. Take a set of vertices $\mathfrak{X}$, boundary Y and adjoin one vertex y^* to the boundary. Denote new boundary by $Y^* = Y \cup \{y^*\}$ and the new set of vertices by $\mathfrak{X}^* = \mathfrak{X} \setminus \{y^*\}$ and define

$$F^*(y) = \begin{cases} F(y) & \text{if } y \in Y \\ u_I(y^*) & \text{if } y = y^*. \end{cases} \tag{3.5}$$

Let $u_I(x)$ be the value of the game with $\mathfrak{X}$ and Y, and $u_I^*(x)$ be the value of the game with $\mathfrak{X}^*$ and Y^*. The goal is to show that

$$u_I^*(x) = u_I(x).$$

Once we prove the above, the main result follows by extending F to the set $S(x)$.

Remark 3.3. The idea of extending F is used in [9, Lemma 3.5]

Hence, we have to show $u_I^*(x) = u_I(x)$. Since we consider only Markovian strategies we can think of them as mappings $S_I : \mathfrak{X} \to \mathfrak{X}$. For the game $\mathfrak{X}^*$ and Y^*, we define S_I^* as a restriction of S_I to $\mathfrak{X}^*$ Here are the steps in detail:

$$\begin{aligned}
u_I^*(x) &= \sup_{S_I^*} \inf_{S_{II}^*} \big(E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*})\big) \\
&= \sup_{S_I^*} \inf_{S_{II}^*} \big(E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c}\big) \\
&= \sup_{S_I^*} \inf_{S_{II}^*} \big(E^x_{S_I^*, S_{II}^*} u_I(y^*) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c}\big) \\
&= \sup_{S_I^*} \inf_{S_{II}^*} \big(E^x_{S_I^*, S_{II}^*} \sup_{S_I} \inf_{S_{II}} E^{y^*}_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}} \\
&\quad + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c}\big) \\
&= \sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \big(E^x_{S_I^*, S_{II}^*} \big(E^{y^*}_{S_I, S_{II}} F(X_\tau)\big) \chi_{\{X_{\tau^*} = y^*\}} \\
&\quad + E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c}\big).
\end{aligned} \tag{3.6}$$

If we can show that

$$\begin{aligned}
&\sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \big(E^x_{S_I^*, S_{II}^*} \big(E^{y^*}_{S_I, S_{II}} F(X_\tau)\big) \chi_{\{X_{\tau^*} = y^*\}} \\
&+ E^x_{S_I^*, S_{II}^*} F^*(X_{\tau^*}) \chi_{\{X_{\tau^*} = y^*\}^c}\big) \\
&= \sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \big(E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}^c}\big).
\end{aligned} \tag{3.7}$$

We can complete the proof in the following way:

$$\begin{aligned}
u_I^*(x) &= \sup_{S_I^*} \inf_{S_{II}^*} \sup_{S_I} \inf_{S_{II}} \big(E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}^c}\big) \\
&= \sup_{S_I} \inf_{S_{II}} \sup_{S_I^*} \inf_{S_{II}^*} \big(E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}^c}\big) \\
&= \sup_{S_I} \inf_{S_{II}} \big(E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}} + E^x_{S_I, S_{II}} F(X_\tau) \chi_{\{X_{\tau^*} = y^*\}^c}\big) \\
&= \sup_{S_I} \inf_{S_{II}} E^x_{S_I, S_{II}} F(X_\tau) = u_I(x).
\end{aligned}$$

Let us clarify (3.7). Actually, we have the following two equalities

$$E^x_{S_I^*,S_{II}^*}E^{y^*}_{S_I,S_{II}}F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}} = E^x_{S_I,S_{II}}F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}}, \tag{3.8}$$

$$E^x_{S_I^*,S_{II}^*}F^*(X_{\tau^*})\chi_{\{X_{\tau^*}=y^*\}^c} = E^x_{S_I,S_{II}}F(X_\tau)\chi_{\{X_{\tau^*}=y^*\}^c} \tag{3.9}$$

Equation (3.8) could be thought of as payoff computed for the trajectories that travel through a point y^*. Roughly speaking we first discount boundary points to the point y^* and then discount value at y^* back to x which is the same as to discount boundary points to x through trajectories that contain y^*, keeping in mind that S_i^* is just a restriction of S_i. Equation (3.9) is a payoff computed for the trajectories that avoid y^*, and, therefore, there is no difference between S_i^* and S_i, since S_i^* is just a restriction of S_i to $\mathfrak{X}\setminus\{y^*\}$. □

The following proposition is an extension of the result stated in [6]. It characterizes optimal strategies. By optimal strategies we mean any pair of strategies $\hat{S}_I$ and $\hat{S}_{II}$ such that

$$E^x_{\hat{S}_I,\hat{S}_{II}}F(X_\tau) = \sup_{S_I}\inf_{S_{II}} E^x_{S_I,S_{II}}F(X_\tau) = u_I = u_{II}. \tag{3.10}$$

Proposition 3.4. *Consider a game on the graph E with finite set of vertices $\mathfrak{X}$. Then the the strategy $\hat{S}_I$ ($\hat{S}_{II}$) under which player I (player II) moves from vertex x to vertex z with*

$$u(z) = \max_{y\in S(x)} u(y), \quad (u(z) = \min_{y\in S(x)} u(y))$$

is optimal.

Proof. Let us start the game at vertex x ($X_0 = x$). We claim that under strategies $\hat{S}_I$ and $\hat{S}_{II}$ $u_I(X_k)$ is a martingale due to following arguments:

$$\begin{aligned}
&\mathbb{E}^x_{\hat{S}_I,\hat{S}_{II}}[u_I(X_k)|X_0,\dots,X_{k-1}]\\
&= \alpha u_I(X_k^I) + \beta u_I(X_k^{II}) + \gamma \fint_{S(X_{k-1})} u_I(y)dy\\
&= \alpha \max_{y\in S(X_{k-1})} u_I(y) + \beta \min_{y\in S(X_{k-1})} u_I(y) + \gamma \fint_{S(X_{k-1})} u_I(y)dy\\
&= u_I(X_{k-1}),
\end{aligned} \tag{3.11}$$

where $v(X_k^I)$ indicates the choice of player I and $v(X_k^{II})$ indicates the choice of player II. Then

$$u_I(X_k^{II}) = \min_{y\in S(X_{k-1})} u_I(y), \quad u_I(X_k^{II}) = \max_{y\in S(X_{k-1})} u_I(y)$$

by choice of strategies $\hat{S}_I$ and $\hat{S}_{II}$. In addition, since u_I is a bounded function, we conclude that $u_I(X_k)$ is a uniformly integrable martingale. Hence, by Doob's Optional Stopping Theorem

$$E^x_{\hat{S}_I,\hat{S}_{II}}F(X_\tau) = E^x_{\hat{S}_I,\hat{S}_{II}}u_I(X_\tau) = E^x_{\hat{S}_I,\hat{S}_{II}}u_I(X_0) = u_I(x), \tag{3.12}$$

□

Example 3.5. We would like to warn the reader that the Proposition 3.4 does not claim that tugging towards that maximum of F on the boundary would be an optimal strategy for player I. Figure 1 shows a counterexample.

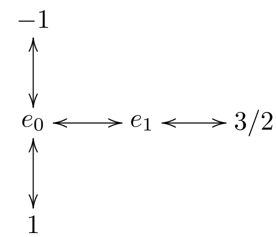

FIGURE 1. Counterexample - tugging towards the boundary

The boundary vertices are indicated by the numbers, which reflect the value of F at each vertex. We consider the game starting at vertex e_0 and require player II always pull towards the vertex labelled -1. For player I we choose S_I^a to be the strategy of always tugging towards vertex 3/2 and let S_I^b be the strategy of moving towards vertex 1. We see that

$$E^{e_0}_{S_I^a,S_{II}} F(X_\tau) = -1\cdot 2/3 + 3/2\cdot 1/3 = -1/6, \tag{3.13}$$
$$E^{e_0}_{S_I^b,S_{II}} F(X_\tau) = -1\cdot 1/2 + 1\cdot 1/2 = 0\,. \tag{3.14}$$

4. UNIQUENESS

Uniqueness will follow from the comparison principle below proven by using Doob's Optional Sampling Theorem.

Theorem 4.1 (via Martingales)**.** *Let v be a solution of*

$$v(x) = \alpha \max_{y\in S(x)} v(y) + \beta \min_{y\in S(x)} v(y) + \gamma \fint_{S(x)} v(y)dy \tag{4.1}$$

on a graph E with a countable set of vertices $\mathfrak{X}$ and boundary Y. Assume

- $F(y) = u_I(y)$, *for all* $y\in Y$,
- $\inf_Y F > -\infty$,
- *v bounded from below, and*
- $v(y)\ge F(y)$, *for all* $y\in Y$

Then u_I is bounded from below on $\mathfrak{X}$ and $v(x)\ge u_I(x)$, for $x\in\mathfrak{X}$.

Proof. Note that we only need "$\le$" in equation (4.1). The theorem says that u_I is the smallest super-solution with given boundary value F. We proceed as in [9, Lemma 2.1]. Since the game ends almost surely,

$$u_I \ge \inf_Y F > -\infty$$

which proves that u_I is bounded from below. Now we have to show that

$$v(x) \ge \sup_{S_I}\inf_{S_{II}} F_-^x(S_I,S_{II}) = u_I(x)$$

If we fix an arbitrary strategy S_I, then we have to show that

$$v(x) \ge \inf_{S_{II}} F_-^x(S_I,S_{II}). \tag{4.2}$$

Consider a game that start at vertex x ($X_0 = x$). We have two cases

Case 1: If our fixed S_I cannot force the game to end a.s. (i.e. $\mathbb{P}^x_{S_I,S_{II}}(\tau < \infty) < 1$), then by the definition of F_-, $\inf_{S_{II}} F^x_-(S_I, S_{II}) = -\infty$ and the inequality (4.2) holds.

Case 2: Now assume that our fixed S_I forces the game to end despite all the efforts of the second player. Let player II choose a strategy of moving to $\min_{y\in S(x)} v(y)$ - denote such a strategy $\hat{S}_{II}$. If we prove that $v(X_k)$ is a supermartigale bounded from below, then we can finish the proof by applying Doob's Optional Stopping Theorem:

$$\begin{aligned}\inf_{S_{II}} \mathbb{E}^x_{S_I,S_{II}} F(X_\tau) &\le \mathbb{E}^x_{S_I,\hat{S}_{II}} F(X_\tau) \le \mathbb{E}^x_{S_I,\hat{S}_{II}} v(X_\tau)\\ &\le \mathbb{E}^x_{S_I,\hat{S}_{II}} v(X_0) = v(X_0) = v(x),\end{aligned}$$

where we have used Fatou's lemma. The result follows upon taking $\sup_{S_I}$. Hence, we only need to prove that $v(X_k)$ is a supermartingale under measure $\mathbb{P}^x_{S_I,\hat{S}_{II}}$:

$$\begin{aligned}&\mathbb{E}^x_{S_I,\hat{S}_{II}}[v(X_k)|X_0,\dots,X_{k-1}]\\ &= \alpha v(X_k^I) + \beta v(X_k^{II}) + \gamma \fint_{S(X_{k-1})} v(y)dy\\ &\le \alpha \max_{y\in S(X_{k-1})} v(y) + \beta \min_{y\in S(X_{k-1})} v(y) + \gamma \fint_{S(X_{k-1})} v(y)dy = v(X_{k-1}),\end{aligned}$$

where $v(X_k^I)$ indicates the choice of player I and $v(X_k^{II})$ indicates the choice of player II. Then $v(X_k^{II}) = \min_{y\in S(X_{k-1})} v(y)$ by choice of strategy for player II. □

In case $\min_{y\in S(X_{k-1})} v(y)$ is not achieved (i.e. graph is not locally finite), we need to modify the above proof by making player II move within ϵ neighborhood of $\min_{y\in S(X_{k-1})} v(y)$. We can prove similar result for u_{II}. The next theorem is the extension of the result obtained in [5].

Theorem 4.2. *If graph E is finite and F is bounded below on Y, then $u_I = u_{II}$, so the game has a value.*

Proof. Clearly, finite E implies that F is bounded below. We included this redundant statement to suggest future possible extensions to an uncountable graph. We know that $u_I \le u_{II}$ always holds, so we only need to show $u_I \ge u_{II}$. Assume F is bounded below. Similar to the proof of Lemma 4.1 we can show that u_I is a supermartingale bounded below by letting player I to choose an arbitrary strategy S_I and requiring player II always move to $\min_{y\in S(x)} u_I(y)$ from x - strategy $\hat{S}_{II}$. For simplicity of the presentation we consider a case when $\min_{y\in S(x)} u_I(y)$ is achievable, for the general case we have to employ ϵ, like in Theorem 4.1. We start the game at x, so $X_0 = x$. Recall $u_{II}(x) = \inf_{S_{II}} \sup_{S_I} F_+(S_I, S_{II})$

$$\begin{aligned}u_{II}(x) &\le \sup_{S_I} \mathbb{E}^x_{S_I,\hat{S}_{II}}[F(X_\tau)] \quad \text{(since E is finite)}\\ &= \sup_{S_I} \mathbb{E}^x_{S_I,\hat{S}_{II}}[u_I(X_\tau)]\\ &\le \sup_{S_I} \mathbb{E}^x_{S_I,\hat{S}_{II}}[u_I(X_0)] = u_I(x).\end{aligned}$$

Due to Doob's Optional Stopping Theorem. □

5. Connections among games, partial differential equations and DPP

This section summarizes some previous results and presents new prospectives on known issues.

Theorem 5.1. *Assume we are given a function u on the set of vertices $\mathfrak{X}$ and consider a strategy $\hat{S}_I$ ($\hat{S}_{II}$) where player I (player II) moves from vertex x to vertex z with*

$$u(z) = \max_{y\in S(x)} u(y) \quad (u(z) = \min_{y\in S(x)} u(y)).$$

Then the following two statements are equivalent:

- *the process $u(X_n)$ is a martingale under the measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$,*
- *the function u is a solution of Dirichlet problem* (1.1).

In addition, $u(X_n)$ is a martingale under the measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$ implies that $\hat{S}_I$ and $\hat{S}_{II}$ are the optimal strategies.

Proof. Suppose that $u(X_n)$ is a martingale under measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$. Fix an arbitrary point $x \in \mathfrak{X}$ and consider a game which starts at $x = X_0$, then

$$\begin{aligned} E^x_{\hat{S}_I,\hat{S}_{II}}[u(X_1)|X_0] &= \alpha u(X_1^I) + \beta u(X_1^{II}) + \gamma \fint_{S(X_0)} u(y)dy \\ &= \alpha \max_{y\in S(X_0)} u(y) + \beta \min_{y\in S(X_0)} +\gamma \fint_{S(X_0)} u(y)dy \\ &= u(X_0). \end{aligned} \tag{5.1}$$

Conversely, assume that u solves Dirichlet problem (1.1), then (5.1) implies that $u(X_n)$ is a martingale under measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$.

Let us show a final implication. The result relies on the fact that our game has a value and value of game function is the solution of the Dirichlet problem (1.1). Since $u(X_n)$ is a martingale under measure induced by strategies $\hat{S}_I$ and $\hat{S}_{II}$ we have

$$E^x_{\hat{S}_I,\hat{S}_{II}} F(X_\tau) = E^x_{\hat{S}_I,\hat{S}_{II}} u(X_\tau) = E^x_{\hat{S}_I,\hat{S}_{II}} u(X_0) = u(x). \tag{5.2}$$

By the uniqueness result (Theorem 4.1)

$$u(x) = \sup_{S_I} \inf_{S_{II}} E^x_{S_I,S_{II}} F(X_\tau). \tag{5.3}$$

□

6. Strong comparison principle

Theorem 6.1. *Assume that u and v are solutions of equation* (1.1) *on $\mathfrak{X} \setminus Y$, $\gamma \neq 0$, $u \le v$ on the boudary Y, and exists $x \in \mathfrak{X}$ such that $u(x) = v(x)$, then $u = v$ through the whole $\mathfrak{X}$.*

Proof. By Theorem 4.1 from the fact that $u \leq v$ on the boundary we know that $u \leq v$ on $\mathfrak{X}$. By definition of p-harmonious function we have

$$v(x) = \alpha \max_{y \in S(x)} v(y) + \beta \min_{y \in S(x)} v(y) + \gamma \fint_{S(x)} v(y)dy, \tag{6.1}$$

$$u(x) = \alpha \max_{y \in S(x)} u(y) + \beta \min_{y \in S(x)} u(y) + \gamma \fint_{S(x)} u(y)dy. \tag{6.2}$$

Since $u \geq v$ on $\mathfrak{X}$ we know that

$$\max_{y \in S(x)} v(y) \leq \max_{y \in S(x)} u(y),$$
$$\min_{y \in S(x)} v(y) \leq \min_{y \in S(x)} u(y),$$
$$\fint_{S(x)} v(y)dy \leq \fint_{S(x)} u(y)U(dy).$$

But since $u(x) = v(x)$, we actually have equalities

$$\max_{y \in S(x)} v(y) = \max_{y \in S(x)} u(y),$$
$$\min_{y \in S(x)} v(y) = \min_{y \in S(x)} u(y), \quad \fint_{S(x)} v(y)dy = \fint_{S(x)} u(y)dy.$$

From equality of average values and the fact that $u \geq v$ we conclude that $u = v$ on $S(x)$. Since our graph is connected, we immediately get the result. □

7. Remarks on unique continuation

We can pose the following question. Let E be a finite graph with the vertex set $\mathfrak{X}$ and let $B_R(x)$ be the ball of radius R contained within this graph. Here we assign to every edge of the graph length one and let

$$d(x, y) = \inf_{x \sim y} \{|x \sim y|\},$$

where $x \sim y$ is the path connecting vertex x to the vertex y and $|x \sim y|$ is the number of edges in this path. Assume that u is a p-harmonious function on $\mathfrak{X}$ and $u = 0$ on $B_R(x)$. Does this mean that $u = 0$ on $\mathfrak{X}$? It seems like the answer to this question depends on the values of u on the boundary Y, as well as properties of the graph E itself. Here we can provide simple examples for particular graph, which shows that u does not have to be zero through the whole $\mathfrak{X}$. See tables 1 and 2.

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TABLE 1. $p = 2$, 8 neighbors

164	−349	80	163	1	−164	1	163	96	−617	74
−349	−52	−19	28	1	−20	1	28	−38	−9	596
80	−19	−4	1	1	−2	1	1	−1	35	−217
163	28	1	0	0	0	0	0	1	−26	−26
1	1	1	0	0	0	0	0	−2	1	1
−164	−20	−2	0	0	0	0	0	1	7	52
1	1	1	0	0	0	0	0	1	1	1
163	28	1	0	0	0	0	0	−2	1	−53
80	−19	−4	1	1	−2	1	1	−1	−19	80
−349	−52	−19	28	1	−20	1	28	−19	2	−160
164	−349	80	163	1	−164	1	163	77	403	461

TABLE 2. $p = \infty$, 8 neighbors

−31	21	−11	−5	1	3	1	−5	11	−21	23
21	−5	5	−3	−1	1	−1	3	−5	1	21
−11	5	0	1	−1	0	1	−1	0	5	−11
−5	−3	1	0	0	0	0	0	1	−3	5
3	−1	−1	0	0	0	0	0	−1	−1	3
1	1	0	0	0	0	0	0	0	1	1
3	−1	1	0	0	0	0	0	1	−1	3
−5	3	−1	0	0	0	0	0	−1	3	−5
11	−5	0	1	−1	0	1	−1	0	−5	11
−21	1	5	−3	−1	1	−1	3	−5	5	−21
23	21	−11	−5	1	3	1	−5	11	−21	31

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