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SOLVABILITY OF A SECOND-ORDER MULTI-POINT BOUNDARY-VALUE PROBLEMS AT RESONANCE ON A HALF-LINE WITH DIM KER L=2

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Abstract. We show the existence of solutions for a second-order multi-point boundary-value problem at resonance on a half-line, where the dimension of the kernel of the differential operator is 2. Our main tools are the coincidence degree theory due to Mawhin, suitable operators, and algebraic methods. Our results are illustrated with an example.

1. Introduction

In this article, we show the existence of solutions for the boundary-value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0, +\infty), \tag{1.1}$$

$$x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \quad \lim_{t\to+\infty} x'(t) = \sum_{j=1}^{n} \beta_j x'(\eta_j), \tag{1.2}$$

where $f : [0, +\infty) \times R^2 \to R$, $e \in L^1[0, +\infty)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < +\infty$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < +\infty$, $m \geq 2$, $n \geq 1$.

Multi-point boundary value problems of ordinary differential equations arise in a variety of different areas of Applied Mathematics and Physics. For example, the vibrations of a guy wire of a uniform cross-section being composed of N parts of different densities can be set up as a multi-point boundary-value problem (see [19]); many problems in the theory of elastic stability can be handled by the method of multi-point problems(see [26]). Bridges of small size are often designed with two supported points, which leads to a standard two-point boundary condition and bridges of large size are sometimes contrived with multi-point supports, which corresponds to a multi-point boundary condition (see [30]).

Boundary-value problem (1.1)-(1.2) is called a problem at resonance if $Lx := x''(t) = 0$ has non-trivial solutions under the boundary condition (1.2); i.e., when $\dim\ker L \geq 1$.

On the finite interval [0,1], the first-order, second-order and high-order multi-point boundary-value problems at resonance have been studied by many authors (see [2, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 18, 20, 21, 22, 24, 25]), where $\dim\ker L = 1$.

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In [11, 28, 29], the second-order multi-point boundary-value problems at resonance have been discussed when $\dim\ker L = 2$ on the finite interval [0,1]. Recently, the boundary-value problems at resonance on the infinite interval with $\dim\ker L = 1$ has been investigated by many authors, see [12, 13, 27, 17]and references cited therein. Although the existing literature on solutions of multi-point boundary-value problems is quite wide, to the best of our knowledge, there is few paper to investigate the resonance case with $\dim\ker L = 2$ on the infinite interval.

Motivated by the above results, by constructing the suitable operators and getting help from the algebraic methods, we will show the existence of solutions for the second-order multi-point boundary-value problem at resonance on a half-line with $\dim\ker L = 2$, which brings many difficulties. And we give an example to illustrate our results. Some methods used in this paper are new and they can be used to solve the nth-order boundary-value problems at resonance with $1 < \dim\ker L \le n$.

This paper is organized as follows. In Section 2, some necessary backgrounds will be stated and some lemmas be proved. In Section 3, the main results will be given and proved. In Section 4, an example is given to illustrate our results.

In this article, we will assume the following conditions:

- (C1) $f : [0,+\infty) \times R^2 \to R$ is a S-Carathéodory function; i.e.,
  - (i) $f(t,\cdot)$ is continuous on R^2 for a.e. $t \in [0,+\infty)$.
  - (ii) $f(\cdot,x)$ is Lebesgue measurable on $[0,+\infty)$ for each $x \in \mathbb{R}^2$.
  - (iii) For each $r > 0$, there exists a function $\varphi_r \in L^1[0,+\infty)$, $\varphi_r(t) \ge 0$, $t \in [0,+\infty)$ satisfying $\int_0^{+\infty} s\varphi_r(s)ds < +\infty$ such that
$$|f(t,x)| \le \varphi_r(t), \quad \text{a. e. } t \in [0,+\infty),\ ||x|| < r.$$
- (C2) $\sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_j = 1$, $\sum_{i=1}^m \alpha_i\xi_i = 0$.
- (C3) $\Delta = \begin{vmatrix} Q_1e^{-t} & Q_2e^{-t} \\ Q_1te^{-t} & Q_2te^{-t} \end{vmatrix} := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \ne 0$, where
$$Q_1y = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} (\xi_i - s)y(s)ds, \quad Q_2y = \sum_{j=1}^n \beta_j \int_{\eta_j}^{+\infty} y(s)ds.$$

2. Preliminary

To obtain our results, we introduce some notation and two theorems.

Let X and Y be real Banach spaces and let $L : \mathrm{dom}(L) \subset X \to Y$ be a Fredholm operator with index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that
$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$
It follows that
$$L|_{\mathrm{dom}\, L\cap\ker P} : \mathrm{dom}\, L \cap \ker P \to \operatorname{Im} L$$
is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X, $\mathrm{dom}\, L \cap \overline{\Omega} \ne \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 (citem4). *Let $L : \mathrm{dom}\, L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

1. *$Lx \ne \lambda Nx$ for every $(x,\lambda) \in [(\mathrm{dom}\, L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;*
2. *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$;*

(3) $deg(QN|_{\ker L},\ \Omega\cap\ker L, 0)\neq 0$, *where* $Q: Y\to Y$ *is a projection such that* $\operatorname{Im} L=\ker Q$.

Then the equation $Lx=Nx$ *has at least one solution in* $\operatorname{dom} L\cap\overline{\Omega}$.

Let
$$X=\{x\in C^1[0,+\infty): \lim_{t\to+\infty}\frac{|x(t)|}{1+t} \text{ and } \lim_{t\to+\infty}x'(t) \text{ exist}\}$$
with norm $\|x\|=\max\{\|x\|_0,\ \|x'\|_\infty\}$, where
$$\|x\|_0=\sup_{t\in[0,\infty)}\frac{|x(t)|}{1+t},\quad \|x\|_\infty=\sup_{t\in[0,+\infty)}|x(t)|.$$
It is easy to prove that $(X,\|\cdot\|)$ is a Banach space.

Theorem 2.2 ([1]). *Let* $M\subset X$. *Then* M *is relatively compact if the following conditions hold:*

(a) M *is bounded in* X;
(b) *the functions belonging to* M *are equi-continuous on any compact interval of* R^+;
(c) *the functions from* M *are equi-convergent at* $+\infty$.

Let $Y=L^1[0,+\infty)$ with the norm $\|y\|_1=\int_0^{+\infty}|y(s)|ds$. Define $Lx=x''$, with domain
$$\operatorname{dom} L=\{x\in X: x''\in L^1[0,+\infty), x(0)=\sum_{i=1}^m\alpha_i x(\xi_i), \lim_{t\to+\infty}x'(t)=\sum_{j=1}^n\beta_j x'(\eta_j)\}.$$
Obviously, $\ker L=\{a+bt: a,b\in\mathbb{R}\}$. Now, we will prove that
$$\operatorname{Im} L=\{y\in Y: Q_1y=Q_2y=0\}.$$
In fact, if $Lx=y$, then $y\in Y$ and
$$x(t)=x(0)+x'(0)t+\int_0^t(t-s)y(s)ds.$$
It follows from (1.2) that $Q_1y=Q_2y=0$.

On the other hand, assume $y\in Y$ satisfying $Q_1y=Q_2y=0$. Take
$$x(t)=\int_0^t(t-s)y(s)ds.$$
Then $x\in X$, $x''(t)=y(t)$ and x satisfies (1.2). So, $x\in\operatorname{dom} L$; i.e., $y\in\operatorname{Im} L$.

Define operators $T_1,\ T_2: Y\to Y$ as follows:
$$T_1y=\frac{1}{\Delta}(\Delta_{11}Q_1y+\Delta_{12}Q_2y)e^{-t},\quad T_2y=\frac{1}{\Delta}(\Delta_{21}Q_1y+\Delta_{22}Q_2y)e^{-t},$$
where Δ_{ij} is the algebraic cofactor of a_{ij}. Define the operator $Q: Y\to Y$ by
$$Qy=T_1y+(T_2y)\cdot t.$$
By a simple calculation, we obtain $T_1(T_1y)=T_1y$, $T_1(T_2yt)=0$, $T_2(T_1y)=0$, $T_2(T_2yt)=T_2y$. So, $Q^2y=Qy$; i.e., $Q: Y\to Y$ is a linear projector. Obviously, Q is continuous.

For $y\in Y$, $y=(y-Qy)+Qy$, we have $Qy\in\operatorname{Im} Q$ and $Q(y-Qy)=0$. It follows from $Q(y-Qy)=0$, the definitions of Q, T_1, T_2 and condition (C3), that $Q_1(y-Qy)=Q_2(y-Qy)=0$; i.e., $y-Qy\in\operatorname{Im} L$. So, $Y=\operatorname{Im} L+\operatorname{Im} Q$.

Take $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, then $y = Qy = 0$; i.e., $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$, So, we have $\dim \ker L = \operatorname{codim} \operatorname{Im} L = 2$, thus L is a Fredholm operator with index zero.

Define the continuous projection $P : X \to \ker L$ by

$$(Px)(t) = x(0) + x'(0)t, \quad t \in [0, +\infty).$$

Then $X = \ker L \oplus \ker P$.

Define the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ by

$$K_P y = \int_0^t (t-s)y(s)ds.$$

Then K_P is the inverse operator of $L|_{\operatorname{dom} L \cap \ker P}$ and

$$\|K_P y\| \le \|y\|_1. \tag{2.1}$$

In fact, for $x \in \operatorname{dom} L \cap \ker P$, $K_P L(x) = \int_0^t (t-s)x''(s)ds = x(t)$. On the other hand, for $y \in \operatorname{Im} L, LK_P(y) = (\int_0^t (t-s)y(s)ds)'' = y(t)$. By

$$\frac{|K_P y|}{1+t} \le \int_0^{+\infty} |y(s)|ds = \|y\|_1,$$
$$|(K_P y)'(t)| = |\int_0^t y(s)ds| \le \|y\|_1,$$

we obtain (2.1).

Let the nonlinear operator $N : X \to Y$ be defined by

$$Nx = f(t, x(t), x'(t)) + e(t), \quad t \in [0, +\infty).$$

Then problem (1.1)–(1.2) is equivalent to

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

Lemma 2.3. *Suppose that Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \overline{\Omega} \neq \Phi$. Then N is L-compact on $\overline{\Omega}$.*

Proof. Since Ω is bounded, there exists a constant $r > 0$ such that $\|x\| \le r$ for any $x \in \overline{\Omega}$. For $x \in \overline{\Omega}$, by (C1), we obtain

$$\begin{aligned} |Q_1 Nx| &= |\sum_{i=1}^m \alpha_i \int_0^{\xi_i} (\xi_i - s)[f(s, x(s), x'(s)) + e(s)]ds| \\ &\le \sum_{i=1}^m |\alpha_i \xi_i| \int_0^{+\infty} \varphi_r(s) + |e(s)|ds := l_1 \end{aligned}$$

and

$$\begin{aligned} |Q_2 Nx| &= |\sum_{j=1}^n \beta_j \int_{\eta_j}^{+\infty} f(s, x(s), x'(s)) + e(s)ds| \\ &\le \sum_{j=1}^n |\beta_j| \cdot \int_0^{+\infty} \varphi_r(s) + |e(s)|ds := l_2. \end{aligned}$$

Thus,

$$\begin{aligned} \|QNx\|_1 &= \int_0^{+\infty} |QNx(s)|ds \\ &\le \int_0^{+\infty} |T_1Nx(s)|ds + \int_0^{+\infty} |T_2Nx(s)|sds \\ &\le \frac{1}{|\Delta|}[|\Delta_{11}|\cdot|Q_1Nx| + |\Delta_{12}|\cdot|Q_2Nx|] \\ &\quad + \frac{1}{|\Delta|}[|\Delta_{21}|\cdot|Q_1Nx| + |\Delta_{22}|\cdot|Q_2Nx|] \\ &\le \frac{1}{|\Delta|}[(|\Delta_{11}|+|\Delta_{21}|)l_1 + (|\Delta_{12}|+|\Delta_{22}|)l_2]. \end{aligned} \tag{2.2}$$

So, $QN(\overline{\Omega})$ is bounded. Now, we will prove that $K_P(I-Q)N(\overline{\Omega})$ is compact.

(a). Obviously, $K_P(I-Q)N : \overline{\Omega} \to Y$ is continuous. For $x \in \overline{\Omega}$, since

$$\|Nx\|_1 = \int_0^{+\infty} |f(s,x(s),x'(s)) + e(s)|ds \le \int_0^{+\infty} \varphi_r(s) + |e(s)|ds := l_3, \tag{2.3}$$

$$\begin{aligned} \frac{|K_P(I-Q)Nx(t)|}{1+t} &= \frac{1}{1+t}\Big|\int_0^t (t-s)(I-Q)Nx(s)ds\Big| \\ &\le \int_0^{+\infty} |Nx(s)| + |QNx(s)|ds \\ &= \|Nx\|_1 + \|QNx\|_1, \end{aligned}$$

and

$$\begin{aligned} |[K_P(I-Q)Nx]'(t)| &= \Big|\int_0^t (I-Q)Nx(s)ds\Big| \\ &\le \int_0^{+\infty} |Nx(s)| + |QNx(s)|ds \\ &= \|Nx\|_1 + \|QNx\|_1, \end{aligned}$$

by (2.2) and (2.3), we obtain that $K_P(I-Q)N(\overline{\Omega})$ is bounded.

(b). For any $T \in [0,+\infty)$, we will prove that functions belonging to $K_P(I-Q)N(\overline{\Omega})$ are equi-continuous on $[0,T]$. In fact, for $x \in \overline{\Omega}$, we have

$$|Nx(s)| \le \varphi_r(s) + |e(s)|. \quad s \in [0,\infty), \tag{2.4}$$

$$|QNx(s)| \le \frac{1}{|\Delta|}[(|\Delta_{11}|l_1 + |\Delta_{12}|l_2) + (|\Delta_{21}|l_1 + |\Delta_{22}|l_2)s]e^{-s}. \tag{2.5}$$

For any $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\begin{aligned} &\Big|\frac{K_P(I-Q)Nx(t_1)}{1+t_1} - \frac{K_P(I-Q)Nx(t_2)}{1+t_2}\Big| \\ &= \Big|\frac{\int_0^{t_1}(t_1-s)(I-Q)Nx(s)ds}{1+t_1} - \frac{\int_0^{t_2}(t_2-s)(I-Q)Nx(s)ds}{1+t_2}\Big| \\ &\le \Big|\frac{t_1}{1+t_1}\int_0^{t_1}(I-Q)Nx(s)ds - \frac{t_2}{1+t_2}\int_0^{t_2}(I-Q)Nx(s)ds\Big| \\ &\quad + \Big|\frac{1}{1+t_1}\int_0^{t_1} s(I-Q)Nx(s)ds - \frac{1}{1+t_2}\int_0^{t_2} s(I-Q)Nx(s)ds\Big| \end{aligned}$$

$$
\begin{aligned}
&\le |\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}|\cdot\int_0^{+\infty}|Nx(s)|+|QNx(s)|ds+\int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds\\
&\quad+|\frac{1}{1+t_1}-\frac{1}{1+t_2}|\cdot T\int_0^{+\infty}|Nx(s)|+|QNx(s)|ds\\
&\quad+T\cdot\int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds\\
&=(|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}|+|\frac{1}{1+t_1}-\frac{1}{1+t_2}|\cdot T)(\|Nx\|_1+\|QNx\|_1)\\
&\quad+(1+T)\int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds.
\end{aligned}
$$

and

$$
\begin{aligned}
|[K_P(I-Q)Nx]'(t_1)-[K_P(I-Q)Nx]'(t_2)|&=|\int_{t_1}^{t_2}(I-Q)Nx(s)ds|\\
&\le\int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds.
\end{aligned}
$$

By (2.2)–(2.5), the continuity of $\frac{t}{1+t}$ and $\frac{1}{1+t}$ and the absolute continuity of integral, we obtain that functions from $K_P(I-Q)N(\overline{\Omega})$ are equi-continuous on $[0,T]$.

(c). Now, we will show that functions in $K_P(I-Q)N(\overline{\Omega})$ are equi-convergent at $+\infty$. For $x\in\overline{\Omega}$, we have

$$
\begin{aligned}
\lim_{t\to+\infty}\frac{K_P(I-Q)Nx(t)}{1+t}&=\int_0^{+\infty}(I-Q)Nx(s)ds.\\
\lim_{t\to+\infty}[K_P(I-Q)Nx]'(t)&=\int_0^{+\infty}(I-Q)Nx(s)ds.
\end{aligned}
$$

By

$$
\begin{aligned}
&|\frac{K_P(I-Q)Nx(t)}{1+t}-\int_0^{+\infty}(I-Q)Nx(s)ds|\\
&\le|\frac{t}{1+t}\int_0^t(I-Q)Nx(s)ds-\int_0^{+\infty}(I-Q)Nx(s)ds|\\
&\quad+\frac{1}{1+t}\int_0^t|s(I-Q)Nx(s)|ds\\
&\le\int_t^{+\infty}|(I-Q)Nx(s)|ds\\
&\quad+\frac{1}{1+t}\Big[\int_0^{+\infty}|(I-Q)Nx(s)|ds+\int_0^{+\infty}|s(I-Q)Nx(s)|ds\Big]\\
&\le\int_t^{+\infty}|Nx(s)|+|QNx(s)|ds+\frac{1}{1+t}\int_0^{+\infty}(1+s)[|Nx(s)|+|QNx(s)|]ds,
\end{aligned}
$$

and

$$
|[K_P(I-Q)Nx]'(t)-\int_0^{+\infty}(I-Q)Nx(s)ds|\le\int_t^{+\infty}|Nx(s)|+|QNx(s)|ds,
$$

From (2.4) and (2.5), we can get that functions from $K_P(I-Q)N(\overline{\Omega})$ are equicontinuous at $+\infty$. By Theorem 2.2, we obtain that $K_P(I-Q)N(\overline{\Omega})$ is compact. Therefore, N is $L-$compact on $\overline{\Omega}$. □

3. Main results

The following theorem is our main result.

Theorem 3.1. *Assume that* (C1)–(C3) *and the following conditions hold:*

- (H1) *There exist functions* $\alpha(t), \beta(t), \gamma(t), \delta(t) \in L^1[0,+\infty)$*, and* $\theta \in [0,1)$ *such that either*
$$|f(t,u,v)| \le \alpha(t) + \beta(t)\frac{|u|}{1+t} + \gamma(t)|v| + \delta(t)(\frac{|u|}{1+t})^\theta$$
or
$$|f(t,u,v)| \le \alpha(t) + \beta(t)\frac{|u|}{1+t} + \gamma(t)|v| + \delta(t)|v|^\theta;$$
- (H2) *There exist constants* $A > 0$, $B > 0$ *such that, if* $|x(t)| > A$ *for every* $t \in [0,B]$ *or* $|x'(t)| > A$ *for every* $t \in [0,+\infty)$*, then either* $Q_1Nx \neq 0$ *or* $Q_2Nx \neq 0$*, where* $\|\beta\|_1 + \|\gamma\|_1 < \frac{1}{2+B}$*;*
- (H3) *There exists a constant* $C > 0$ *such that, if* $|a| > C$ *or* $|b| > C$*, then either*
  (1) $aQ_1N(a+bt) + bQ_2N(a+bt) < 0$*, or*
  (2) $aQ_1N(a+bt) + bQ_2N(a+bt) > 0$.

Then the boundary-value problem (1.1)–(1.2) *has at least one solution in* X*.*

Proof. We divide the proof into four steps.

Step1. Let
$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda Nx, \text{ for some } \lambda \in [0,1]\}.$$
We will prove that Ω_1 is bounded. In fact, $x \in \Omega_1$ means $\lambda \neq 0$ and $Nx \in \operatorname{Im} L$. Thus
$$Q_1Nx = Q_2Nx = 0.$$
By (H2), there exist $t_0 \in [0,B]$, $t_1 \in [0,+\infty)$ such that
$$|x(t_0)| \le A, \quad |x'(t_1)| \le A.$$
So,
$$|x'(t)| = |x'(t_1) - \int_t^{t_1} x''(s)ds| \le A + \int_t^{t_1} |Nx(s)|ds \le A + \|Nx\|_1;$$
i.e., $\|x'\|_\infty \le A + \|Nx\|_1$. Considering
$$\begin{aligned}|x(0)| &= |x(t_0) - \int_0^{t_0} x'(s)ds| \le A + |\int_0^{t_0} x'(s)ds| \\ &\le A + \|x'\|_\infty \cdot B \le A(1+B) + B\cdot\|Nx\|_1,\end{aligned}$$
we have
$$\|Px\| \le |x(0)| + |x'(0)| \le A(2+B) + (1+B)\|Nx\|_1.$$
By $LPx = 0$, (2.1) and (H1), we obtain
$$\begin{aligned}\|x\| &= \|Px + (I-P)x\| \le \|Px\| + \|K_PL(I-P)x\| \\ &\le \|Px\| + \|Lx\|_1 \le \|Px\| + \|Nx\|_1 \\ &\le (2+B)(A + \|Nx\|_1) \\ &\le (2+B)(A + \|\alpha\|_1 + \|\beta\|_1\cdot\|x\| + \|\gamma\|_1\cdot\|x\| + \|\delta\|_1\cdot\|x\|^\theta + \|e\|_1).\end{aligned}$$

So,

$$\|x\| \le \frac{2+B}{1-(2+B)(\|\beta\|_1+\|\gamma\|_1)}(A+\|\alpha\|_1+\|e\|_1+\|\delta\|_1\cdot\|x\|^\theta).$$

It follows from $\theta \in [0,1)$ that Ω_1 is bounded.

Step2. Set $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$. Then Ω_2 is bounded. In fact, $x \in \Omega_2$ implies $x = a+bt$ and $Q_1N(a+bt) = Q_2N(a+bt) = 0$. By (H_3), we obtain $|a| \le C$, $|b| \le C$. So, Ω_2 is bounded.

Step3. Define the isomorphism $J : \ker L \to \operatorname{Im} Q$ by

$$J(a+bt) = \frac{1}{\Delta}[\Delta_{11}a + \Delta_{12}b + (\Delta_{21}a + \Delta_{22}b)t]e^{-t}.$$

Assume (H3)(1) holds. Let

$$\Omega_3 = \{x \in \ker L : -\lambda Jx + (1-\lambda)QNx = 0,\ \text{for some } \lambda \in [0,1]\}.$$

Then Ω_3 is bounded.

In fact, $x \in \Omega_3$ means that there exist constants $a, b \in \mathbb{R}$, $\lambda \in [0,1]$ such that $x = a+bt$ and $\lambda Jx = (1-\lambda)QNx$. If $\lambda = 0$, then $QNx = 0$. So,

$$\begin{aligned} \Delta_{11}Q_1Nx + \Delta_{12}Q_2Nx = 0, \\ \Delta_{21}Q_1Nx + \Delta_{22}Q_2Nx = 0. \end{aligned}$$

It follows from $\Delta \ne 0$ that $Q_1Nx = Q_2Nx = 0$. By (H3), we obtain $|a| \le C$, $|b| \le C$.

If $\lambda = 1$, we can similarly get $a = b = 0$. For $\lambda \in (0,1)$, by $\lambda Jx = (1-\lambda)QNx$, we obtain

$$\begin{aligned} \lambda\Delta_{11}a + \lambda\Delta_{12}b = (1-\lambda)\Delta_{11}Q_1N(a+bt) + (1-\lambda)\Delta_{12}Q_2N(a+bt), \\ \lambda\Delta_{21}a + \lambda\Delta_{22}b = (1-\lambda)\Delta_{21}Q_1N(a+bt) + (1-\lambda)\Delta_{22}Q_2N(a+bt). \end{aligned}$$

It follows from $\Delta \ne 0$ that

$$\begin{aligned} \lambda a = (1-\lambda)Q_1N(a+bt), \\ \lambda b = (1-\lambda)Q_2N(a+bt). \end{aligned}$$

If $|a| > C$, $|b| > C$, by (H3)(1), we obtain

$$\lambda(a^2+b^2) = (1-\lambda)[aQ_1N(a+bt) + bQ_2N(a+bt)] < 0,$$

a contradiction. So, Ω_3 is bounded.

Remark 3.2. If (H3)(2) holds, take

$$\Omega_3 = \{x \in \ker L : \lambda Jx + (1-\lambda)QNx = 0,\ \text{for some } \lambda \in [0,1]\}.$$

We can similarly prove that Ω_3 is bounded.

Step4. Take an open bounded set $\Omega \supset \bigcup_{i=1}^3 \overline{\Omega_i} \bigcup \{0\}$. We will prove that (1.1)–(1.2) has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

By Step1 and Step2, we obtain

(1) $Lx \ne \lambda Nx$, for every $(x,\lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;
(2) $Nx \notin \operatorname{Im} L$, for every $x \in \ker L \cap \partial\Omega$.

Now we will show that

(3) $\deg(QN|_{\ker L}, \Omega \cap \ker L,\ 0) \ne 0$.

Let $H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$. By step 3, we know that $H(x,\lambda) \neq 0$, for every $(x,\lambda) \in (\ker L \cap \partial\Omega) \times [0,1]$. Thus, by the homotopy property of degree, we obtain

$$\begin{aligned}\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(H(\cdot,0), \Omega \cap \ker L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \ker L, 0)\\ &= \deg(\pm J,\ \Omega \cap \ker L, 0) = \pm 1 \neq 0.\end{aligned}$$

By Theorem 2.1, we can get that $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$; i.e. , (1.1)–(1.2) has at least one solution in X. The prove is completed. □

4. Example

Let's consider the boundary-value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in [0,\infty), \tag{4.1}$$

$$x(0) = 2x(1) - x(2), \quad x'(\infty) = x'(2), \tag{4.2}$$

where

$$f(t, x(t), x'(t)) = \begin{cases} -e^{-10t}x(0), & 0 \le t \le 2,\\ e^{-10t} sin x'(t) + e^{-t}\sqrt[3]{x'(t)}, & t > 2.\end{cases}$$

$$e(t) = \begin{cases} 0, & 0 \le t \le 2,\\ te^{-t}, & t > 2.\end{cases}$$

Corresponding to problem (1.1)-(1.2), we have that $m = 2$, $n = 1$, $\alpha_1 = 2$, $\alpha_2 = -1$, $\xi_1 = 1$, $\xi_2 = 2$, $\beta_1 = 1$, $\eta_1 = 2$. Obviously, (C1) and (C2) are satisfied. By simple calculation, we obtain $a_{11} = -(1-e^{-1})^2$, $a_{21} = 6e^{-1} - 2 - 4e^{-2}$, $a_{12} = e^{-2}$, $a_{22} = 3e^{-2}$.

$$\Delta = \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = e^{-4} - e^{-2} \neq 0\,.$$

So, (C3) is satisfied. Take $\alpha(t) = 0$, $\theta = \frac{1}{3}$,

$$\beta(t) = \begin{cases} (1+t)e^{-10t}, & 0 \le t \le 2,\\ 0, & t > 2,\end{cases} \quad \gamma(t) = \begin{cases} 0, & 0 \le t \le 2,\\ e^{-10t}, & t > 2,\end{cases}$$

$$\delta(t) = \begin{cases} 0, & 0 \le t \le 2,\\ e^{-t}, & t > 2.\end{cases}$$

Then f satisfies (H1). We can easily get that $\|\beta\|_1 = \frac{1}{10}[\frac{11}{10} - \frac{31}{10}e^{-20}]$, $\|\gamma\|_1 = \frac{1}{10}e^{-20}$. So, we have $\|\beta\|_1 + \|\gamma\|_1 < 1/5$.

Let $B = 2$, $A = e^{-54}/1000$. We get that $Q_1Nx \neq 0$ if $|x(t)| > A$, for any $t \in [0,2]$ and $Q_2Nx \neq 0$ if $|x'(t)| > A$, for any $t \in [0,\infty)$. This means that (H2) is satisfied.

Set $C = 100$. We can easily get that

$$aQ_1N(a+bt) + bQ_2N(a+bt) > 0$$

if $|a| > C$ or $|b| > C$. So, (H3) is satisfied.

By theorem 3.1, we obtain that problem (4.1)–(4.2) has at least one solution.

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