Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 121, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

COMPUTATION OF RATIONAL SOLUTIONS FOR A FIRST-ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this article, we study differential equations of the form $y' = \sum A_i(x)y^i / \sum B_i(x)y^i$ which can be elliptic, hyperbolic, parabolic, Riccati, or quasi-linear. We show how rational solutions can be computed in a systematic manner. Such results are most likely to find applications in the theory of limit cycles as indicated by Giné et al [4].

1. Introduction

When confronting an unfamiliar differential equation, it is natural to try to find the simplest type of solutions such as polynomial and rational solutions. Indeed, exact solutions (such as polynomial and rational solutions) for the nonlinear differential equation

$$\frac{dy}{dx} = P(x,y) \tag{1.1}$$

are of great interests, in particular understanding the whole set of solutions and their dynamical properties. In 1936, Rainville [7] determined all the Riccati differential equations of the form

$$\frac{dy}{dx} = y^2 + A_1(x)y + A_0(x)$$

with A_0 and A_1 polynomials, which have polynomial solutions. In 1954, Campbell and Golomb [4] provided an algorithm for finding all the polynomial solutions of the differential equation

$$B_0(x)\frac{dy}{dx} = A_2(x)y^2 + A_1(x)y + A_0(x)$$

with B_0, A_0, A_1 and A_2 polynomials. In 2006, Behloul and Cheng [1] (see also [2]) gave another algorithm for looking for the rational solutions of the equation

$$B_0(x)\frac{dy}{dx} = A_n(x)y^n + A_{n-1}(x)y^{n-1} + \cdots + A_0(x)$$

2000 *Mathematics Subject Classification.* 34A05.
Key words and phrases. Polynomial solution; rational solution; nonlinear differential equation.

Submitted August 23, 2011. Published September 19, 2011.
D. Behloul is supported by a national PNR (2011-2013) grant.

with B_0 and A_i polynomials. In 2011, Giné et al [4] developed new results about 'periodic' polynomial solutions for

$$\frac{dy}{dx} = A_n(x)y^n + A_{n-1}(x)y^{n-1} + \dots + A_0(x) \tag{1.2}$$

with A_i polynomials. Such results give rise to sharp information on the number of polynomial limit cycles. These conclusions are important since the theory of limit cycles is an active and difficult research area. For a concise list of references related to limit cycles of (1.2), including the works by Abel, Briskin, Gasull, Llibre, Neto, Lloyd, the readers is referred to [4].

Clearly, equations of the form (1.2) are among the easiest of equations of the form (1.1). The next level of difficulty will come from studying the case when $P(x, y)$ is a rational function. In this paper, we are concerned with the rational solutions of the differential equation

$$y' = \frac{A_n(x)y^n + A_{n-1}(x)y^{n-1} + \dots + A_0(x)}{B_m(x)y^m + B_{m-1}(x)y^{m-1} + \dots + B_0(x)}, \tag{1.3}$$

where $A_0, A_1, \dots, A_n$ and $B_0, B_1, \dots, B_m$ are (complex valued) polynomials (of one independent complex variable) such that A_n and B_m are not identically zero.

By providing a systematic scheme for computing all the rational solutions of (1.3), we hope that our results lead to estimates of the number of 'rational limit cycles', more general than those in Giné et al [4], and to qualitative results for non-linear equations of the form (1.3) but with an additional nonlinear perturbations.

As another motivation for our study, we quote a result by Malmquist [6] which states: If the differential equation (1.3) is not one of the two forms

$$B_0(x)\frac{dy}{dx} = A_1(x)y + A_0(x)$$

or

$$B_0(x)\frac{dy}{dx} = A_2(x)y^2 + A_1(x)y + A_0(x)$$

then all its one-valued solutions must be *rational*. For example the equation $\frac{dy}{dx} = y$ admits e^x as a one-valued solution which is not rational and the equation $\frac{dy}{dx} = 1+y^2$ admits $\tan x$ as a one-valued solution which is not rational.

Clearly, equation (1.3) is only defined at places where the denominator does not vanish. However, a root of the denominator may also be a root of the numerator and (1.3) may still be meaningful by assigning proper values to the rational function on the right-hand side. To avoid such technical details, we will define a polynomial solution to be a polynomial function $y = y(x)$ such that

$$\begin{aligned} &y'(x)\{B_m(x)y^m + B_{m-1}(x)y^{m-1} + \dots + B_0(x)\} \\ &\equiv A_n(x)y^n + A_{n-1}(x)y^{n-1} + \dots + A_0(x); \end{aligned} \tag{1.4}$$

and a rational solution to be a pair of polynomials $(U(x), V(x))$ such that the degree of V is greater than or equal to 1 and

$$\begin{aligned} &(V(x)U'(x) - U(x)V'(x))\{B_m(x)y^m + \dots + B_0(x)t\} \\ &\equiv V^2(x)\{A_n(x)y^n + \dots + A_0(x)\}. \end{aligned} \tag{1.5}$$

Since the right-hand sides and the left-hand sides are polynomials, singularities are thus avoided.

To motivate what follows, let us consider the specific example

$$y' = \frac{y^3 + 2x}{2x^2 y + x}. \tag{1.6}$$

Suppose we try to find a constant (polynomial) solution of the form $y(x) = \lambda$. Then substituting it into the above equation, we see that

$$0 = 0 \cdot \{2x^2\lambda + x\} \equiv \lambda^3 + 2x$$

for all x, which is impossible. Next, we try polynomial solutions with degree 1. Then $y''(x) \equiv 0$, such that

$$0 = y'' = \Big(\frac{y^3 + 2x}{2x^2 y + x}\Big)' = \frac{y'(-4x^2 + 4xy^3 + 3y^2)}{x(2xy+1)^2} - \frac{1}{x^2}\frac{y(4x^2 + 4xy^3 + y^2)}{(2xy+1)^2}.$$

Replacing y' by $\frac{y^3+2x}{2x^2y+x}$ in the above equation and rearranging term,

$$\begin{aligned} &[(-4x)y^5 + (8x^2 - 3)y^4 + 6xy^3 + (1 - 4x^2)y^2 + (8x^3 - 6x)y + (4x^2)]y \\ &\equiv -8x^3. \end{aligned} \tag{1.7}$$

Thus $y(x)$ is a factor of the polynomial x^3. Hence $y = \lambda x$ for a nonzero number λ. Then from (1.6),

$$\lambda(2x^2\lambda + 1) \equiv \lambda^3 x^2 + 2,$$

so that $\lambda = 2$. We may easily check that $y(x) = 2x$ is indeed a solution of (1.6).

Next we may try polynomials with higher degrees of course. But we should stop for a while and consider the existence and uniqueness of all polynomial and rational solutions as well as schematic methods for computing them. To this end, we first settle on a convenient notation. We will let $\mathbb{N}$ be the set of nonnegative integers, $\mathbb{N}^*$ the set of positive integers, and $\mathbb{C}$ the set of complex numbers. When $G = G(x)$ is a nontrivial polynomial, its degree is denoted by $\deg G(x)$, and when it is the zero polynomial, its degree is defined to be $-\infty$. When $H = H(x,y)$ is a bivariate polynomial of the form

$$H(x,y) = h_n(x)y^n + h_{n-1}(x)y^{n-1} + \dots + h_0(x)$$

where $h_0, \dots, h_n$ are polynomials with h_n not identically zero, then $\deg_y H(x,y)$ is taken to be n (e.g., if $H(x,y) = 3xy^2 + y$ then $\deg_y H(x,y) = 2$, although $H(0,y) = y$). We will set $a_i = \deg A_i(x)$ for $i = 0, 1, \dots, n$ and $b_i = \deg B_i(x)$ for $i = 0, 1, 2, \dots, m$,

$$P(x,y) = A_n(x)y^n + A_{n-1}(x)y^{n-1} + \dots + A_0(x), \tag{1.8}$$

$$Q(x,y) = B_m(x)y^m + B_{m-1}(x)y^{m-1} + \dots + B_0(x). \tag{1.9}$$

Let us also write A_n and B_m in the form

$$\begin{aligned} A_n(x) &= Ax^{a_n} + \dots, \\ B_m(x) &= Bx^{b_m} + \dots \end{aligned}$$

where $A, B \neq 0$.

The derivative of a function $g(x)$ of one variable is denoted by $g'(x)$ or $g^{(1)}(x)$ and the higher order derivatives by $g^{(2)}(x), g^{(3)}(x), \dots$ as usual and partial derivatives of a function $H(x,y)$ of two variables are denoted respectively by $H'_x(x,y)$ and $H'_y(x,y)$.

Let $G = G(x)$ be a polynomial. We recall that the multiplicity of G at α is defined to be 0 if α is not a root of G, and be the positive integer s if α is a root of G with multiplicity s. Let $H = H(x)$ be another polynomial which is not identically zero. For the rational function $F(x) = G(x)/H(x)$, if F is not identically zero, its valuation $v_\alpha(F)$ at α is the difference of the multiplicity of F at α and the multiplicity of G at α; otherwise, its valuation is $+\infty$. For example, if $F(x) = x(x+1)/(x^3 - 2x^2)$, then $v_0(F) = 1 - 2 = -1$, $v_{-1}(F) = 1 - 0 = 1$, $v_2(F) = 0 - 1 = -1$ and $v_\alpha(F) = 0$ if $\alpha \notin \{-1, 0, 2\}$.

In the rest of our discussions, we will assume that P and Q are coprime; i.e., $\gcd(P, Q) = 1$. Since $n, m \in \mathbb{N}$, we may classify (1.3) into five mutually distinct and exhaustive cases:

Case I: If $n > m + 2$, then (1.3) is said to be *elliptic*.

Case II: If $n < m + 2$ and $m \neq 0$, then (1.3) is said *to be hyperbolic.*

Case III: If $n = m + 2$ and $m \neq 0$, then (1.3) is said to be *parabolic.*

Case IV: If $(n, m) = (2, 0)$, then (1.3) is said to be Riccati.

Case V: If $(n, m) = (0, 0)$ or $(1, 0)$, then (1.3) is said to be quasi-linear.

We intend to show the following results:

- If (1.3) is not quasi-linear, then it has a finite number of polynomial solutions, and they can be computed in a systematic manner.
- If (1.3) is neither quasi-linear nor Riccati, then it has a finite number of rational solutions.
- If (1.3) is hyperbolic or elliptic, then all its rational solutions can be generated by polynomial solutions of another differential equation of the same form.
- If (1.3) is parabolic, we can compute all the rational solutions of (1.3) provided we have at least one particular rational solution.
- If (1.3) is quasi-linear, then we can compute all its polynomial and rational solutions in a systematic manner, although the number of polynomial or rational solutions may be infinite.
- If (1.3) is Riccati, we can compute all its rational solutions provided we have at least one particular rational solution, although the number of rational solutions may be infinite.

2. Polynomial solutions

It is easy to determine the set of all constant polynomial solutions of (1.3). We simply substitute $y(z) = \lambda$ into (1.3) to obtain

$$A_n(x)\lambda^n + A_{n-1}(x)\lambda^{n-1} + \dots + A_0(x) \equiv 0.$$

By expanding the left-hand side into a polynomial in x, and then comparing coefficients on both sides of the resulting equation, we may then obtain a finite system of polynomial equations in λ:

$$H_i(\lambda) = 0, \quad i = 1, 2, \dots, a = \max\{a_0, a_1, \dots, a_n\}.$$

If an H_i is a nonzero constant polynomial, then λ cannot exist. Else, we may let H be the greatest common divisor of $H_0, H_1, \dots, H_a$. Then λ equals to one of the roots of H.

Next, we seek nonconstant polynomial solutions. First note that if $y = y(x)$ is a polynomial solution of (1.4) with degree $d \geq 1$, then $\deg(A_i y^i) = a_i + id$ for

$i = 0, 1, \dots, n$, $\deg(B_i y^i y') = b_i + id + (d-1)$ for $i = 0, 1, \dots, m$. This motivates us to define $n+m+1$ indices $f_0(d), f_1(d), \dots, f_{n+m+1}(d)$ by

$$f_i(d) = a_i + id, \quad i = 0, \dots, n;$$
$$f_{i+n+1}(d) = b_i + id + d - 1, \; i = 0, \dots, m.$$

We will also let

$$f(d) = \max\{f_0(d), f_1(d), \dots, f_{n+m+2}(d)\}, \quad d = 1, 2, \dots .$$

Lemma 2.1. *If $y = y(x)$ is a polynomial solution of (1.4) with degree $d \geq 1$, then there exists $i, j \in \{0, 1, \dots, n+m+1\}$ such that $i < j$ and*

$$f_i(d) = f_j(d) = f(d).$$

Proof. Let

$$y(x) = y_d x^d + y_{d-1} x^{d-1} + \dots + y_1 x + y_0, \; y_d \neq 0, \tag{2.1}$$

be a polynomial solution of (1.3) with degree $d \geq 1$. Then $\deg(A_i y^i) = f_i(d)$ for $i = 0, 1, \dots, n$, and $\deg(B_i y^i y') = f_{i+n+1}$ for $i = 0, 1, \dots, m$. Let i be the least nonnegative integer such that $f_i(d) = f(d)$. By substituting y into (1.4), we see that

$$B_m(x) y'(x) y^m(x) + \dots + B_0(x) y'(x) \equiv A_n(x) y^n(x) + \dots + A_0(x).$$

Suppose $f_j(d) < f_i(d)$ for all $j \neq i$. If $i \in \{0, 1, \dots, n\}$, then by rearranging the above identity, we see that

$$Q x^{f_i(d)} + W(x) \equiv 0$$

where $W(x)$ is a polynomial of degree strictly less than $f_i(d)$, and Q is the product of y_d^i and the leading coefficient of the polynomial A_i. This is impossible since y_d and the leading coefficient of A_i are nonzero. If $i = n+t+1 \in \{n+1, \dots, n+m+1\}$, then by rearranging the above identity, we see that

$$\bar{Q} x^{f_i(d)} + \bar{W}(x) \equiv 0$$

where $\bar{W}(x)$ is a polynomial of degree strictly less than $f_i(d)$, and $\bar{Q}$ is the product of $d y_d y_d^t$ and the leading coefficient of the polynomial B_t. Again, this is impossible. The proof is complete. □

In view of Lemma 2.1, we may say that a positive integer d is feasible if $f(d)$ is attained by two of the indices $f_0(d), f_1(d), \dots, f_{n+m+1}(d)$. Let us denote the set of feasible integers by Ω.

Lemma 2.2. *The set Ω of feasible integers is bounded from above.*

Proof. There are several cases. First suppose $n > m+1$. Then for all sufficiently large d, $nd + a_n > md + b_m + d - 1$ and

$$\begin{aligned}
& f(d) \\
&= \max\big\{a_0, a_1 + d, \dots, a_n + nd; b_0 + d - 1, b_1 + d + d - 1, \dots, b_m + md + d - 1\big\} \\
&= \max\{a_n + nd, b_m + md - 1\} \\
&= \max\{a_n + nd\} \\
&= f_n(d) \\
&> \max\{f_0(d), f_1(d), \dots, f_{n-1}(d); f_{n+1}(d), \dots, f_{n+m+1}(d)\}.
\end{aligned}$$

Thus we may let d_0 be the first positive integer such that the above chain of inequalities hold for all $d \geq d_0$. If t is feasible, then by Lemma 2.1, $t < d_0$.

Suppose $n < m+1$. Then for all sufficiently large d, $nd + a_n < md + b_m + d - 1$ and

$$f(d) = f_{n+m+1}(d) > \max\{f_0(d), f_1(d), \dots, f_{n+m}(d)\} \tag{2.2}$$

for sufficiently large d. Let d_0 be the first positive integer such that the above chain of inequalities hold for all $d \geq d_0$. If t is feasible, then by Lemma 2.1, $t < d_0$.

Suppose $n = m+1$ and $a_n > b_m - 1$. Then $nd + a_n > md + b_m + d - 1$ for all d, and for all sufficiently large d,

$$f(d) = f_n(d) > \max\{f_0(d), f_1(d), \dots, f_{n-1}(d); f_{n+1}(d), \dots, f_{n+m+1}(d)\}.$$

As in the first case, we let d_0 be the first positive integer such that the above chain of inequalities hold for all $d \geq d_0$. If t is feasible, then by Lemma 2.1, $t < d_0$.

Suppose $n = m+1$ and $a_n < b_m - 1$. Then $nd + a_n < md + b_m + d - 1$ for all d, and for all sufficiently large d, (2.2) holds. By letting d_0 be the first positive integer such that the above chain of inequalities hold for all $d \geq d_0$, we see that a feasible integer t satisfies $t < d_0$.

Finally, suppose $n = m+1$ and $a_n = b_m - 1$. If $y(x)$ defined by (2.1) is a solution, then the leading coefficient y_d satisfies the equation $By_d^m dy_d = Ay_d^n$. Thus $y_d = 0$ or $Bd = A$. The former case is not possible, and therefore $A/B = d$. In other words, d is feasible only if $d = A/B$. The proof is complete. ◻

Lemma 2.3. *Let $y = y(x)$ be a polynomial solution of* (1.3). *Then for each $k \in \mathbb{N}^*$, we have*

$$y^{(k)}(x) = \frac{P_k(x, y(x))}{(B_m y^m(x) + \dots + B_0)^{r_k}},$$

where each P_k is a bivariate polynomial, $P_1 = P$ and $r_k \in \mathbb{N}$.If (1.3) *is not quasi-linear, then $P_k(x,y)$ is not identically zero for each k.*

Proof. There are several cases.

Case 1: Equation (1.3) is quasi-linear. Then either

$$y' = \frac{A_0(x)}{B_0(x)}, \quad \text{or} \quad y' = \frac{A_1(x)y + A_0(x)}{B_0(x)}.$$

In either cases, we may easily find $y^{(k)}$ by induction and show that it is not necessarily of the required form, i.e. $P_k(x, y(x)) \equiv 0$. (For example, from the equation $xy' = y$ one has $y' + xy'' = y'$, so that $xy'' = 0$.)

Case 2: Equation (1.3) has the form

$$B_0 y' = A_n y^n + \dots + A_0, \quad n \geq 2, \ B_0 \neq 0$$

and $A_n \neq 0$. Then

$$B_0^k y^{(k)} = \alpha_k A_n^k y^{k(n-1)+1} + R_k(x, y),$$

where $\deg_y R_k < k(n-1) + 1$ and $\alpha_k = \prod_{i=0}^{k-1}(i(n-1)+1)$.

The proof is by induction on k. For $k = 1$,

$$B_0 y' = \alpha_1 A_n y^n + R_1(x, y),$$

where $R_1(x,y) = A_{n-1}y^{n-1} + \dots + A_0$ and $\alpha_1 = 1$. Let us suppose our assertion is true for k; i.e.,

$$B_0^k y^{(k)} = \alpha_k A_n^k y^{k(n-1)+1} + R_k(x, y), \quad \deg_y R_k < k(n-1) + 1.$$

By differentiating the two members with respect to x, we obtain

$$kB_0'B_0^{k-1}y^{(k)}+B_0^ky^{(k+1)}=(kn-k+1)\alpha_k y'y^{kn-k}+\frac{d}{dx}(R_k(x,y))$$

while multiplying by B_0,

$$kB_0'B_0^ky^{(k)}+B_0^{k+1}y^{(k+1)}=\alpha_{k+1}B_0y'y^{kn-k}+B_0\frac{d}{dx}(R_k(x,y)).$$

Using the induction hypothesis and (1.3),

$$\begin{aligned} &kB_0'\alpha_k(A_n^ky^{k(n-1)+1}+R_k(x,y))+B_0^{k+1}y^{(k+1)}\\ &=\alpha_{k+1}(A_ny^n+\dots+A_0)y^{kn-k}+B_0\frac{d}{dx}(R_k(x,y)). \end{aligned}$$

It follows that

$$B_0^{k+1}y^{(k+1)}=\alpha_{k+1}A_n^{k+1}y^{kn+n-k}+R_{k+1}(x,y),$$

where

$$\begin{aligned} R_{k+1}(x,y)=B_0\frac{d}{dx}(R_k(x,y))&+\alpha_{k+1}(A_{n-1}y^{kn+n-k-1}+\dots+A_0y^{kn-k})\\ &-kB_0'(\alpha_ky^{kn-k+1}+R_k(x,y)). \end{aligned}$$

We may now conclude that

$$\deg_y(R_{k+1}(x,y))<kn+n-k.$$

Case 3: Equation (1.3) has the form $y'=P(x,y)/Q(x,y)$ where $\gcd(P,Q)=1$and $\deg_y Q\geq 1$. We can write $y'=P/R^sU$, where $Q=R^sU$, R is irreducible, $\deg_y R\geq 1$, $\gcd(R,U)=1$ and $s\in\mathbb{N}^*$. We will prove (by induction) that for all $k\in\mathbb{N}$, one has

$$y^{(k)}=\frac{P_k}{R^{t_k}U^{r_k}} \quad\text{and}\quad \gcd(R,P_k)=1 \tag{2.3}$$

where $t_k\in\mathbb{N}^*$ and $r_k\in\mathbb{N}^*$. Then, since $\gcd(R,P_k)=1$, $P_k(x,y)$ is not identically zero for all k.

Now, for $k=0$, since R is irreducible and $\gcd(P,Q)=1$, we see that $\gcd(R,P)=1$. We take $P_0=P$, $t_0=s$ and $r_0=1$. Then the result is true for $k=0$.

Let us suppose that our result is true for the order k, i.e.,

$$y^{(k)}=\frac{P_k}{R^{t_k}U^{r_k}} \text{ and } \gcd(R,P_k)=1$$

where $t_k\in\mathbb{N}^*$ and $r_k\in\mathbb{N}^*$. By differentiating both sides with respect to x, and replacing y' by P/R^sU, we have

$$\begin{aligned} y^{(k+1)}&=\Big(((P_k)'_yRU-t_kP_kR'_yU+r_kP_kRU'_y)P\\ &\quad+R^sU((P_k)'_xRU-t_kP_kR'_xU+r_kP_kRU'_x)\Big)/(R^{t_k+1+s}U^{r_k+2})\\ &\equiv\frac{P_{k+1}}{R^{t_{k+1}}U^{r_{k+1}}}. \end{aligned}$$

It remains to prove that $\gcd(R, P_{k+1}) = 1$. We have

$$\begin{aligned} P_{k+1} &= ((P_k)'_y RU - t_k P_k R'_y U + r_k P_k RU'_y)P \\ &\quad + R^s U((P_k)'_x RU - t_k P_k R'_x U + r_k P_k RU'_x) \\ &= \big\{((P_k)'_y U + r_k P_k U'_y)P + R^{s-1}U((P_k)'_x RU - t_k P_k R'_x U + r_k P_k RU'_x)\big\}R \\ &\quad - t_k P_k R'_y UP. \end{aligned}$$

Let

$$T = ((P_k)'_y U + r_k P_k U'_y)P + R^{s-1}U((P_k)'_x RU - t_k P_k R'_x U + r_k P_k RU'_x)$$

which is a bivariate polynomial $T(x, y)$, and $W = -t_k P_k R'_y UP$ which is also a bivariate polynomial $W(x, y)$. Then we may write

$$P_{k+1} = TR + W.$$

First $\gcd(R, W) = 1$, because $\gcd(R, P_k) = 1$ is the induction hypothesis, then $\gcd(R, R'_y) = 1$ since R is irreducible, and $\gcd(R, U) = 1$, $\gcd(R, P) = 1$ by definition of R and U.

Second $\gcd(R, P_{k+1}) = 1$, for otherwise if $\gcd(R, P_{k+1}) \neq 1$, then in view of the fact that R is irreducible, R divides P_{k+1}. But $W = P_{k+1} - TR$, thus R divides W, which is a contradiction. We may now conclude that $\gcd(R, P_{k+1}) = 1$. The proof is complete. □

We are now able to prove the following fundamental theorem.

Theorem 2.4. *If the differential equation* (1.3) *is not quasi-linear, then it admits a finite number of polynomial solutions, and they can be computed in a systematic manner.*

Proof. As explained before, we may easily determine the constant polynomial solutions of (1.3). Next, by Lemma 2.2, the set of feasible integers is bounded above, say, by δ. For each polynomial $y = y(x)$ of the form (2.1) and of degree $d \leq \delta$, we calculate P_{d+1} in Lemma 2.3. Then we are led to the algebraic identity $P_{d+1}(x, y(x)) \equiv 0$. This algebraic equation can be written as $D_\sigma y^\sigma(x) + \cdots + D_1 y(x) \equiv D_0$, where each D_i is a polynomial in x, $\sigma \in \mathbb{N}^*$, and D_0 as well as D_σ are not identically zero. Thus the polynomial y is a factor of D_0. Now we may replace all possible factors of D_0 into (1.3), and apply the method of undetermined coefficients to find y. The proof is complete. □

An example will illustrate the above proof.

Example 2.5. Consider the equation

$$y' = \frac{y^3 + 2x}{2x^2 y + x}, \tag{2.4}$$

where $A_3(x) = 1$, $A_2(x) = 0$, $A_1(x) = 0$, $A_0(x) = 2x$, $B_1(x) = 2x^2$ and $B_0(x) = x$. This equation is not quasi-linear, we can find all its polynomial solutions. First of all, constant solutions are not possible since substituting $y = \lambda$ into it yielding

$$\lambda^3 + 2x \equiv 0,$$

which is impossible. Next, we may easily see that

$$
\begin{aligned}
f_0(d) &= a_0 + 0 \cdot d = 1,\\
f_1(d) &= a_1 + d = 0 + d = d,\\
f_2(d) &= a_2 + 2d = 0 + 2d = 2d,\\
f_3(d) &= a_3 + 3d = 0 + 3d = 3d,\\
f_4(d) &= b_0 + 0 \cdot d + d - 1 = d,\\
f_5(d) &= b_1 + d + d - 1 = 1 + 2d,\\
f(d) &= \max\{3d, 1 + 2d\} = 3d.
\end{aligned}
$$

Since $1 + 2d < 3d$ for $d > 1$, we see further that $\Omega = \{1\}$. Let y be a polynomial solution of degree 1. Then as we have already seen in the Introduction, (1.7) must hold, and $y = 2x$ is a polynomial solution and hence is also the unique polynomial solution of (2.4).

3. Rational solutions

We now turn to rational solutions of (1.3).

Theorem 3.1. *If (1.3) is elliptic, then any rational solution of (1.3) is of the form $y = u/A_n$ where u is a polynomial; and if (1.3) is hyperbolic or $(n, m) = (1, 0)$, then there exists $\varrho \in \mathbb{N}$ (which can be determined) such that any rational solution of (1.3) is of the form $y = u/B_m^{\varrho}$, where u is a polynomial.*

Before we turn to the proof, recall from Taylor's expansion that

$$
\begin{aligned}
A_n(x) &= A_n(x_0) + \dots + A_n^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + \dots + A_n^{(a_n)}(x_0)\frac{(x-x_0)^{a_n}}{a_n!},\\
B_m(x) &= B_m(x_0) + \dots + B_m^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + \dots + B_m^{(b_m)}(x_0)\frac{(x-x_0)^{b_m}}{b_m!}.
\end{aligned}
$$

Furthermore, if x_0 is a root of A_n or B_m, then we can write

$$
\begin{aligned}
A_n(x) &= A_n^{(\alpha)}(x_0)\frac{(x-x_0)^\alpha}{\alpha!} + \dots + A_n^{(a_n)}(x_0)\frac{(x-x_0)^{a_n}}{a_n!}, \ \alpha = v_{x_0}(A_n),\\
B_m(x) &= B_m^{(\beta)}(x_0)\frac{(x-x_0)^\beta}{\beta!} + \dots + B_m^{(b_m)}(x_0)\frac{(x-x_0)^{b_m}}{b_m!}, \ \beta = v_{x_0}(B_m).
\end{aligned}
$$

Proof of Theorem 3.1. First note that if u is a rational solution of an elliptic equation (1.3), then a pole of u is a root of A_n. Indeed, let α be a pole of u with order $k > 0$. If $A_n(\alpha)$ is not null, then the valuation of $P(x, u)$ (as a function of x) at α is exactly $-nk$ and the valuation of $Q(x, y)y'$ (as a function of x) at α is at least $-mk - k - 1$. Since $n > m + 2$, the equality $Q(x, u)u' = P(x, u)$ is then impossible.

Now let y be a rational solution of (1.3). Then it can be written as u/A_n where u is rational. From (1.3), we have

$$
\big(B_m(\frac{u}{A_n})^m + \dots + B_0\big)\big(\frac{u'A_n - uA_n'}{A_n^2}\big) = A_n(\frac{u}{A_n})^n + \dots + A_1\frac{u}{A_n} + A_0.
$$

But $n - 1 \geq m + 2$, thus

$$
(A_n^{n-m-3}B_m u^m + \dots + A_n^{n-3}B_0)(u'A_n - uA_n') = u^n + \dots + A_n^n A_1\frac{u}{A_n} + A_n^{n-1}A_0,
$$

and

$$\begin{aligned}&(A_n^{n-m-2}B_m u^m+\dots+A_n^{n-2}B_0)u'\\&=u^n+\dots+(A_n^{n-m-3}B_m u^{m+1}+\dots+A_n^{n-3}B_0u)A_n'+\dots+A_n^{n-1}A_0,\end{aligned}$$

so that (1.3) becomes the so called "reduced equation"

$$(\tilde{B}_m u^m+\dots+\tilde{B}_0)u'=u^n+\tilde{A}_{n-1}u^{n-1}+\dots+\tilde{A}_0 \tag{3.1}$$

where $\tilde{B}_i$, $\tilde{A}_i$ are polynomials and $\tilde{B}_m$ is not identically zero. Note that (3.1) is also elliptic. Thus by what we have discussed above, a pole α of u as a solution of (3.1) must be a root of the leading coefficient of the right hand side. But since this coefficient is 1, u cannot have any poles. We conclude that u is a polynomial.

Suppose (1.3) is hyperbolic. Let y be a rational function and α a pole of order $k > 0$ of y. If $B_m(\alpha)$ is not null, then the valuation of $Q(x,y)y'$ (as a function of x) at α is exactly $-mk-k-1$ and the valuation of $P(x,y)$ (as a function of x) at α is at least $-nk$. Since $n\le m+1$, the equality $Q(x,y)y'=P(x,y)$ is then impossible, unless $B_m(\alpha)=0$. We may conclude that any rational solution of (1.3) is of the form $u/B_m^r$where u is a polynomial and $r\in\mathbb{N}$.

Let x_0 a root of B_m of order $v_{x_0}(B_m)\in\mathbb{N}^*$, and y a rational solution with the pole x_0:

$$y=\frac{c}{(x-x_0)^{-v_{x_0}(y)}}+R,$$

where $c\in\mathbf{C}\backslash\{0\}$, R is rational and $v_{x_0}(R)>v_{x_0}(y)$.

Let us show that there exists $k'_{x_0}\in\mathbb{N}$ (which can be determined) such that

$$-v_{x_0}(y)\le k'_{x_0}.$$

First there exists a least integer $k_{x_0}\in\mathbb{N}^*$ which can easily be determined, such that for any integer $k\ge k_{x_0}$, we have

$$nk-v_{x_0}(A_n)>ik-v_{x_0}(A_i)$$

for $i=0,\dots,n-1$, and

$$mk-v_{x_0}(B_m)+k+1>ik-v_{x_0}(B_i)+k+1$$

for $i=0,\dots,m-1$. (In practice one uses $mk-v_{x_0}(B_m)>ik-v_{x_0}(B_i)$.)

Next, if $m+1>n$, then $mk-v_{x_0}(B_m)+k+1>nk-v_{x_0}(A_n)$ so that $(m+1-n)k+1>v_{x_0}(B_m)-v_{x_0}(A_n)$ for sufficiently large k.

If $n<m+1$ then $-v_{x_0}(y)\le k_{x_0}$, and we may take $k'_{x_0}=k_{x_0}$

If $n=m+1$ and $v_{x_0}(A_n)\ne v_{x_0}(B_m)-1$, then $-v_{x_0}(y)\le k_{x_0}$ and we may take $k'_{x_0}=k_{x_0}$

If $n=m+1$ and $v_{x_0}(A_n)=v_{x_0}(B_m)-1$, then we put $v_{x_0}(A_n)=\alpha$, $v_{x_0}(B_m)=\beta$ and $v_{x_0}(y)=\gamma$, so that replacing y by $(c(x-x_0)^\gamma+R)$in (1.3) and using Taylor's expansion of A_n and B_m at x_0, we have $(B_my^m+\dots)y'=A_ny^n+\dots$. Hence

$$\begin{aligned}&\Big(\big(B_m^{(\beta)}(x_0)\frac{(x-x_0)^\beta}{\beta!}+\dots\big)\big(c^m(x-x_0)^{m\gamma}+\dots\big)+\dots\Big)\big(c\gamma(x-x_0)^{\gamma-1}+R'\big)\\&=\big(A_n^{(\alpha)}(x_0)\frac{(x-x_0)^\alpha}{\alpha!}+\dots\big)\big(c^n(x-x_0)^{n\gamma}\big)+\dots\big)+\dots\end{aligned}$$

and

$$\begin{aligned} &B_m^{(\beta)}(x_0)\frac{(x-x_0)^\beta}{\beta!}c^m(x-x_0)^{m\gamma}c\gamma(x-x_0)^{\gamma-1}+\dots \\ &=A_n^{(\alpha)}(x_0)\frac{(x-x_0)^\alpha}{\alpha!}c^n(x-x_0)^{n\gamma}+\dots, \end{aligned}$$

so that

$$\frac{B_m^{(\beta)}(x_0)}{\beta!}c^{m+1}\gamma(x-x_0)^{m\gamma+\beta+\gamma-1}+\dots=\frac{A_n^{(\alpha)}(x_0)}{\alpha!}c^n(x-x_0)^{\alpha+n\gamma}+\dots.$$

But $n=m+1$ and $\alpha=\beta-1$, thus

$$\frac{B_m^{(\alpha+1)}(x_0)}{(\alpha+1)!}c^n\gamma(x-x_0)^{\alpha+n\gamma}+\dots=\frac{A_n^{(\alpha)}(x_0)}{\alpha!}c^n(x-x_0)^{\alpha+n\gamma}+\dots.$$

Comparing coefficients of $(x-x_0)^{\alpha+n\gamma}$, we see that

$$\frac{B_m^{(\alpha+1)}(x_0)}{(\alpha+1)!}c^n\gamma=\frac{A_n^{(\alpha)}(x_0)}{\alpha!}c^n,$$

which, in view of $c\neq 0$, implies that

$$\gamma=(\alpha+1)\frac{A_n^{(\alpha)}(x_0)}{B_m^{(\alpha+1)}(x_0)}.$$

If $-\gamma$ is an integer and is greater than k_{x_0}, then we may take $k'_{x_0}=-\gamma$, else we take $k'_{x_0}=k_{x_0}$. Let us show that $\varrho=\max\{k'_{x_i}:x_i \text{ is a root of } B_m\}$, where k'_{x_i} are defined as above.

Let $x_0,x_1,\dots,x_h$ be the roots of $B_m$and y a rational solution of (1.3). We know that any pole of y is a root of B_m. Then

$$y=\frac{p_1(x)}{(x-x_0)^{-v_{x_0}(y)}(x-x_1)^{-v_{x_1}(y)}\dots(x-x_h)^{-v_{x_h}(y)}}$$

where $p_1(x)$ is a polynomial (eventually some $v_{x_i}(y)$ can be equal to zero). Since $-v_{x_i}(y)\leq k'_{x_i}$ for $i=1,\dots,h$, multiplying the last fraction by

$$\frac{(x-x_0)^{v_{x_0}(y)+k'_{x_0}}(x-x_1)^{v_{x_1}(y)+k'_{x_1}}\dots(x-x_h)^{v_{x_h}+k'_{x_h}}}{(x-x_0)^{v_{x_0}(y)+k'_{x_0}}(x-x_1)^{v_{x_1}(y)+k'_{x_1}}\dots(x-x_h)^{v_{x_h}+k'_{x_h}}}\equiv 1,$$

we obtain

$$y=\frac{p_2(x)}{(x-x_0)^{k'_{x_0}}(x-x_1)^{k'_{x_1}}\dots(x-x_h)^{k'_{x_h}}}$$

where $p_2(x)$ is a polynomial.

Multiplying the above fraction by

$$\frac{(x-x_0)^{\varrho'-k'_{x_0}}(x-x_1)^{\varrho'-k'_{x_1}}\dots(x-x_h)^{\varrho'-k'_{x_h}}}{(x-x_0)^{\varrho'-k'_{x_0}}(x-x_1)^{\varrho'-k'_{x_1}}\dots(x-x_h)^{\varrho'-k'_{x_h}}}\equiv 1$$

where $\varrho'=\max\{k'_{x_i}\}$, we obtain

$$y=\frac{p_3(x)}{[(x-x_0)(x-x_1)\dots(x-x_h)]^{\varrho'}},$$

where $p_3(x)$ is a polynomial. But

$$B_m(x)=B(x-x_0)^{v_{x_0}(B_m)}(x-x_1)^{v_{x_1}(B_m)}\dots(x-x_h)^{v_{x_h}(B_m)},$$

if we multiply the last fraction by

$$\frac{B^{\varrho'}(x-x_0)^{(v_{x_0}(B_m)-1)\varrho'}(x-x_1)^{(v_{x_1}(B_m)-1)\varrho'}\dots(x-x_h)^{(v_{x_h}(B_m)-1)\varrho'}}{B^{\varrho'}(x-x_0)^{(v_{x_0}(B_m)-1)\varrho'}(x-x_1)^{(v_{x_1}(B_m)-1)\varrho'}\dots(x-x_h)^{(v_{x_h}(B_m)-1)\varrho'}} \equiv 1,$$

we obtain

$$y = \frac{p_4(x)}{B_m^{\varrho'}},$$

where $p_4(x)$ is a polynomial. We now take $\varrho = \varrho' = \max\{k'_{x_i} : x_i \text{ is a root of } B_m\}$. $\square$

We remark that we can also take $\varrho = LCM\{k'_{x_i} : x_i \text{ is a root of } B_m\}$ or any integer s greater than ϱ'. When multiplying the last fraction by

$$\frac{B_m^{s-\varrho'}}{B_m^{s-\varrho'}} \equiv 1,$$

we obtain

$$y = \frac{p_5(x)}{B_m^s},$$

where $p_5(x)$ is a polynomial.

Example 3.2. Equation (2.4) is elliptic. Hence any rational solution is of the form $y(x) = u(x)/A_3(x) = u(x)$ for some polynomial u. By Example 2.5, $y(x) = 2x$ is the unique rational solution of (2.4).

Example 3.3. Consider the equation

$$y' = \frac{xy^2 + y}{y^3 + x}. \tag{3.2}$$

which is hyperbolic. Since $B_3 = 1$, its rational solutions are equal to its polynomial solutions. The only constant polynomial solution is $y(x) = 0$. Furthermore, since $\Omega = \{1\}$, then if $y(x)$ is a polynomial solution of degree 1, we obtain from (3.2) that

$$y'' = y\frac{y^4 - 1}{(y^3 + x)^2} + \frac{y'}{(y^3 + x)^2}(2x^2y - xy^4 + x - 2y^3). \tag{3.3}$$

Replacing y' by $\frac{xy^2+y}{y^3+x}$ in (3.3), we see that

$$-y^6 + x^2y^4 + 2xy^3 + 3y^2 + (-2x^3)y + (-3x^2) = 0; \tag{3.4}$$

that is,

$$[-y^5 + x^2y^3 + 2xy^2 + 3y + (-2x^3)]y = 3x^2.$$

One concludes that y divides $3x^2$. Thus $y = \lambda x$ where λ is some nonzero number. Replacing y by λx in (3.2), we have

$$\lambda = \frac{x^2\lambda^2 + \lambda}{\lambda^3 x^2 + 1}.$$

Thus $\lambda^4 = \lambda^2$ i.e. $\lambda = 1, -1$. In conclusion, $0, x$ and $-x$ are all the rational solutions of (3.2).

Example 3.4. Consider the equation

$$y' = \frac{y^3 - 1}{xy^2 - 1}$$

which is a hyperbolic equation, we can then compute all its rational solutions. By Theorem 3.1, since $x_0 = 0$ is the only root of order 1 of $B_m(x) = x$, we know that $y = u/x^{\varrho}$ where u is a polynomial and ϱ is determined as in the proof of Theorem 3.1. More precisely, let

$$y = \frac{c}{x^{-v_0(y)}} + R,$$

where $c \in \mathbb{C}\backslash\{0\}$, R is rational and $v_0(R) > v_0(y)$.

Let us find $k_0 \in \mathbb{N}$ such that for any integer $k \geq k_0$, we have

$$3k - v_0(A_3) > ik - v_0(A_i)$$

for $i = 0, 1, 2$ and

$$2k - v_0(B_2) > ik - v_0(B_i)$$

for $i = 0, 1$.

Since $A_2 = A_1 = B_1 \equiv 0$, we see that $v_0(A_2) = v_0(A_1) = v_0(B_1) = +\infty$, and $v_0(A_3) = 0 = \alpha$, $v_0(A_0) = 0$, $v_0(B_2) = 1 = \beta$, $v_0(B_0) = 0$. Therefore, it is clear that $k_0 = 1$.

Here $n = m + 1 = 3$ and $v_0(A_3) = v_0(B_2) - 1 = 0$, then put $v_0(y) = \gamma$, so that replacing y by $(cx^{\gamma} + R)$in (1.3) , as in proof of Theorem 3.1, we obtain

$$\gamma = (\alpha + 1)\frac{A_n^{(\alpha)}(x_0)}{B_m^{(\alpha+1)}(x_0)} = 1$$

since $\alpha = 0,\ n = 3,\ m = 2,\ x_0 = 0,\ A_3(0) = 1$ and $B_2^{(1)}(0) = 1$. Since $\gamma > 0$, one takes $\varrho = k_0 = 1$.

The reduced equation satisfied by the polynomial u is

$$u' = \frac{2u^3 - xu - x^3}{xu^2 - x^2}. \tag{3.5}$$

Here $\Omega = \{1, 2\}$. Then $u^{(3)} = 0$. By differentiating both sides of (3.5), we obtain

$$u'' = \frac{(2u^4 - 5u^2x + 2ux^3 + x^2)u'}{x(x - u^2)^2} - \frac{(2u^5 - 4u^3x + 2u^2x^3 + ux^2 - x^4)}{x^2(x - u^2)^2}.$$

Replacing u' by $\frac{2u^3 - xu - x^3}{xu^2 - x^2}$, we see that

$$u'' = \frac{2(-u^7 + 3u^5x - u^3x^2 - 3u^2x^4 + ux^6 + x^5)}{x^2(x - u^2)^3}.$$

By differentiating both sides again, we obtain

$$\begin{aligned} u^{(3)} &= \frac{(u^8 - 4u^6x + 12u^4x^2 - 12u^3x^4 + 5u^2x^6 - 3u^2x^3 + x^7)u'}{x^2(x - u^2)^4} \\ &\quad + \frac{2u(-2u^8 + 8u^6x - 12u^4x^2 + 6u^3x^4 - 4u^2x^6 + 3u^2x^3 + x^7)}{x^3(x - u^2)^4}. \end{aligned}$$

If we replace u' by $\frac{2u^3 - xu - x^3}{xu^2 - x^2}$ and $u^{(3)}$ by 0 in the above equation, we see that

$$\begin{aligned} 0 &= 2u^{11} - 11xu^9 + x^3u^8 + 12x^2u^7 + 8x^4u^6 - (2x^6 + 12x^3)u^5 \\ &\quad + 12x^5u^4 + (3x^4 - 19x^7)u^3 + (5x^9 - 3x^6)u^2 + 3x^8u + x^{10}. \end{aligned}$$

We may conclude that u divides x^{10}. Thus y is a constant function or $u = \lambda x$ or $u = \lambda x^2$, where $\lambda \in \mathbb{C}\backslash\{0\}$. It is clear that (3.5) has non-constant solutions only, because replacing y by a constant λ in (3.5), we obtain for all $x \in \mathbb{C}$ that $2\lambda^3 - x\lambda - x^3 = 0$ which is impossible. If $u = \lambda x$ then

$$\lambda = \frac{2(\lambda x)^3 - x(\lambda x) - x^3}{x(\lambda x)^2 - x^2};$$

i.e., $\lambda^3 = 1$. One concludes that $u = x, xe^{\frac{2i\pi}{3}}, xe^{\frac{4i\pi}{3}}$. If $u = \lambda x^2$ then

$$2\lambda x = \frac{2(\lambda x^2)^3 - x(\lambda x^2) - x^3}{x(\lambda x^2)^2 - x^2};$$

i.e., $\lambda = 1$. One concludes that $u = x^2$. Finally $y = 1, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}$ or x.

4. The parabolic case

Theorem 4.1. *Let us consider the differential equation*

$$y' = \frac{A_{m+2}y^{m+2} + \dots + A_0}{B_m y^m + \dots + B_0}, \tag{4.1}$$

where $m \in \mathbb{N}^*$, A_i, B_i are polynomials such that A_{m+2} and B_m are not identically zero, and $A_{m+2}y^{m+2} + \dots + A_0$ and $B_m y^m + \dots + B_0$ are coprime. Then (4.1) admits a finite number of rational solutions.

Proof. There are two cases.

Case 1: Suppose $A_0 = 0$, (i.e. $y = 0$ is a solution). Then $B_0 \neq 0$. Let $z = 1/y$, equation (4.1) becomes

$$z' = -\frac{A_{m+2} + \dots + A_1 z^{m+1}}{B_m + \dots + B_0 z^m}, \tag{4.2}$$

which is hyperbolic. Thus equation (4.1) admits a finite number of rational solutions.

Case 2: Suppose $A_0 \neq 0$. If (4.1) admits a rational solution f. If $z = y - f$, equation (4.1) becomes

$$z' = \frac{C_{m+2}z^{m+2} + \dots + C_1 z}{D_m z^m + \dots + D_0}, \tag{4.3}$$

where C_i, D_i are polynomials, $C_0 = 0$ and D_0 is not identically zero. This is exactly the first case; i.e., $z = 0$ is a solution. Let $\varphi = \frac{1}{z}$ then (4.1) becomes hyperbolic which has a finite number of rational solutions φ. But $\varphi = \frac{1}{z} = \frac{1}{y-f}$, thus $y = \frac{1}{\varphi} + f$. $\square$

As a corollary, we can compute all the rational solutions of (4.1) if we have at least one particular rational solution of (4.1).

Example 4.2. Consider the equation

$$y' = \frac{y^4 - y}{-y^2 + x}. \tag{4.4}$$

This equation is parabolic, we can compute all its polynomial solutions. Furthermore, if we find a polynomial solution of (4.4), we can compute all its rational solutions. Since $\Omega = \emptyset$, we only look for constant solutions. This leads us to $y' = 0$, and $y^4 - y = 0$. Thus $y = 0, 1, e^{2i\pi/3}, e^{4i\pi/3}$. We have four constant solutions of

(4.4). Let $z = 1/y$ (as in the proof of Theorem 4.1, $z = 1/(y - f)$ with $f = 0$). From (4.4), we have

$$z' = \frac{z^3 - 1}{xz^2 - 1}, \tag{4.5}$$

which is the same equation in Example 3.4. In conclusion, $0, 1, e^{2i\pi/3}, e^{4i\pi/3}$ and $1/x$ are the rational solutions of (4.4).

5. The quasi-linear and Riccati cases

Suppose first that (1.3) is quasi-linear. It suffices to consider the equation

$$B_0 y' = A_1 y + A_0. \tag{5.1}$$

We may first determine δ (the upper bound of $\deg y$) by Lemma 2.1. Replacing y by $y_\delta x^\delta + \cdots + y_0$ in (5.1), we obtain

$$B_0(\delta y_\delta x^{\delta-1} + \cdots + y_1) = A_1(y_\delta x^\delta + \cdots + y_0) + A_0.$$

Then rearranging terms in the resulting equation, we obtain

$$K_l x^l + K_{l-1} x^{l-1} + \cdots + K_0 = 0,$$

where each K_i may depend on $y_0, y_1, \ldots, y_\delta$, which is equivalent to following linear system: $K_l = K_{l-1} = \cdots = K_0 = 0$. After solving this system, we obtain y_i.

Next let us compute rational solutions of (5.1). If $A_1 \equiv 0$, by means of the (classical) partial fraction decomposition, we see that

$$y' = \frac{A_0}{B_0} = p(x) + \sum_{d,\alpha} \frac{c(d, \alpha)}{(x - \alpha)^d},$$

where $p(x)$ is a polynomial, $c(d, \alpha) \in \mathbb{C}$ and the sum is over the set of roots α of B_0 with multiplicity d. Using direct integration, we see that a solution y is rational if and only if $c(1, \alpha) = 0$ for all α.

If $A_1 \neq 0$, by Theorem 3.1, there exits $\varrho \in \mathbb{N}$ such that $y = u/B_0^\varrho$, where u is a polynomial. Replacing y by u/B_0^ϱ in (5.1), we obtain

$$B_0 u' = (A_1 + \varrho B_0')u + B_0^\varrho A_0.$$

We may then determine u.

Note that the number of polynomial or rational solutions of (5.1) may not be finite. As an example, the equation

$$xy' = y + x^2,$$

has polynomial solutions of the form $x^2 + \lambda x$ where λ is an arbitrary complex number. As another example, the equation

$$y' = \frac{1}{x^2}$$

has rational solutions of the form $-\frac{1}{x} + \lambda$ where λ is an arbitrary complex number.

We now suppose (1.3) is Riccati. It suffices to consider the equation

$$B_0 y' = A_2 y^2 + A_1 y + A_0 \tag{5.2}$$

By Theorem 2.4, we can compute all its polynomial solutions (finite number). If we have a rational solution f of (5.2), then by letting $z = 1/(y - f)$, we obtain

$$-B_0 z' = (2f A_2 + A_1)z + A_2,$$

which is a quasi-linear equation. Again, we remark that the number of rational solutions of (5.2) may not be finite. For example, all solutions of the Riccati equation $y' = -y^2$ are of the form

$$y = \frac{1}{x+\lambda}$$

where λ is an arbitrary complex number.

As our final remark. we can find in [5, Chapter I] elementary methods of integration of classical ODE which can be used to find the desired solutions in this section.

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