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UNIQUENESS OF POSITIVE SOLUTIONS FOR AN ELLIPTIC SYSTEM

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Abstract. We prove the uniqueness of positive solutions for an elliptic system that appears in the study of solutions for a degenerate predator-prey model in the strong-predator case.

1. Introduction

This article is devoted to showing the uniqueness of positive solutions for the elliptic system

$$
\begin{gathered}
-\Delta u = \lambda u - buv \quad \text{in } \Omega, \\
-\Delta v = \mu v\big(1 - \xi \frac{v}{u}\big) \quad \text{in } \Omega, \\
\partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, ν is the outward unit normal vector on $\partial\Omega$, $\partial_\nu = \frac{\partial}{\partial\nu}$, λ, b, μ and ξ are positive constants.

Problem (1.1) appears in the study of positive solutions of the degenerate predator-prey model in the strong-predator case

$$
\begin{gathered}
-\Delta u = \lambda u - a(x)u^2 - \beta uv \quad \text{in } \Omega, \\
-\Delta v = \mu v\big(1 - \frac{v}{u}\big) \quad \text{in } \Omega, \\
\partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.2}
$$

where β is a positive constant, and $a(x)$ is a continuous function satisfying $a(x) = 0$ on $\overline{\Omega}_0$ and $a(x) > 0$ in $\overline{\Omega} \setminus \overline{\Omega}_0$, where Ω_0 is a smooth domain with $\overline{\Omega}_0 \subset \Omega$. Recently, problem (1.2) has been studied in [2, 3]. Under the condition $\mu > \lambda \geq \lambda_1$, where λ_1 denotes the first eigenvalue of the Laplace equation on Ω_0 with homogenous Dirichlet boundary condition, Du and Wang [3] described spatial patterns of positive solutions of problem (1.2) by studying asymptotic behavior of positive solutions as $\beta \to 0^+$ (weak-predator), $\beta \to +\infty$ (strong-predator) and $\mu \to +\infty$ (small-predator diffusion), respectively. For related work on problem (1.2), please refer to [8].

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Clearly, problem (1.1) has a positive solution $(u,v) = (\frac{\xi\lambda}{b}, \frac{\lambda}{b})$. In [3, Remark 3.2], the authors pointed out that when the spatial dimension $N = 1$, the positive solution of problem (1.1) is unique for any $\mu > 0$ by a simple variation of the arguments in [6]. In the present paper, we prove the uniqueness for all sufficiently large μ in the high dimensional case, which can be stated as follows

Theorem 1.1. *Let $N \geq 2$. Then there exists a positive constant μ_0 depending only on λ and Ω such that problem* (1.1) *admits a unique positive solution for any $\mu \geq \mu_0$.*

Remark 1.2. The proof to Theorem 1.1 is based on the fact that $(\hat{u}, \hat{v})$ is a positive solution of problem (1.1) if and only if $(\frac{b}{\xi}\hat{u}, b\hat{v})$ is a positive solution of

$$\begin{gathered} -\Delta u = u(\lambda - v) \quad \text{in } \Omega, \\ -\Delta v = \mu v\big(1 - \frac{v}{u}\big) \quad \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Remark 1.3. As a result of Theorem 1.1 and [3, Remarks 3.1-3.2], one can prove that if (u_β, v_β) is a solution of problem (1.2), then for any $\mu \geq \mu_0$, we have, as $\beta \to +\infty$,

$$\begin{gathered} \Big(\frac{u_\beta}{\|u_\beta\|_\infty}, \frac{v_\beta}{\|v_\beta\|_\infty}\Big) \rightharpoonup (1,1) \quad \text{in } [H^1(\Omega)]^2, \\ \Big(\frac{u_\beta}{\|u_\beta\|_\infty}, \frac{v_\beta}{\|v_\beta\|_\infty}\Big) \to (1,1) \quad \text{in } [L^p(\Omega)]^2, \forall p > 1. \end{gathered}$$

2. Proof of Theorem 1.1

First recall several preliminary results.

Lemma 2.1 (Harnack Inequality [5])**.** *Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\overline{\Omega})$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant C which depends only on B where $\|c\|_\infty \leq B$ such that* $\max_{\overline{\Omega}} w \leq C \min_{\overline{\Omega}} w$.

Lemma 2.2 (Maximum Principle [7])**.** *Suppose that $g \in C^1(\Omega \times \mathbb{R}^1)$. Then*

(i) *if $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\Delta w(x) + g(x,w) \geq 0$ in Ω, $\partial_\nu w \leq 0$ on $\partial\Omega$, and $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$.*

(ii) *if $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\Delta w(x) + g(x,w) \leq 0$ in Ω, $\partial_\nu w \geq 0$ on $\partial\Omega$, and $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.*

The following lemma can be inferred from [2, Lemma 3.7] (see also [8]).

Lemma 2.3. *Let $\{u_n\} \subset H^1(\Omega)$ satisfy, in the weak sense,*

$$-\Delta u_n \leq A u_n, \quad u_n \geq 0, \quad \partial_\nu u_n|_{\partial\Omega} = 0, \quad \|u_n\|_\infty \leq B, \ \forall n \geq 1,$$

where A and B are positive constants. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a nonnegative function $u \in H^1(\Omega) \cap L^p(\Omega)$ for all $p > 1$, such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\Omega), \quad u_n \to u \quad \text{in } L^p\Omega).$$

If we further assume that $\|u_n\|_\infty \geq \delta > 0$ for all $n \geq 1$, then $u \neq 0$.

The following lemma gives the uniform bounds of the positive solutions for problem (1.3).

Lemma 2.4. *Let (u_μ, v_μ) be a positive solution of problem (1.3). Then there exist a positive constant $\mu_0 = \mu_0(\lambda, \Omega)$ and two positive constants C_2, C_1 independent of μ such that for all $\mu \geq \mu_0$,*

$$C_1 \leq u_\mu, \quad v_\mu \leq C_2 \quad on\ \overline{\Omega}. \tag{2.1}$$

Moreover, as $\mu \to +\infty$,

$$u_\mu \to \lambda \quad in\ C^1(\overline{\Omega}). \tag{2.2}$$

Proof. By Lemma 2.2 and the definition of v_μ, it follows that

$$\max_{\overline{\Omega}} u_\mu \geq \max_{\overline{\Omega}} v_\mu, \quad \min_{\overline{\Omega}} v_\mu \geq \min_{\overline{\Omega}} u_\mu. \tag{2.3}$$

Hence, to prove (2.1), it suffices to show that there exist a positive constant $\mu_0 = \mu_0(\lambda, \Omega)$ and two positive constants C_2, C_1 independent of μ such that

$$C_1 \leq \min_{\overline{\Omega}} u_\mu, \quad \max_{\overline{\Omega}} u_\mu \leq C_2, \quad \forall \mu \geq \mu_0. \tag{2.4}$$

We first prove the second inequality of (2.4). Assume on the contrary that there exist a sequence $\{\mu_n\}$ converging to $+\infty$ and the corresponding solution (u_{μ_n}, v_{μ_n}), such that

$$\|u_{\mu_n}\|_\infty \to +\infty \quad \text{as } n \to +\infty.$$

Denote

$$\hat{u}_{\mu_n} = \frac{u_{\mu_n}}{\|u_{\mu_n}\|_\infty + \|v_{\mu_n}\|_\infty}, \quad \hat{v}_{\mu_n} = \frac{v_{\mu_n}}{\|u_{\mu_n}\|_\infty + \|v_{\mu_n}\|_\infty}.$$

Then $\hat{u}_{\mu_n}$ and $\hat{v}_{\mu_n}$ satisfy $\|\hat{u}_{\mu_n}\|_\infty + \|\hat{v}_{\mu_n}\|_\infty = 1$, $\|\hat{u}_{\mu_n}\|_\infty \geq \frac{1}{2}$ by (2.3), and

$$\begin{gathered} -\Delta \hat{u}_{\mu_n} = \hat{u}_{\mu_n}(\lambda - v_{\mu_n}) \quad \text{in } \Omega, \\ -\Delta \hat{v}_{\mu_n} = \mu_n \hat{v}_{\mu_n}\big(1 - \frac{\hat{v}_{\mu_n}}{\hat{u}_{\mu_n}}\big) \quad \text{in } \Omega, \\ \partial_\nu \hat{u}_{\mu_n} = \partial_\nu \hat{v}_{\mu_n} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.5}$$

In particular, we have

$$-\Delta \hat{u}_{\mu_n} \leq \lambda \hat{u}_{\mu_n} \quad \text{in } \Omega, \quad \partial_\nu \hat{u}_{\mu_n} = 0 \quad \text{on } \partial\Omega. \tag{2.6}$$

By Lemma 2.3 and $\|\hat{v}_{\mu_n}\|_\infty \leq 1$, there exist a subsequence of $\{(\hat{u}_{\mu_n}, \hat{v}_{\mu_n})\}$, still denoted by itself, and a pair of non-negative functions $(\hat{u}, \hat{v}) \in \big(H^1(\Omega) \cap L^p(\Omega)\big) \times L^\infty(\Omega)$ for all $p > 1$, $\hat{u} \not\equiv 0$, such that

$$\hat{u}_{\mu_n} \rightharpoonup \hat{u} \quad \text{in } H^1(\Omega), \quad \hat{u}_{\mu_n} \to \hat{u} \quad \text{in } L^p(\Omega), \quad \hat{v}_{\mu_n} \rightharpoonup \hat{v} \quad \text{in } L^2(\Omega).$$

Integrating the first equation of (2.5) over Ω yields

$$\lambda \int_\Omega \hat{u}_{\mu_n} dx = \int_\Omega v_{\mu_n} \hat{u}_{\mu_n} dx = (\|u_{\mu_n}\|_\infty + \|v_{\mu_n}\|_\infty) \int_\Omega \hat{u}_{\mu_n} \hat{v}_{\mu_n} dx.$$

From $\|u_{\mu_n}\|_\infty \to +\infty$ $(n \to +\infty)$, we have

$$\int_\Omega \hat{u}\hat{v} dx = \lim_{n\to+\infty} \int_\Omega \hat{u}_{\mu_n} \hat{v}_{\mu_n} dx = \lim_{n\to+\infty} \frac{\lambda}{\|u_{\mu_n}\|_\infty + \|v_{\mu_n}\|_\infty} \int_\Omega \hat{u}_{\mu_n} dx = 0. \tag{2.7}$$

By the second equation in (2.5), $\hat{v}_{\mu_n}$ is a positive solution of

$$-\Delta w + \mu_n \frac{\hat{v}_{\mu_n}}{\hat{u}_{\mu_n}} w = \mu_n w \quad \text{in } \Omega, \quad \partial_\nu w = 0 \quad \text{on } \partial\Omega. \tag{2.8}$$

From the variational characterization of the first eigenvalue it follows that

$$\int_\Omega |\nabla\phi|^2 dx + \mu_n \int_\Omega \frac{\hat{v}_{\mu_n}}{\hat{u}_{\mu_n}}\phi^2 dx \geq \mu_n \int_\Omega \phi^2 dx$$

for any $\phi \in \{w \in H^2(\Omega); \partial_\nu w = 0 \text{ on } \partial\Omega\}$ (cf. [1]). Taking $\phi = \hat{u}_{\mu_n}$ yields

$$\frac{1}{\mu_n}\int_\Omega |\nabla \hat{u}_{\mu_n}|^2 dx + \int_\Omega \hat{v}_{\mu_n}\hat{u}_{\mu_n} dx \geq \int_\Omega \hat{u}_{\mu_n}^2 dx.$$

Passing to the limit and using (2.7), we obtain $\int_\Omega \hat{u}^2 dx = 0$, so $\hat{u} = 0$, which is a contradiction. Thus there exist a positive constant $\mu_0 = \mu_0(\lambda, \Omega)$ and a positive constant C_2 independent of μ such that

$$\max_{\overline{\Omega}} u_\mu \leq C_2, \quad \forall \mu \geq \mu_0. \tag{2.9}$$

Next we prove the first inequality in (2.4). Suppose that this is not so. Then there exist $\{\mu_n\}$ converging to $+\infty$ and the corresponding solution (u_{μ_n}, v_{μ_n}) such that

$$\lim_{n\to+\infty} \min_{\overline{\Omega}} u_{\mu_n} = 0. \tag{2.10}$$

Now rewrite the equation of u_{μ_n} as

$$\Delta u_{\mu_n} + f(x) u_{\mu_n} = 0 \quad \text{in } \Omega, \quad \partial_\nu u_{\mu_n} = 0 \quad \text{on } \partial\Omega,$$

where $f(x) = \lambda - v_{\mu_n}$. By the first estimate of (2.3) and (2.9), we have, for all sufficiently large n,

$$\|f\|_\infty \leq \lambda + \|v_{\mu_n}\|_\infty \leq \lambda + C_2,$$

by Lemma 2.1, there exists a positive constant C_3 independent of n such that for all sufficiently large n,

$$\max_{\overline{\Omega}} u_{\mu_n} \leq C_3 \min_{\overline{\Omega}} u_{\mu_n}.$$

Therefore, it follows from (2.10) and the first estimate of (2.3) that

$$\lim_{n\to+\infty} \max_{\overline{\Omega}} u_{\mu_n} = 0, \quad \lim_{n\to+\infty} \max_{\overline{\Omega}} v_{\mu_n} = 0. \tag{2.11}$$

Denote $\tilde{u}_{\mu_n} = u_{\mu_n}/\|u_{\mu_n}\|_\infty$. Then $\tilde{u}_{\mu_n}$ satisfies $\|\tilde{u}_{\mu_n}\|_\infty = 1$, and

$$-\Delta \tilde{u}_{\mu_n} = \tilde{u}_{\mu_n}(\lambda - v_{\mu_n}) \quad \text{in } \Omega, \quad \partial_\nu \tilde{u}_{\mu_n} = 0 \quad \text{on } \partial\Omega.$$

By (2.3), (2.9) and the definition of $\tilde{u}_{\mu_n}$, both $\{-\Delta\tilde{u}_{\mu_n}\}$ and $\{\tilde{u}_{\mu_n}\}$ are bounded sets in $L^\infty(\Omega)$. By the standard elliptic theory (cf. [4, Theorem 9.9]), $\{\tilde{u}_{\mu_n}\}$ is bounded in $W^{2,p}(\Omega)$ for any $p > 1$. Therefore, there exist a subsequence of $\{\tilde{u}_{\mu_n}\}$, still denoted by itself, and a nonnegative function $\tilde{u} \in C^1(\overline{\Omega})$ with $\|\tilde{u}\|_\infty = 1$, such that

$$\tilde{u}_{\mu_n} \to \tilde{u} \quad \text{in } C^1(\overline{\Omega}),$$

by (2.11) and the definition of $\tilde{u}_{\mu_n}$, we derive that

$$-\Delta\tilde{u} = \lambda\tilde{u} \quad \text{in } \Omega, \quad \partial_\nu \tilde{u} = 0 \quad \text{on } \partial\Omega.$$

This implies $\tilde{u} = 0$, which is a contradiction. This proves (2.1).

Next we show (2.2). By (2.1) and the equation of u_μ, $\{-\Delta u_\mu\}_{\mu\geq\mu_0}$, $\{u_\mu\}_{\mu\geq\mu_0}$ and $\{v_\mu\}_{\mu\geq\mu_0}$ are bounded sets in $L^\infty(\Omega)$. By the standard elliptic theory, there exist a sequence $\{\mu_n\}$ converging to $+\infty$, the corresponding solution (u_{μ_n}, v_{μ_n}) of problem (1.1) and a pair of functions $(u, v) \in C^1(\overline{\Omega}) \times L^\infty(\Omega)$ with $C_1 \leq u, v \leq C_2$, such that

$$u_{\mu_n} \to u \quad \text{in } C^1(\overline{\Omega}), \quad v_{\mu_n} \rightharpoonup v \quad \text{in } L^2(\Omega).$$

Clearly, (u,v) satisfies, in the weak sense,

$$-\Delta u = u(\lambda - v) \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

Multiplying the equation of v_{μ_n} by $\phi \in C_0^\infty(\Omega)$ and integrating over Ω, we get

$$-\frac{1}{\mu_n}\int_\Omega v_{\mu_n}\Delta\phi dx = \int_\Omega v_{\mu_n}\Big(1-\frac{v_{\mu_n}}{u_{\mu_n}}\Big)\phi dx.$$

Passing to the limit yields

$$\int_\Omega v\big(1-\frac{v}{u}\big)\phi dx = 0,$$

which implies that $v\big(1-\frac{v}{u}\big) = 0$. Since $v \neq 0$, we must have $v = u$. By the regularity theory of elliptic equation, $u \in C^2(\overline{\Omega})$ and satisfies

$$-\Delta u = u(\lambda - u) \quad \text{in } \Omega, \quad \partial_\nu u = 0 \quad \text{on } \partial\Omega.$$

Then $u = \lambda$. The proof is complete. □

Proof of Theorem 1.1. Let (u_μ, v_μ) be a positive solution of problem (1.3). By (2.2), there exists a constant $\mu_0 = \mu_0(\lambda,\Omega)$ such that for all $\mu \geq \mu_0$,

$$u_\mu \leq 2\lambda \quad \text{on } \overline{\Omega}. \tag{2.12}$$

Multiplying the equations of u_μ and v_μ by $\frac{\lambda-u_\mu}{u_\mu^2}$ and $\frac{1}{\mu}\frac{\lambda-v_\mu}{v_\mu}$, respectively, we obtain

$$-2\lambda\int_\Omega \frac{|\nabla u_\mu|^2}{u_\mu^3}dx + \int_\Omega \frac{|\nabla u_\mu|^2}{u_\mu^2}dx = \int_\Omega \frac{(\lambda-u_\mu)(\lambda-v_\mu)}{u_\mu}dx,$$

and

$$\begin{aligned}
-\frac{\lambda}{\mu}\int_\Omega \frac{|\nabla v_\mu|^2}{v_\mu^2}dx &= \int_\Omega \frac{(u_\mu-v_\mu)(\lambda-v_\mu)}{u_\mu}dx \\
&= \int_\Omega \frac{(u_\mu-\lambda)(\lambda-v_\mu)}{u_\mu}dx + \int_\Omega \frac{(\lambda-v_\mu)^2}{u_\mu}dx.
\end{aligned}$$

Adding these two equalities yields

$$-2\lambda\int_\Omega \frac{|\nabla u_\mu|^2}{u_\mu^3}dx + \int_\Omega \frac{|\nabla u_\mu|^2}{u_\mu^2}dx - \frac{\lambda}{\mu}\int_\Omega \frac{|\nabla v_\mu|^2}{v_\mu^2}dx = \int_\Omega \frac{(\lambda-v_\mu)^2}{u_\mu}dx. \tag{2.13}$$

Noting (2.12), for all $\mu \geq \mu_0$, we obtain

$$-2\lambda\int_\Omega \frac{|\nabla u_\mu|^2}{u_\mu^3}dx + \int_\Omega \frac{|\nabla u_\mu|^2}{u_\mu^2}dx = \int_\Omega (u_\mu - 2\lambda)\frac{|\nabla u_\mu|^2}{u_\mu^3}dx \leq 0,$$

which and (2.13) implies that $\int_\Omega \frac{(\lambda-v_\mu)^2}{u_\mu}dx \leq 0$, hence $v_\mu = \lambda$ for all $\mu \geq \mu_0$, so $u_\mu = \lambda$ for all $\mu \geq \mu_0$. Combining this and Remark 1.2 completes the proof. □

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