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PERIODIC BOUNDARY-VALUE PROBLEMS FOR FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT. We study the periodic boundary-value problem

$$x^{(iv)}(t) + f(\ddot{x})\dddot{x}(t) + b\ddot{x}(t) + g(t, \dot{x}(t-\tau)) + dx = p(t)$$
$$x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \quad \ddot{x}(0) = \ddot{x}(2\pi), \quad \dddot{x}(0) = \dddot{x}(2\pi),$$

Under some resonant conditions on the asymptotic behaviour of the ratio $g(t, y)/(by)$ for $|y| \to \infty$. Uniqueness of periodic solutions is also examined.

1. INTRODUCTION

In this article we study the periodic boundary-value problem

$$\begin{gathered} x^{(iv)}(t) + f(\ddot{x})\dddot{x}(t) + b\ddot{x} + g(t, \dot{x}(t-\tau)) + dx = p(t) \\ x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \quad \ddot{x}(0) = \ddot{x}(2\pi), \quad \dddot{x}(0) = \dddot{x}(2\pi), \end{gathered} \tag{1.1}$$

with fixed delay $\tau \in [0, 2\pi)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $P : [0, 2\pi] \to \mathbb{R}$ and $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ are $2\pi-$periodic in t and g satisfies Caratheodory conditions with b and d real constants. The unknown function $x : [0, 2\pi] \to \mathbb{R}$ is defined for $0 < t \leq \tau$ by $x(t-\tau) = [2\pi - (t-\tau)]$. We are concerned with the existence and uniqueness of periodic solution of equation (1.1) under some resonant conditions on g.

It is pertinent to note that fourth-order differential equations with time delay are used to model problems in engineering and biological or physiological systems. For instance, the oscillatory movements of muscles that occur as a result of the interaction of a muscle with its load (see [5]). For other papers dealing with the study of fourth order differential equations with time delay see [2, 3] and references therein.

In what follows, we shall use the spaces $C([0, 2\pi]), C^k([0, 2\pi])$ and $L^k([0, 2\pi])$ of continuous, k times continuously differentiable or measurable real functions whose kth power of the absolute value are lebesgue integrable. We shall use the following

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Sobolev spaces:

$$W^{4,2}_{2\pi} = \big\{x : [0,2\pi] \to \mathbb{R} : x, \dot{x}, \ddot{x}, \dddot{x} \text{ are absolutely continuous on } [0,2\pi] \text{ and } x(0) = x(2\pi),\ \dot{x}(0) = \dot{x}(2\pi),\ \ddot{x}(0) = \ddot{x}(2\pi),\ \dddot{x}(0) = \dddot{x}(2\pi)\big\}$$

with the norm

$$|x|^2_{W^{4,2}_{2\pi}} = \sum_{i=0}^{4} \frac{1}{2\pi} \int_0^{2\pi} |x^i(t)|^2 dt$$

and

$$H^1_{2\pi} = \{x : [0,2\pi] \to \mathbb{R} : x \text{ is absolutely continuous on } [0,2\pi] \text{ and } \dot{x} \in L^2_{2\pi}\}$$

with the norm

$$|x|^2_{W^{4,2}_{2\pi}} = \Big(\frac{1}{2\pi} \int_0^{2\pi} x(t)dt\Big)^2 + \frac{1}{2\pi} \int_0^{2\pi} |\dot{x}|^2 dt.$$

2. The Linear Problem

We consider here the linear delay equation

$$\begin{gathered} x^{(iv)}(t) + a\dddot{x}(t) + b\ddot{x}(t) + c\dot{x}(t-\tau)) + dx = 0 \\ x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \quad \ddot{x}(0) = \ddot{x}(2\pi), \quad \dddot{x}(0) = \dddot{x}(2\pi), \end{gathered} \tag{2.1}$$

where c is a real constant.

Lemma 2.1. *Let $b < 0$, $d > 0$ and*

$$0 < \frac{c}{b} < n \tag{2.2}$$

where n is an integer $n \geq 1$. Then (2.1) has no non-trivial periodic solution for any fixed $\tau \in [0, 2\pi)$.

Proof. We consider a solution of the form $x(t) = e^{\lambda t}$ where $\lambda = in$ with $i^2 = -1$. Then Lemma 2.1 will follow if

$$\psi(n,\tau) = n^4 - bn^2 + cn \sin n\tau + d \neq 0$$

for all $n \geq 1$ and $\tau \in [0,2\pi)$. By (2.2), we obtain

$$\begin{aligned} b_{-1}\psi(n,\tau) &= \frac{n^4}{b} - n^2 + \frac{c}{b} n \sin n\tau + \frac{d}{b} \\ &\leq \frac{n^4}{b} - n^2 + \frac{c}{b} n + \frac{d}{b} \\ &< \frac{n^4}{b} + \frac{d}{b} < 0. \end{aligned}$$

Therefore, $\psi(n,\tau) \neq 0$ and the result follows. If $x \in L^1_{2\pi}$ we shall write

$$\bar{x} = \frac{1}{2\pi} \int_0^{2\pi} x(t)dt, \quad \tilde{x}(t) = x(t) - \bar{x}$$

such that $\int_0^{2\pi} \tilde{x}(t)dt = 0$. □

We consider next the delay equation

$$\begin{gathered} x^{(iv)}(t) + a\dddot{x}(t) + b\ddot{x}(t) + c(t)\dot{x}(t-\tau)) + dx = 0 \\ x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \quad \ddot{x}(0) = \ddot{x}(2\pi), \quad \dddot{x}(0) = \dddot{x}(2\pi), \end{gathered} \tag{2.3}$$

where a, b are constants and $c(t) \in L^2_{2\pi}$.

Theorem 2.2. *Let $b < 0$, $d > 0$ and $\Gamma(t) = b^{-1}c(t) \in L^2_{2\pi}$. Suppose that*

$$0 < \Gamma(t) < 1\,. \tag{2.4}$$

Then (2.3) has no non-trivial periodic solution for every fixed $\tau \in [0, 2\pi)$.

Proof. Let $x(t)$ be any solution of (2.3). Then

$$\begin{aligned}
0 &= \frac{1}{2\pi}\int_0^{2\pi} \dddot{x}(t)\Big[\frac{b^{-1}}{2\pi}\Big\{x^{(iv)} + a\dddot{x} + dx + \{\ddot{x} + \Gamma(t)\dot{x}(t-\tau)\}\Big\}\Big]dt \\
&= -\frac{b^{-1}}{2\pi}\int_0^{2\pi} \dddot{x}^2(t)dt - \frac{db^{-1}}{2\pi}\int_0^{2\pi} \dot{x}^2(t)dt + \frac{1}{2\pi}\int_0^{2\pi} \dddot{x}(t)[\dddot{x}(t) + \Gamma(t)\dot{x}(t-\tau)]dt \\
&\geq \frac{1}{2\pi}\int_0^{2\pi} \dddot{x}(t)[\ddot{x}(t) + \Gamma(t)\dot{x}(t-\tau)]dt \\
&= \int_0^{2\pi} [\dddot{x}^2(t) + \Gamma(t)\dddot{x}(t)\dot{x}(t-\tau)]dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big[\dddot{x}^2(t) - \frac{\Gamma(t)}{2}\dddot{x}^2(t) - \frac{\Gamma(t)}{2}\dot{x}^2(t-\tau)\Big]dt \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{\Gamma(t)}{2}\Big[\dddot{x}(t) + \dot{x}(t-\tau)\Big]^2 dt.
\end{aligned}$$

In the above expression we used the equality

$$ab = (\frac{a+b}{2})^2 - \frac{a^2}{2} - \frac{b^2}{2}.$$

From the periodicity of $\dot{x}(t)$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t)dt = \frac{1}{2\pi}\int_0^{2\pi} \dddot{x}^2(t-\tau)dt.$$

Hence,

$$\begin{aligned}
0 &\geq \frac{1}{2}\Big[\frac{1}{2\pi}\int_0^{2\pi} [\dddot{x}^2(t) - \Gamma(t)\dddot{x}^2(t)]dt\Big] \\
&= \frac{1}{2}\Big[\frac{1}{2\pi}\int_0^{2\pi} [\dddot{x}^2(t-\tau) - \Gamma(t)\dot{\dddot{x}}^2(t-\tau)]dt \\
&\geq \delta|\dot{\dddot{x}}|^2_{H^1_{2\pi}} = \delta|\dot{x}|_{H^1_{2\pi}}.
\end{aligned}$$

By [4, Lemma 1] where $\delta > 0$ is a constant. This implies that x is constant a.e. But since $d \neq 0$ we must have $x = 0$, a. e. $\square$

3. The non-linear problem

We shall consider here a preliminary Lemma which will enable us obtain a priori estimates required for our results.

Lemma 3.1. *Let all the conditions of Lemma 2.1 hold and let δ be related to $\Gamma(t)$ by Theorem 2.2. Suppose that $v \in L^2_{2\pi}$ and*

$$0 < v(t) < \Gamma(t) + \epsilon \quad a.e.\ t \in [0, 2\pi]$$

holds for any $v \in L^2_{2\pi}$, where $\epsilon > 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \dddot{x}(t)\Big[b^{-1}\{x^{(iv)} + a\dddot{x} + dx\} + \ddot{x} + \Gamma(t)\dot{x}(t-\tau)\Big]dt \geq (\delta - \epsilon)|\dot{x}|^2_{H^1_{2\pi}}.$$

Proof. From the proof of Theorem 2.2, we have

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\dddot{x}(t)\Big[b^{-1}\{x^{(iv)}+a\dddot{x}+dx\}+\ddot{x}+v(t)\dot{x}(t-\tau)\Big]dt\\
&\geq \frac{1}{2}\Big[\frac{1}{2\pi}\int_0^{2\pi}[\dddot{x}^2(t)-\Gamma(t)\dddot{x}^2(t)]dt\Big]+\frac{1}{2}\Big[\frac{1}{2\pi}\int_0^{2\pi}[\dddot{x}^2(t-\tau)-\Gamma(t)\dot{\ddot{x}}^2(t-\tau)]dt\Big]\\
&\quad -\epsilon\frac{1}{2\pi}\int_0^{2\pi}(\dot{\ddot{x}}^2(t-\tau)+\dddot{x}^2(t))dt\\
&\geq \frac{1}{2}\Big[\frac{1}{2\pi}\int_0^{2\pi}[\dddot{x}^2(t-\tau)-\Gamma(t)\dot{\ddot{x}}^2(t-\tau)]dt\Big]-\frac{\epsilon}{2\pi}\int_0^{2\pi}\dot{x}^2(t-\tau)\\
&\quad -\frac{\epsilon}{2\pi}\int_0^{2\pi}\dddot{x}^2(t-\tau)dt\\
&\geq \delta|\dot{\ddot{x}}|^2_{H^1_{2\pi}}-\epsilon|\dot{\ddot{x}}|^2_{H^1_{2\pi}}\\
&\geq (\delta-\epsilon)|\dot{\ddot{x}}|^2_{H^1_{2\pi}}.
\end{aligned}
$$

∎

We shall consider the non-linear delay equation

$$
x^{(iv)}+f(\ddot{x})\dddot{x}+b\ddot{x}+g(t,\dot{x}(t-\tau))+dx=p(t) \tag{3.1}
$$

where $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and $g:[0,2\pi]\times\mathbb{R}\to\mathbb{R}$ are 2π periodic in t and g satisfies Caratheodory condition; that is, $g(\cdot,x)$ is measurable on $[0,2\pi]$ for each $x\in\mathbb{R}$ and $g(t,\cdot)$ is continuous on $\mathbb{R}$ for almost each $t\in[0,2\pi]$. We assume moreover that for $r>0$ there exists $Y_r\in L^2_{2\pi}$ such that $|g(t,y)|\leq Y_r(t)$ for a.e. $t\in[0,2\pi]$ and $x\in[-r,r]$.

Theorem 3.2. *Let $b<0$ and $d>0$. Suppose that g is Caratheodory function satisfying the inequality*

$$
g(t,y)\geq 0,\quad |y|\leq r \tag{3.2}
$$

$$
\lim_{|y|\to\infty}\sup\frac{g(t,y)}{by}\leq\Gamma(t) \tag{3.3}
$$

uniformly a.e., $t\in[0,2\pi]$ where $r>0$ is a constant and $\Gamma(t)\in L^2_{2\pi}$ is such that

$$
0<\Gamma(t)<1 \tag{3.4}
$$

Then for arbitrary continuous function f, the boundary-value problem (3.1) *has at least one 2π-periodic solution.*

Proof. Let $\delta>0$ be associated to the function Γ by Theorem 2.2. Then by (3.2), (3.3) there exists a constant $R_1>0$ such that

$$
0\leq\frac{g(t,y)}{by}<\Gamma(t)+\frac{\delta}{2} \tag{3.5}
$$

if $|y|\geq R_1$ for a. e., $t\in[0,2\pi]$ and all $y\in\mathbb{R}$. Define $\bar{Y}:[0,2\pi]\times\mathbb{R}\to\mathbb{R}$ by

$$
\overline{Y}=\begin{cases}
y^{-1}g(t,y), & |y|\geq R_1\\
R^{-1}g(t,R), & 0<y<R_1\\
-R_1^{-1}g(t,-R_1), & -R_1<y<0\\
\Gamma(t), & y=0.
\end{cases} \tag{3.6}
$$

Then by (3.5), we have

$$0 \le \overline{Y}(t,y) < \Gamma(t) + \frac{\delta}{2} \tag{3.7}$$

for a. e. $t \in [0,2\pi]$ for all $y \in \mathbb{R}$. Moreover the function $\overline{Y}(t,y)$ satisfies Caratheodory conditions and

$$\tilde{g}(t,\dot{x}(t-\tau)) = b^{-1}g(t,\dot{x}(t-\tau)) - \overline{Y}(t,\dot{x}(t-\tau))\dot{x}(t-\tau)$$

is such that a. e. $t \in [0,2\pi]$ and all $x \in \mathbb{R}$, we have

$$|\tilde{g}(t,\dot{x}(t-\tau))| \le \alpha(t) \tag{3.8}$$

for some $\alpha(t) \in L^2_{2\pi}$. To prove that (3.1) has at least one periodic solution, it suffices to show that the possible solution of the family of equations

$$\begin{aligned} &b^{-1}[x^{(iv)} + \lambda f(\ddot{x})\dddot{x}] + \ddot{x} + (1-\lambda)\Gamma(t)\dot{x}(t-\tau) + \lambda Y(t,\dot{x}(t-\tau)) \\ &+ b^{-1}dx + \lambda\tilde{g}(t,\dot{x}(t-\tau)) + \overline{Y}(t,\dot{x}(t-\tau)) = \lambda b^{-1}p(t) \end{aligned} \tag{3.9}$$

are a-priori bounded in $W^{4,2}_{2\pi}$ independently of $\lambda \in [0,1]$. By inequality (3.7) one has

$$0 \le (1-\lambda)\Gamma(t) + \lambda\overline{Y}(t,\dot{x}(t-\tau)) \le \Gamma(t) + \frac{\delta}{2} \tag{3.10}$$

for a. e. $t \in [0,2\pi]$ and all $x \in \mathbb{R}$. From Theorem 2.2, we can derive that for $\lambda = 0$ equation (3.9) has only the trivial solution. Then using Lemma 3.1 and Cauchy Schwarz inequality we obtain

$$\begin{aligned} 0 &= \frac{1}{2\pi}\int_0^{2\pi} \ddot{x}\Big\{b^{-1}[x^{(iv)} + f(\ddot{x})\dddot{x}] + \ddot{x} + (1-\lambda)\Gamma(t)\dot{x}(t-\tau) \\ &\quad + \lambda\overline{Y}(t,\dot{x}(t-\tau))\dot{x}(t-\tau) + \lambda\tilde{g}(t,\dot{x}(t-\tau)) + b^{-1}dx - \lambda p(t)\Big\}dt \\ &\ge \frac{\delta}{2}|\dot{x}|^2_{H^1_{2\pi}} - (|\alpha|_2 + |b^{-1}||p|_2)|\dot{\bar{x}}|_2 + |b^{-1}|d|\dot{\bar{x}}|_2 \\ &\ge \frac{\delta}{2}|\dot{x}|^2_{H^1_{2\pi}} - \beta|\dot{x}|_{H^2_{2\pi}} - b^{-1}|\dddot{x}|^2_{2\pi} \\ &\ge \frac{\delta}{2}|\dot{x}|^2_{H^1_{2\pi}} - \beta|\dot{x}|_{H^1_{2\pi}} \end{aligned}$$

for some $\beta > 0$. Hence,

$$|\dot{x}|_{H^1_{2\pi}} \le \frac{2\beta}{\delta} = c_1, \tag{3.11}$$

with $c_1 > 0$. This implies

$$|\ddot{x}|_2 \le c_2 \tag{3.12}$$

$$|\ddot{x}|_\infty \le c_3 \tag{3.13}$$

where $c_2 > 0$ and $c_3 > 0$. Using Wirtinger's inequality in (3.12), we obtain

$$|\dot{x}|_2 \le c_4 \tag{3.14}$$

with $c_4 > 0$. Multiplying (3.9) by $-\ddot{x}(t)$ and integrating over $[0,2\pi]$, we obtain

$$|\dddot{x}|^2_2 \le |\ddot{x}|^2_2|1 + \frac{\delta}{2}|\ddot{x}|_2 + |\alpha|_2 + d|\dot{x}|_2 + |p|_2|\ddot{x}|_2$$

Applying Wirtingers inequality we obtain

$$|\dddot{x}|^2_2 \le c_5 \tag{3.15}$$

with $c_5 > 0$ and hence
$$|\ddot{x}|_\infty \le c_6$$
with $c_6 > 0$. We multiply (3.9) by $x^{(iv)}(t)$ and integrate over $[0, 2\pi]$ to get
$$\begin{aligned}
-b^{-1}|x^{(iv)}|_2^2 &\le |f(\ddot{x})|_\infty |\ddot{x}|_2 |x^{(iv)}|_2 |b^{-1}| + |\ddot{x}|_2 |x^{(iv)}|_2 + |1 + \frac{\delta}{2}||\dot{x}|_2 |x^{i(iv)}|_2 \\
&\quad + |b^{-1}||d||\ddot{x}|_2 + |\alpha|_2 |x^{(iv)}|_2 + |p|_2 |x^{i(iv)}|_2 \\
&\le |f(\ddot{x})|_\infty |\ddot{x}|_2 |x^{(iv)}|_2 |b^{-1}| + |\ddot{x}|_2 |x^{(iv)}|_2 \\
&\quad + |1 + \frac{\delta}{2}||\dot{x}|_2 |x^{(iv)}|_2 |b^{-1}|d| x^{(iv)}|_2 + |\alpha|_2 |x^{(iv)}|_2 + |p|_2 |x^{i(iv)}|_2 |b^{-1}|,
\end{aligned}$$
where we used the Wirtinger's inequality. Thus
$$|x^{(iv)}|_2 \le c_7 \tag{3.16}$$
with $c_7 > 0$. Finally multiplying (3.9) by $x(t)$ and integrating over $[0, 2\pi]$ we obtain
$$|x|_2 \le c_8 \tag{3.17}$$
with $c_8 > 0$. Hence,
$$|x|_{W_{2\pi}^{4,2}} = |x|_2 + |\dot{x}|_2 + |\ddot{x}|_2 + |\dddot{x}|_2 + |x^{(iv)}|_2 \le c_8 + c_4 + c_2 + c_5 + c_7 = C_9$$
Taking $R > C_9 > 0$, the required a priori bound in $W_{2\pi}^{4,2}$ is obtained independently of x and λ. □

4. Uniqueness Result

For $f(x) = a$, a constant, in (1.1), we have the following uniqueness result.

Theorem 4.1. *Let a, b, d be constants with $b < 0$ and $d > 0$. Suppose g is a Caratheodory function satisfying*
$$0 < \frac{g(t, \dot{x}_1) - g(t, \dot{x}_2)}{b(\dot{x}_1 - \dot{x}_2)} < \Gamma(t) \tag{4.1}$$
for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \ne x_2$ where $\Gamma(t) \in L_{2\pi}^2$ is such that $0 < \Gamma(t) < 1$. Then for all arbitrary constant a and every $\tau \in [0, 2\pi)$ the boundary-value problem
$$\begin{gathered}
x^{(iv)}(t) + a\dddot{x} + b\ddot{x} + g(t, \dot{x}(t - \tau)) + dx = p(t) \\
x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi), \ddot{x}(0) = \ddot{x}(2\pi), \dddot{x}(0) = \dddot{x}(2\pi),
\end{gathered} \tag{4.2}$$
has at most one solution.

Proof. Let x_1, x_2 be any two solutions of (4.2). Set $x = x_1 - x_2$. Then x satisfies the boundary value problem
$$\begin{gathered}
b^{-1}x^{(iv)}(t) + a\dddot{x} + \Gamma(t)\dot{x}(t - \tau) + b^{-1}dx = 0 \\
x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \quad \ddot{x}(0) = \ddot{x}(2\pi), \quad \dddot{x}(0) = \dddot{x}(2\pi),
\end{gathered}$$
where the function $\Gamma(t) \in L_{2\pi}^2$ is defined by
$$\Gamma(t) = \begin{cases} \frac{g(t, \dot{x}_1(t-\tau)) - g(t, \dot{x}_2(t-\tau))}{\dot{x}(t)} & \text{if } \ddot{x}(t) \ne 0 \\ \frac{1}{2} & \text{if } \ddot{x}(t) = 0 \end{cases}$$
if $\dot{x}(t)$ on every subset of $[0, 2\pi]$ of positive measure, then x is constant Since $d \ne 0$ we must have $x = 0$ and hence $x_1 = x_2$ a.e. Suppose on the other hand that

$\dot{x}(t) \neq 0$ on a certain subset of $[0, 2\pi]$ of positive measure, then using the arguments of Theorem 2.2 we obtain that $x = 0$ and hence $x_1 = x_2$ a .e. $\square$

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