Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 131, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR A COUPLED QUASILINEAR SYSTEM ON TIME SCALES

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Abstract. In this work we study a quasilinear system of equations on time scales. Using the Krasnoselskii fixed-point theorem, sufficient conditions are given for the existence of positive solutions.

1. Introduction

In this paper we consider the existence of positive solutions for the system of dynamic equations

$$\begin{aligned} -\Big(u^{\Delta}(t)\Big)^{\Delta} &= \lambda f(v(t)), \quad \forall t \in [0,T]_{\mathbb{T}}, \\ -\Big(v^{\Delta}(t)\Big)^{\Delta} &= \lambda g(u(t)), \quad \forall t \in [0,T]_{\mathbb{T}}, \end{aligned} \tag{1.1}$$

satisfying the boundary conditions

$$\begin{aligned} u^{\Delta}(0) - \beta u^{\Delta}(\eta) = 0, \quad u(T) - \beta u(\eta) = 0, \\ v^{\Delta}(0) - \beta v^{\Delta}(\eta) = 0, \quad v(T) - \beta v(\eta) = 0, \end{aligned} \tag{1.2}$$

where $\eta \in [0,T]_{\mathbb{T}}, \beta \in \mathbb{R}$, such that $0 < \beta < 1$. We are seeking for a pair (u, v) of solutions for the system (1.1)-(1.2). Our general assumptions are:

- (H1) the functions f, g belong to $C([0,\infty),[0,\infty))$;
- (H2) the following limits exist as real numbers:

$$f_0 := \lim_{x\to 0^+} f(x)/x, \quad g_0 := \lim_{x\to 0^+} g(x)/x,$$
$$f_\infty := \lim_{x\to\infty} f(x)/x, \quad g_\infty := \lim_{x\to\infty} g(x)/x, \quad f_\infty g_\infty \neq 0.$$

The theory of dynamic equations on time scales (more generally, on measure chains) was introduced in 1988 by Stefan Hilger in his PhD thesis (see [10, 11]). The theory presents a structure where, once a result is established for a general time scale, then special cases can be obtained by taking the particular time scale. If $\mathbb{T} = \mathbb{R}$, then we have the result for differential equations. Choosing $\mathbb{T} = \mathbb{Z}$ we immediately get the result for difference equations. A great deal of work has been done since 1988, unifying and extending the theories of differential and difference equations, and many results are now available in the general setting of time scales,

2000 *Mathematics Subject Classification.* 37C25, 34B18, 47H10.
Key words and phrases. Time scales; fixed point theorem; positive solution; cone.

Submitted March 1, 2011. Published October 11, 2011.

see [1, 2, 3, 4] and references therein. In this paper we prove existence of positive solutions to the problem (1.1)-(1.2) on a general time scale $\mathbb{T}$.

The outline of this paper is as follows. In section 2, we give some preliminary results with respect to the calculus on time scales. For more details see for example [7, 8]. Section 3 is devoted to the existence of positive solutions using fixed-point theory. We are concerned with determining values of λ (eigenvalues) for which there exist positive solutions of (1.1)–(1.2). In [12], a Green function plays a fundamental role to define an appropriate operator on a suitable cone and to prove existence of solutions for a system of dynamic equations. Differently, here we will use the well known Guo-Krasnoselskii fixed point theorem without introducing the Green function.

2. Preliminary results on time scales

Now, we introduce some basic concepts of time scales, preliminaries and lemmas that will be needed later. For a deep details, the reader can see [1, 2, 3, 4, 6, 7] and references therein. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$ which are defined in [10, 11],

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}$$

are called the forward jump operator and the backward jump operator, respectively.

The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$ (assume t is not left-scattered if $t = \sup \mathbb{T}$), we define $f^\Delta(t)$ to be the number (provided it exists) such that for all $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le |\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t. If $\mathbb{T} = \mathbb{R}$, then f^Δ coincides with the usual derivative f'. If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta = \Delta f := f(t + 1) - f(t)$ is the forward difference.

Similarly, for $t \in \mathbb{T}$ (assume t is not right-scattered if $t = \inf \mathbb{T}$), the nabla derivative of f at the point t is defined in [5] to be the number $f^\nabla(t)$ (provided it exists) with the property that for all $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \le |\rho(t) - s|, \quad \text{for all } s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$. If $\mathbb{T} = Z$, then $f^\nabla(t) = \nabla f := f(t) - f(t-1)$ is the backward difference operator.

We say that a function f is left-dense continuous (i.e., ld-continuous), provided f is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists at each right-dense point in $\mathbb{T}$. It is well-known that if f is ld-continuous and if $F^\nabla(t) = f(t)$, then we can define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a)$$

for all $a, b \in \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. If f is

rd-continuous and $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a)$$

for all $a, b \in \mathbb{T}$.

From now on, $\mathbb{T}$ is a closed subset of $\mathbb{R}$ such that $0 \in \mathbb{T}_k$, $T \in \mathbb{T}^k$. Let $\mathbb{E} = \mathbb{C}_{ld}([0,T],\mathbb{R})$ which is a Banach space of ld-continuous functions with the maximum norm $\|u\| = \max_{[0,T]} |u(t)|$.

The main tool in this paper is an application of the Guo-Krasnoselskii fixed point theorem for operators leaving a Banach space cone invariant.

Theorem 2.1 (Krasnosel'skii [13]). *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$G : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $\|Gu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Gu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$; *or*
(ii) $\|Gu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Gu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, G has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main Results

By a positive solution of the eigenvalue problem (1.1)-(1.2), we understand a pair of function $(u(t), v(t))$ which is positive on $[0,T]_{\mathbb{T}}$, and satisfies (1.1)-(1.2).

Lemma 3.1. *Assume that hypotheses* (H1)–(H2) *are satisfied. Then a pair of functions $(u(t), v(t))$ is a solution of* (1.1)-(1.2) *if and only if $u(t), v(t) \in \mathbb{E}$ and $(u(t), v(t))$ satisfies the system integral equations:*

$$u(t) = u(0) + \int_0^t \Big(A_1 - \lambda \int_0^s f(v(r))\Delta r\Big)\Delta s, \tag{3.1}$$

$$v(t) = v(0) + \int_0^t \Big(A_2 - \lambda \int_0^s g(u(r))\Delta r\Big)\Delta s, \tag{3.2}$$

where

$$\begin{gathered}
u(0) = \frac{1}{1-\beta}\Big(\beta \int_0^\eta h_1(s)\Delta s - \int_0^T h_1(s)\Delta s\Big), \\
v(0) = \frac{1}{1-\beta}\Big(\beta \int_0^\eta h_2(s)\Delta s - \int_0^T h_2(s)\Delta s\Big), \\
-h_1(s) = \int_0^s \lambda f(v(r))\Delta r - A_1, \\
A_1 = u^{\Delta}(0) = -\frac{\lambda\beta}{1-\beta} \int_0^\eta f(v(r))\Delta r, \\
-h_2(s) = \int_0^s \lambda g(u(r))\Delta r - A_2, \\
A_2 = u^{\Delta}(0) = -\frac{\lambda\beta}{1-\beta} \int_0^\eta g(u(r))\Delta r.
\end{gathered}$$

Proof. Necessity. Integrating (1.1) we have

$$u^\Delta(s) = u^\Delta(0) - \int_0^s \lambda f(v(r))\Delta r.$$

On the other hand, by the boundary condition (1.2) we have

$$u^\Delta(0) = \beta u^\Delta(\eta) = \beta\Big(u^\Delta(0) - \int_0^\eta \lambda f(v(r))\Delta r\Big).$$

Then

$$A_1 = u^\Delta(0) = \frac{-\lambda\beta}{1-\beta}\int_0^\eta f(v(r))\Delta r.$$

It follows that

$$u^\Delta(s) = -\lambda\int_0^s f(v(r))\Delta r + A_1.$$

Integrating the above equation we obtain

$$u(t) = u(0) + \int_0^t \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s. \tag{3.3}$$

Moreover, by (3.3) and the boundary condition (1.2), we have

$$\begin{aligned}
u(0) &= u(T) - \int_0^T \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s\\
&= \beta u(\eta) - \int_0^T \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s\\
&= \beta\Big(u(0) + \int_0^\eta \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s\Big)\\
&\quad - \int_0^T \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s.
\end{aligned}$$

Then

$$\begin{aligned}
u(0) = \frac{1}{1-\beta}\Big(&\beta\int_0^\eta \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s\\
&- \int_0^T \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s\Big).
\end{aligned}$$

Then we have (3.1). Similar arguments need to be done to prove (3.2).

Sufficiency. Simple calculations by taking the delta derivative of $u(t)$ and $v(t)$ lead to the result. □

Lemma 3.2. *Suppose that* (H1)–(H2) *hold, then a solution* (u, v) *of* (1.1)-(1.2) *satisfies* $u(t) \geq 0$ *and* $v(t) \geq 0$, *for* $t \in [0, T]_{\mathbb{T}}$.

Proof. Since $0 < \beta < 1$, we have

$$\begin{aligned}
&u(0)\\
&= \frac{\beta}{1-\beta}\int_0^\eta \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s - \frac{1}{1-\beta}\int_0^T \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s\\
&\geq \frac{\beta}{1-\beta}\Big\{\int_0^\eta \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s - \int_0^T \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s\Big\}\\
&\geq 0,
\end{aligned}$$

and

$$
\begin{aligned}
u(T) &= u(0) + \int_0^T h_1(s)\Delta s \\
&= \frac{\beta}{1-\beta}\int_0^\eta h_1(s)\Delta s - \frac{1}{1-\beta}\int_0^T h_1(s)\Delta s + \int_0^T h_1(s)\Delta s \\
&\geq \frac{\beta}{1-\beta}\int_0^\eta h_1(s)\Delta s - \frac{\beta}{1-\beta}\int_0^T h_1(s)\Delta s \\
&\geq \frac{\beta}{1-\beta}\Big(\int_0^\eta h_1(s)\Delta s - \int_0^T h_1(s)\Delta s\Big) \\
&\geq \frac{-\beta}{1-\beta}\int_\eta^T h_1(s)\Delta s \\
&\geq \frac{\beta}{1-\beta}\int_\eta^T (-h_1(s))\Delta s \geq 0.
\end{aligned}
$$

If $t \in [0,T]_{\mathbb{T}}$, then

$$
\begin{aligned}
u(t) &= u(0) + \int_0^t \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s \\
&\geq u(0) + \int_0^T \Big(A_1 - \lambda\int_0^s f(v(r))\Delta r\Big)\Delta s \\
&= u(T) \geq 0\,.
\end{aligned}
$$

Arguing exactly as above, we have $v(t) \geq 0$. The proof is now complete. □

Lemma 3.3. *Suppose that* (H1)-(H2) *are satisfied, then*

$$
u(T) = \inf_{t\in\mathbb{T}} u(t) \geq \rho u(0) = \rho\|u\|,
$$

where $1 \geq \rho = \beta\frac{T-\eta}{T-\beta\eta} \geq 0$.

Proof. Since $u^\Delta(0) \leq 0$ we have $u^\Delta(s) = u^\Delta(0) - \lambda\int_0^s f(v(r))\Delta r \leq 0$, then $u^\Delta \leq 0$, which implies that u is a non-increasing function on $[0,T]_{\mathbb{T}}$. Moreover, we have $\|u\| = u(0)$, $\inf_{t\in(0,T)_{\mathbb{T}}} u(t) = u(T)$. Hence, from the concavity of $u(t)$ that each point on chord between $(0,u(0))$ and $(T,u(T))$ is below the graph of $u(t)$, we have

$$
u(T) \geq u(0) + T\frac{u(T)-u(\eta)}{T-\eta}.
$$

On other terms

$$
Tu(\eta) - \eta u(T) \geq (T-\eta)u(0).
$$

Using the boundary condition (1.2), it follows

$$
(\frac{T}{\beta} - \eta)u(T) \geq (T-\eta)u(0).
$$

Then

$$
u(T) \geq \beta\frac{T-\eta}{T-\beta\eta}u(0).
$$

□

For our construction of an operator G, define a cone $\mathbb{K} \subset \mathbb{E}$ by

$$\mathbb{K} = \big\{u \in \mathbb{E} : u(t) \geq 0 \text{ on } \mathbb{T} \text{ and } u(t) \geq \rho\|u\|, \text{ for } t \in \mathbb{T}\big\}. \tag{3.4}$$

Also define the positive numbers

$$\lambda_1 = \max\Big(\Big(\frac{\beta(T-\eta)\eta}{(1-\beta)}\sqrt{\rho(f_\infty - \varepsilon)(g_\infty - \varepsilon)}\Big)^{-1}, \Big(\frac{\beta}{1-\beta}(g_\infty - \epsilon)\eta\Big)^{-1}\Big),$$
$$\lambda_2 = \min\Big(\Big(\frac{2\sqrt{2}T^2}{(1-\beta)^2}\sqrt{(f_0 + \varepsilon)(g_0 + \varepsilon)}\Big)^{-1}, \Big(\frac{2T^2}{(1-\beta)^2}(g_0 + \epsilon)\Big)^{-1}\Big).$$

Theorem 3.4. *Assume that conditions* (h1)–(H2) *are satisfied. Then, for each λ satisfying $\lambda_1 < \lambda < \lambda_2$, there exists a pair (u, v) satisfying* (1.1)-(1.2) *such that $u(t) \geq 0$ and $v(t) \geq 0$ on $[0, T]_{\mathbb{T}}$.*

Proof. The main idea of the proof is to use Theorem 2.1. First observe that we seek suitable fixed points in the cone $\mathbb{K}$ 3.2 of the integral operator

$$Gu(t) = u(0) + \int_0^t \Big(A_1 - \lambda \int_0^s f(v(r))\Delta r\Big)\Delta s, \tag{3.5}$$

where v is a function of u given by (3.2).

Lemma 3.5. *Let G defined by* (3.5)*, then*

- (i) $G(\mathbb{K}) \subseteq \mathbb{K}$.
- (ii) $G : \mathbb{K} \to \mathbb{K}$ *is completely continuous.*

Notice that by Lemma 3.2 and Lemma 3.3, (i) is satisfied. On the other hand standard arguments show that G is completely continuous.

Since $\beta < 1$ we have

$$\begin{aligned} Gu(t) \leq u(0) &= \frac{1}{1-\beta}\Big(-\beta\int_0^\eta h_1(s)\Delta s + \int_0^T h_1(s)\Delta s\Big) \\ &\leq \frac{2}{1-\beta}\int_0^T |h_1(s)|\Delta s. \end{aligned} \tag{3.6}$$

Now, from the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \leq (f_0 + \epsilon)x, \ g(x) \leq (g_0 + \epsilon)x, \quad \text{for all } x \text{ such } 0 < x \leq H_1. \tag{3.7}$$

Let $u \in \mathbb{K}$ with $\|u\| = H_1$. We have

$$\begin{aligned} |v(r)| &= \big|v(0) + \int_0^r h_2(s)\Delta s\big| \\ &\leq |v(0)| + \int_0^r |h_2(s)|\Delta s \\ &\leq |v(0)| + \int_0^\eta |h_2(s)|\Delta s. \end{aligned} \tag{3.8}$$

Furthermore,

$$
\begin{aligned}
|h_2(s)| &= \big|A_2 - \lambda \int_0^s g(u(r))\Delta r\big| \\
&\le |A_2| + \lambda \int_0^s |g(u(r))|\Delta r \\
&\le |A_2| + \lambda(g_0+\epsilon)|s|\|u\| \\
&\le \frac{\lambda\beta}{1-\beta}\int_0^\eta |g(u(r))|\Delta r + \lambda(g_0+\epsilon)|s|\|u\| \\
&\le \frac{\lambda\beta}{1-\beta}(g_0+\epsilon)\eta\|u\| + \lambda(g_0+\epsilon)|s|\|u\| \\
&\le \frac{\lambda T}{1-\beta}(g_0+\epsilon)\|u\|.
\end{aligned} \tag{3.9}
$$

Then

$$
\int_0^T |h_2(s)|\Delta s \le \frac{\lambda T^2}{1-\beta}(g_0+\epsilon)\|u\|. \tag{3.10}
$$

Then using (3.10), we have

$$
\begin{aligned}
|v(0)| &= |\frac{\beta}{1-\beta}\int_0^\eta h_2(s)\Delta s - \frac{1}{1-\beta}\int_0^T h_2(s)\Delta s| \\
&\le \frac{\beta}{1-\beta}\int_0^\eta |h_2(s)|\Delta s + \frac{1}{1-\beta}\int_0^T |h_2(s)|\Delta s \\
&\le \frac{1+\beta}{1-\beta}\int_0^T |h_2(s)|\Delta s \\
&\le \lambda T^2\frac{1+\beta}{(1-\beta)^2}(g_0+\epsilon)\|u\|.
\end{aligned} \tag{3.11}
$$

Then from (3.8)-(3.11), we obtain

$$
\begin{aligned}
|v(r)| &\le \lambda T^2\frac{1+\beta}{(1-\beta)^2}(g_0+\epsilon)\|u\| + \frac{\lambda T^2}{(1-\beta)}(g_0+\epsilon)\|u\| \\
&\le \frac{2\lambda T^2}{(1-\beta)^2}(g_0+\epsilon)\|u\|.
\end{aligned} \tag{3.12}
$$

The choice of λ_2 it yields

$$
\|v\| \le \frac{2\lambda T^2}{(1-\beta)^2}(g_0+\epsilon)\|u\| \le \frac{2\lambda_2 T^2}{(1-\beta)^2}(g_0+\epsilon)\|u\| \le \|u\| = H_1. \tag{3.13}
$$

Then by using (3.7) and (3.13) we obtain

$$
|A_1| \le \frac{\lambda\beta}{1-\beta}\int_0^\eta |f(v(r))|\Delta r \le \frac{\lambda\beta}{1-\beta}(f_0+\epsilon)\int_0^\eta |v(r)|\Delta r. \tag{3.14}
$$

It follows from (3.12)-(3.14) that

$$
\begin{aligned}
|A_1| &\le \frac{\lambda\beta}{1-\beta}(f_0+\epsilon)\eta\frac{2\lambda T^2}{(1-\beta)^2}(g_0+\epsilon)\|u\| \\
&= \frac{2\lambda^2\beta T^2\eta}{(1-\beta)^3}(f_0+\epsilon)(g_0+\epsilon)\|u\|.
\end{aligned} \tag{3.15}
$$

Since $h_1(s) = A_1 - \lambda\int_0^s f(v(r))\Delta r$, it follows that

$$|h_1(s)| \le |A_1| + \lambda\int_0^s |f(v(r))|\Delta r.$$

We obtain from (3.7) and (3.12), that

$$\begin{aligned}\int_0^s |f(v(r))|\Delta r &\le \int_0^s (f_0+\epsilon)\|v\|\Delta r \le T(f_0+\epsilon)\|v\| \\ &\le \frac{2\lambda T^3}{(1-\beta)^2}(f_0+\epsilon)(g_0+\epsilon)\|u\|.\end{aligned} \tag{3.16}$$

It follows from (3.15)-(3.16) that

$$\begin{aligned}|h_1(s)| &\le \Big(\frac{2\lambda^2\beta T^2\eta}{(1-\beta)^3}(f_0+\epsilon)(g_0+\epsilon)\|u\| + \frac{2\lambda^2T^3}{(1-\beta)^2}(f_0+\epsilon)(g_0+\epsilon)\|u\|\Big) \\ &\le \frac{4\lambda^2T^3}{(1-\beta)^3}(f_0+\epsilon)(g_0+\epsilon)\|u\|.\end{aligned} \tag{3.17}$$

Combining inequalities (3.12)-(3.16), we have, from (3.6) and the choice of λ_2,

$$\begin{aligned}\|Gu\| &\le \frac{2}{1-\beta}\int_0^T |h_1(s)|\Delta s \\ &\le \frac{8\lambda^2T^4}{(1-\beta)^4}(f_0+\epsilon)(g_0+\epsilon)\|u\| \\ &\le \lambda_2^2\frac{8T^4}{(1-\beta)^4}(f_0+\epsilon)(g_0+\epsilon)\|u\| = \|u\|.\end{aligned}$$

So $\|Gu\| \le \|u\|$. If we set $\Omega_1 = \{u \in \mathbb{E} : \|u\| < H_1\}$, then

$$\|Gu\| \le \|u\| \quad \text{for } u \in \mathbb{K}\cap\partial\Omega_1.$$

Now, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2 > 0$ such that

$$f(x) \ge (f_\infty - \varepsilon)x, \quad g(x) \ge (g_\infty - \varepsilon)x, \quad \text{for } x \ge \overline{H}_2 > 0. \tag{3.18}$$

Let $H_2 = \max\{2H_1, \frac{\overline{H}_2}{\rho}\}$. Let $u \in \mathbb{K}$ with $\|u\| = H_2, \min u(t) \ge \rho\|u\| \ge \overline{H}_2$.

On the other hand,

$$\begin{aligned}Gu(t) &= u(0) + \int_0^t h_1(s)\Delta s \\ &= \frac{\beta}{1-\beta}\int_0^\eta h_1(s)\Delta s - \frac{1}{1-\beta}\int_0^T h_1(s)\Delta s + \int_0^t h_1(s)\Delta s \\ &\ge \frac{\beta}{1-\beta}\int_0^\eta h_1(s)\Delta s - \frac{\beta}{1-\beta}\int_0^T h_1(s)\Delta s \\ &\ge \frac{\beta}{1-\beta}\Big(\int_0^\eta h_1(s)\Delta s - \int_0^T h_1(s)\Delta s\Big) \\ &\ge \frac{-\beta}{1-\beta}\int_\eta^T h_1(s)\Delta s \\ &\ge \frac{\beta}{1-\beta}\int_\eta^T (-h_1(s))\Delta s.\end{aligned} \tag{3.19}$$

Furthermore, from (3.2) and as in (3.19) we have

$$v(t) \geq \frac{\beta}{1-\beta} \int_\eta^T (-h_2(s))\Delta s.$$

Since $A_2 \leq 0$, we have from (3.18)

$$\begin{aligned}
-h_2(s) &\geq \lambda \int_0^s g(u(r))\Delta r - A_2 \\
&\geq \lambda \int_0^s g(u(r))\Delta r \\
&\geq \lambda(g_\infty - \epsilon) \int_0^s u(r)\Delta r \\
&\geq \lambda(g_\infty - \epsilon) \int_0^s \rho\|u\|\Delta r \\
&\geq \lambda(g_\infty - \epsilon)\rho\|u\|s.
\end{aligned}$$

Then by the choice of λ_1,

$$\begin{aligned}
v(t) &\geq \frac{\lambda\beta}{1-\beta}(g_\infty - \epsilon)\rho\|u\| \int_\eta^T s\Delta s \\
&\geq \lambda \frac{\beta}{1-\beta}(g_\infty - \epsilon)\rho\|u\|\eta \\
&\geq \lambda_1 \frac{\beta}{1-\beta}(g_\infty - \epsilon)\rho\eta\|u\| \\
&\geq \rho\|u\| \geq \overline{H_2}.
\end{aligned} \tag{3.20}$$

Since $A_1 \leq 0$, then by (3.20) we have

$$-h_1(s) = \lambda \int_0^s f(v(r))\Delta r - A_1 \geq \lambda \int_0^s f(v(r))\Delta r \geq \lambda(f_\infty - \epsilon) \int_0^\eta v(r)\Delta r.$$

We also have

$$v(r) = v(0) + \int_0^s h_2(s)\Delta s \geq \frac{\beta}{1-\beta} \int_\eta^T (-h_2(s))\Delta s.$$

Applying Lemma 3.3, we have

$$\begin{aligned}
-h_2(s) &= \lambda \int_0^s g(u(r))\Delta r - A_2 \\
&\geq \lambda \int_0^s g(u(r))\Delta r \\
&\geq \lambda \int_0^\eta g(u(r))\Delta r \\
&\geq \lambda(g_\infty - \epsilon) \int_0^\eta u(r)\Delta r \\
&\geq \lambda(g_\infty - \epsilon)\rho\eta\|u\|.
\end{aligned}$$

Consequently,

$$v(r) \geq \lambda \frac{\beta}{1-\beta}(T-\eta)(g_\infty - \epsilon)\rho\eta\|u\|.$$

Then

$$-h_1(s) \geq \lambda^2 (T-\eta) \frac{\beta \rho \eta^2}{1-\beta} (f_\infty - \epsilon)(g_\infty - \epsilon)\|u\|.$$

Finally, for $\lambda \geq \lambda_1$, we obtain

$$\begin{aligned} Gu(t) &= \frac{\beta}{1-\beta} \int_\eta^T (-h_1(s))\Delta s \\ &\geq \lambda^2 \frac{\beta^2 \eta^2 (T-\eta)^2}{(1-\beta)^2} \rho (f_\infty - \epsilon)(g_\infty - \epsilon)\|u\| \\ &\geq \lambda_1^2 \frac{\beta^2 \eta^2 (T-\eta)^2}{(1-\beta)^2} \rho (f_\infty - \epsilon)(g_\infty - \epsilon)\|u\| \\ &\geq \|u\| = H_2. \end{aligned}$$

Hence, $\|Gu\| \geq \|u\|$. So, if we set $\Omega_2 = \{u \in \mathbb{E} : \|u\| < H_2\}$, then

$$\|Gu\| \geq \|u\|, \quad \text{for } u \in \mathbb{K} \bigcap \partial\Omega_2.$$

Applying Theorem 2.1, we obtain that G has a fixed point $u \in \mathbb{K} \bigcap (\overline{\Omega_2}/\Omega_1)$. With v being defined by

$$v(t) = v(0) + \int_0^t \Big(A_2 - \lambda \int_0^s g(u(r))\Delta r \Big) \Delta s,$$

the pair (u, v) is a solution of (1.1)-(1.2) for the given λ. The proof is now complete. □

We point out that, under technical calculus, results of this paper can be extended to the following p-Laplacian case with $2 \leq p \leq +\infty$:

$$\begin{aligned} -\Big(\phi_p(u^\Delta(t))\Big)^\Delta &= \lambda f(v(t)), \quad \forall t \in [0,T]_{\mathbb{T}}, \\ -\Big(\phi_p(v^\Delta(t))\Big)^\Delta &= \lambda g(u(t)), \quad \forall t \in [0,T]_{\mathbb{T}}, \end{aligned} \tag{3.21}$$

where $\phi_p(\xi) = |\xi|^{p-2}\xi$. If $p = 2$ we obtain problem (1.1).

Acknowledgements. This work was supported by the project *New Explorations in Control Theory Through Advanced Research* (NECTAR) cofinanced by *Fundação para a Ciência e a Tecnologia* (FCT), Portugal, and the *Centre National de la Recherche Scientifique et Technique* (CNRST), Morocco. A.B. Malinowska is a senior researcher at the *Center for Research and Development in Mathematics and Applications* (CIDMA) at the University of Aveiro, Portugal, under the support of BUT grant S/WI/02/2011.

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