Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 133, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LIMIT CYCLES AND BOUNDED TRAJECTORIES FOR A NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATION

HENRY GONZÁLEZ

ABSTRACT. In this article, we determine the trajectories of maximum deviation, and the closed trajectories of maximum deviation for nonlinear differential equations of the form

$$\ddot{y} + 2a(t, y, \dot{y})\dot{y} + b(t, y, \dot{y})y = c(t, y, \dot{y})$$

where the coefficients and the right-hand side are piecewise continuous functions in t and continuous in $y, \dot{y}$. Also we find necessary and sufficient conditions for the boundedness of all the trajectories.

1. Introduction

In the application of mathematical models to real problems, after linearization, simplifications and other adaptations of the models, frequently we have to investigate the solutions of differential systems which contain uncertain parameters. One of the methods to attack this problem is the extremal principle which consists in the determination of trajectories that are the solution of some optimal problems and with their help we can determine the considered properties of all the trajectories of our differential system. In 1946, Bulgakov [1] apply this method for an n-order differential equation with constant coefficients and with uncertainty in the right side. In the problem, the maximum deviation was computed for solutions with a fixed end time, and then calculated the supremum of the maximum deviation when the end time tends to infinity. In the previous decades have appeared a great number of publications related with the application of the extremal principle in the solutions of problems of absolute stability [2, 3], stability radius [4], and others. In the present article, using the solutions of maximum deviation for a second-order nonlinear differential equation with uncertainty, we give necessary and sufficient condition for the boundedness of all the trajectories.

This article is organized as follows: In section 2 we formulate the main problem and determine the trajectories of maximum deviation of our differential system. In section 3 we determine the limit cycles of maximum deviation as function of the parameters of the system. Also we show the stability of these cycles and using these results we obtain two qualitative behaviours of phase portrait of the trajectories of

2000 *Mathematics Subject Classification.* 34C05, 70K05.
Key words and phrases. Ordinary differential equations; phase plane analysis; limit cycles; maximum deviation trajectories.

Submitted April 15, 2011. Published October 13, 2011.

maximum deviation; which allows us to give the necessary and sufficient condition for the boundedness of all the trajectories.

2. Problem formulation and the trajectories of maximum deviation

Let $\underline{a}, \overline{a}, \underline{b}, \overline{b}, c_0$ be given real numbers, which satisfy the inequalities

$$0 < \underline{a} \leq \overline{a}, \quad 0 < \underline{b} \leq \overline{b}, \quad \overline{a}^2 - \underline{b} < 0, \quad c_0 > 0, \tag{2.1}$$

and let $\mathcal{E}$ be the family of nonlinear second-order differential equations

$$\ddot{y} + 2a(t, y, \dot{y})\dot{y} + b(t, y, \dot{y})y = c(t, y, \dot{y}), \tag{2.2}$$

where $a(t, y, \dot{y})$, $b(t, y, \dot{y})$, $c(t, y, \dot{y})$ are piecewise continuous functions in the variable t and continuous in $y, \dot{y}$, satisfying the inequalities

$$\underline{a} \leq a(t, y, \dot{y}) \leq \overline{a}, \quad \underline{b} \leq b(t, y, \dot{y}) \leq \overline{b}, \quad |c(t, y, \dot{y})| \leq c_0. \tag{2.3}$$

In this article, we give necessary and sufficient condition in terms of the parameters $\underline{a}, \overline{a}, \underline{b}, \overline{b}, c_0$, for the boundedness of all the trajectories of the family $\mathcal{E}$ of equations. A similar problem has been analyzed for the particular case of linear inhomogeneous equations (2.2) with constant coefficients ($\underline{a} = \overline{a}$, $\underline{b} = \overline{b}$) in [6].

It follows from condition (2.1) for the parameters that for all real number δ there is a trajectory of the family of equations $\mathcal{E}$, which begins in the point $(\delta, 0)$ and realizes infinitely rotating turns in positive direction (counterclockwise) around the origin in the phase plane $y, \dot{y}$. So we can consider for every real number δ the extremal problem

$$\begin{gathered} |y(T)| \to \sup \\ y(.) \text{ is a solution of } \mathcal{E} \\ y(0) = \delta, \quad \dot{y}(0) = 0 \\ \dot{y}(T) = 0, \quad \dot{y}(t) \neq 0, \quad t \in (0, T). \end{gathered} \tag{2.4}$$

This problem has been named problem of maximum deviation of the solutions of the family of equations $\mathcal{E}$. Note that if we consider the same problem with the functions $a(.), b(.), c(.)$ depend only on t, so taking into account that according with (2.3) the bounds for this functions: $\underline{a}, \overline{a}$, $\underline{b}, \overline{b}, c_0$, are constants, we have that the solution of the corresponding linear problem is the same that the nonlinear one.

The extremal problem (2.4) can be interpreted as an optimal control problem with variable time T in which the role of the control is played by the functions $a(t), b(t), c(t), t \in [0, T]$. We say that a trajectory of the family $\mathcal{E}$ is of maximum deviation if it is a continuous solution of an equation of $\mathcal{E}$ and if it is the union of trajectories, which are optimal for the extremal problem (2.4). To solve the optimal problem (2.4) we apply the Pontryagin Maximum Principle [7]-[8]. In the variables $x = (x_1, x_2) := (y, \dot{y})$ the considered differential equations of the family $\mathcal{E}$ are equivalent to the system

$$\begin{gathered} \dot{x}_1 = x_2 \\ \dot{x}_2 = -b(t)x_1 - 2a(t)x_2 + c(t) \end{gathered} \tag{2.5}$$

So the conjugate system and the Pontryagin function are

$$\begin{gathered} \dot{\psi}_0 = 0 \\ \dot{\psi}_1 = b(t)\psi_2 \\ \dot{\psi}_2 = -\psi_0 - \psi_1 - 2a(t)\psi_2, \end{gathered}$$

and

$$H(x,\psi,a,b,c)=\psi_0x_2+\psi_1x_2+\psi_2(-bx_1-2ax_2+c). \tag{2.6}$$

The necessary conditions for the optimality have the form:

(1) $\max H(x(t),\psi(t),a,b,c)=H(x(t),\psi(t),a(t),b(t),c(t))\equiv 0$,
(2) $\psi_0\equiv -1$,
(3) $\psi_1(T)\theta_1+\psi_2(T)\theta_2=0$,

where θ is a vector colinear with $(1,0)$. According with condition (1) and the expression of the Pontryagin function (2.6), we have that the optimal control $(a^0(t),b^0(t),c^0(t))$ satisfies

$$\begin{gathered}a^0(t)=\underline{a}\quad \text{for all } t\\ b^0(t)=\begin{cases}\underline{b} & \text{if } x_1(t)\psi_2(t)>0\\ \overline{b} & \text{if } x_1(t)\psi_2(t)<0\end{cases}\\ c^0(t)=\begin{cases}c_0 & \text{if } \psi_2(t)>0\\ -c_0 & \text{if } \psi_2(t)<0\end{cases}\end{gathered} \tag{2.7}$$

From (3) follows that $\psi_1(T)=0$, so if we write $\overline{\psi}_1=\psi_1+1$, then $\overline{\psi}_1(T)=1$ and from the fact that $(\overline{\psi}_1(t),\psi_2(t))$ and $(x_1(t),x_2(t))$ are conjugate variables we have: $x_1(t)\overline{\psi}_1(t)+x_2(t)\psi_2(t)\equiv C$, where C is a constant. Evaluating the last equality for $t=T$ we obtain $C=x_1(T)$. From all that and the necessary condition (1), (2.6) and (2.5) we have

$$\begin{gathered}\dot x_1(t)\overline{\psi}_1(t)+\dot x_2(t)\psi_2(t)\equiv 0\\ x_1(t)\overline{\psi}_1(t)+x_2(t)\psi_2(t)\equiv x_1(T)\end{gathered}$$

. Solving with respect to $\psi_2(t)$, we have

$$\psi_2(t)=\frac{-x_1(t)\dot x_1(t)}{x_2(t)\dot x_1(t)-x_1(t)\dot x_2(t)}. \tag{2.8}$$

It is well known that the trajectories of maximum deviation move all the time in a unique sense around the origin of coordinates, because if such a trajectory moves in the two possible directions around the origin as illustrate the figure **??**, then we can easily construct a trajectory (by example with the help of the dashed line in figure 1 which contradicts the optimality of the trajectory of maximum deviation.

So the trajectories of maximum deviation move all the time in positive sense around the origin and then in the expression (2.8) the denominator is positive for all $t\in[0,T]$, and the numerator have the same sign as $-x_1(t)\dot x_1(t)=-x_1(t)\dot y(t)$, so from this fact and (2.7) we conclude that the synthesis of the optimal control for the problem (2.4) is

$$\begin{gathered}a^0(y,\dot y)=\underline{a}\quad \text{for all } (y,\dot y)\in\mathbb{R}^2\\ b^0(y,\dot y)=\begin{cases}\underline{b} & \text{if } y\dot y>0\\ \overline{b} & \text{if } y\dot y<0\end{cases}\\ c^0(y,\dot y)=\begin{cases}c_0 & \text{if } \dot y>0\\ -c_0 & \text{if } \dot y<0\end{cases}\end{gathered} \tag{2.9}$$

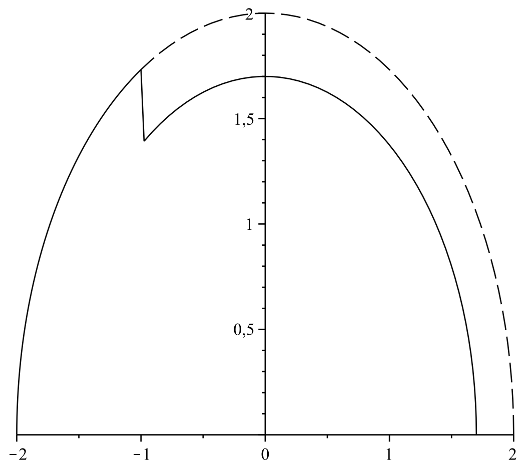

figure 1

Therefore, the trajectories of maximum deviation of $\mathcal{E}$ are the solutions of the equation

$$\ddot{y} + 2a^0(y,\dot{y})\dot{y} + b^0(y,\dot{y})y = c^0(y,\dot{y}), \tag{2.10}$$

where the functions $a^0(y,\dot{y}), b^0(y,\dot{y}), c^0(y,\dot{y})$ are given by (2.9).

3. Closed trajectories of maximum deviation and main results

Now we consider a trajectory of maximum deviation of $\mathcal{E}$ and denote by $(-\delta, 0)$, $(0,\alpha)$, $(\gamma, 0)$, $(0,-\theta)$, $(-\epsilon, 0)$ the points of interception of this trajectory with the axes of coordinate; see figure 2.

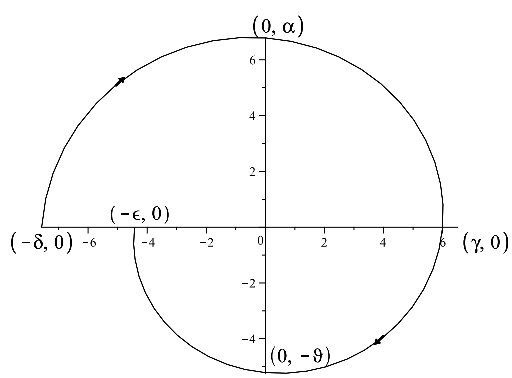

Figure 2: maximum deviation trajectory

The numbers $\alpha, \gamma, \theta, \epsilon$ are functions of the number δ. In the case $\epsilon(\delta) = \delta$ the considered trajectory is closed and we will denote it by C_δ. The bounded region limited by the closed trajectory C_δ is invariant for the family of equations $\mathcal{E}$.

In the qualitative study of the solutions of $\mathcal{E}$ it is fundamental to determine the closed trajectories of maximum deviation of $\mathcal{E}$ and their stability in the sense of limit cycles.

From the expressions (2.9) it follows that the optimal trajectories for (2.4) corresponding to opposite values of δ are symmetric with respect to the origin of coordinates and so the equality $\epsilon(\delta)=\delta$ is fulfilled if and only if the equality $\gamma(\delta)=\delta$ is fulfilled. Thus to determine the closed trajectories of maximum deviation of $\mathcal{E}$ we must solve the equation $\gamma(\delta)=\delta$ with respect to the unknown number δ. Define the functions $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\varphi(\delta)=\alpha, \quad \text{and} \quad \psi(\alpha)=\gamma.$$

From the theorem of existence and uniqueness of solutions for differential equations it follows that the functions φ and ψ are correctly defined on $\mathbb{R}^+=\{x\in\mathbb{R}, x>0\}$, they are injective functions and so the inverse to φ: $\delta=\varphi^{-1}(\alpha)$ exists. Then solving the equation (2.10)-(2.9), with initial condition $y(0)=0, \dot{y}(0)=\alpha$ we obtain for the functions $\varphi^{-1}(\alpha)$ and $\psi(\alpha)$ the expressions:

$$\begin{aligned} \varphi^{-1}(\alpha) &= \frac{e^{\underline{a}\tau(\alpha)}}{\overline{b}}\sqrt{\overline{b}\alpha^2-2\underline{a}c_0\alpha+{c_0}^2}-\frac{c_0}{\overline{b}} \\ \psi(\alpha) &= \frac{e^{-\underline{a}s(\alpha)}}{\underline{b}}\sqrt{\underline{b}\alpha^2-2\underline{a}c_0\alpha+{c_0}^2}+\frac{c_0}{\underline{b}}, \end{aligned} \tag{3.1}$$

where

$$\tau(\alpha)=\begin{cases} \frac{1}{\overline{\beta}}\arctan(\frac{\overline{\beta}\alpha}{c_0-\underline{a}\alpha}) & \text{if } \alpha<\frac{c_0}{\underline{a}} \\ \frac{1}{\overline{\beta}}[\pi-\arctan(\frac{\overline{\beta}\alpha}{-c_0+\underline{a}\alpha})] & \text{if } \alpha>\frac{c_0}{\underline{a}} \\ \frac{\pi}{2\overline{\beta}} & \text{if } \alpha=\frac{c_0}{\underline{a}}, \end{cases} \tag{3.2}$$

$$s(\alpha)=\begin{cases} \frac{1}{\underline{\beta}}[\pi-\arctan(\frac{\underline{\beta}\alpha}{c_0-\underline{a}\alpha})] & \text{if } \alpha<\frac{c_0}{\underline{a}} \\ \frac{1}{\underline{\beta}}\arctan(\frac{\underline{\beta}\alpha}{-c_0+\underline{a}\alpha}) & \text{if } \alpha>\frac{c_0}{\underline{a}} \\ \frac{\pi}{2\underline{\beta}} & \text{if } \alpha=\frac{c_0}{\underline{a}}, \end{cases} \tag{3.3}$$

$$\overline{\beta}=\sqrt{\overline{b}-\underline{a}^2}, \quad \underline{\beta}=\sqrt{\underline{b}-\underline{a}^2}. \tag{3.4}$$

Note that from the conditions on the parameters (2.1) it follows that all the square roots that appear in the expressions (3.1)-(3.4) are well defined.

A number $\delta_0>0$ satisfies the equation $\gamma(\delta)=\delta$ if and only if it satisfies the equation $\psi(\alpha)=\varphi^{-1}(\alpha)$. So if we define the function

$$f(\alpha)=-\psi(\alpha)+\varphi^{-1}(\alpha), \quad \alpha>0. \tag{3.5}$$

Then using (3.1) we have

$$f(\alpha)=\underline{b}e^{\underline{a}\tau(\alpha)}\sqrt{\overline{b}\alpha^2-2\underline{a}c_0\alpha+{c_0}^2}-\overline{b}e^{-\underline{a}s(\alpha)}\sqrt{\underline{b}\alpha^2-2\underline{a}c_0\alpha+{c_0}^2}-c_0(\underline{b}+\overline{b}) \tag{3.6}$$

and the closed trajectories of maximum deviation of the family $\mathcal{E}$ are determined by the positive roots α of the equation

$$f(\alpha)=0. \tag{3.7}$$

Now we put

$$\overline{M}(\alpha)=\overline{b}\alpha^2-2\underline{a}c_0\alpha+{c_0}^2,\ \underline{M}(\alpha)=\underline{b}\alpha^2-2\underline{a}c_0\alpha+{c_0}^2. \tag{3.8}$$

then the derivative of $f(\alpha)$ is

$$f'(\alpha)=\frac{\alpha\underline{b}\overline{b}}{\sqrt{\underline{M}(\alpha)\overline{M}(\alpha)}}\big[e^{\underline{a}\tau(\alpha)}\sqrt{\underline{M}(\alpha)}-e^{-\underline{a}s(\alpha)}\sqrt{\overline{M}(\alpha)}\big]. \tag{3.9}$$

Consider now the function $g:\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$g(\alpha)=\sqrt{\frac{\overline{M}(\alpha)}{\underline{M}(\alpha)}}e^{-\underline{a}(s(\alpha)+\tau(\alpha))}. \tag{3.10}$$

It is easy to see that

$$\begin{aligned} &f'(\alpha)>0 \text{ if and only if } g(\alpha)<1\\ &f'(\alpha)<0 \text{ if and only if } g(\alpha)>1\\ &f'(\alpha)=0 \text{ if and only if } g(\alpha)=1. \end{aligned} \tag{3.11}$$

Easy computations give

$$g'(\alpha)=\frac{(\overline{b}-\underline{b})c_0^2\alpha e^{-\underline{a}(s(\alpha)+\tau(\alpha))}}{\underline{M}(\alpha)\sqrt{\underline{M}(\alpha)\overline{M}(\alpha)}}\geq 0 \quad \text{for all } \alpha>0. \tag{3.12}$$

Note that $\overline{M(\alpha)}/\underline{M(\alpha)}$ is a rational function, both the numerator and denominator are polynomials in α of second degree. So, there exists the limit

$$\lim_{\alpha\to\infty}\frac{\overline{M}(\alpha)}{\underline{M}(\alpha)}=\frac{\overline{b}}{\underline{b}}. \tag{3.13}$$

From expressions (3.2) and (3.3) follows that there exist the limits:

$$\begin{aligned} \tau_\infty &:= \lim_{\alpha\to\infty}\tau(\alpha)=\frac{1}{\overline{\beta}}(\pi-\arctan\frac{\overline{\beta}}{\underline{a}})\\ s_\infty &:= \lim_{\alpha\to\infty}s(\alpha)=\frac{1}{\underline{\beta}}\arctan\frac{\underline{\beta}}{\underline{a}}. \end{aligned} \tag{3.14}$$

Now from (3.13), (3.14) and (3.10) we conclude that there exists the limit

$$G(\underline{a},\underline{b},\overline{b}):=\lim_{\alpha\to\infty}g(\alpha)=\sqrt{(\overline{b}/\underline{b})}\,e^{-\underline{a}(s_\infty+\tau_\infty)}. \tag{3.15}$$

We now consider two cases:

Case 1. $G(\underline{a},\underline{b},\overline{b})\leq 1$.

In this case the function $g(\alpha),\alpha>0$, according with (3.12) increases monotonically, so $g(\alpha)<1$ for all $\alpha>0$, from what follows according with (3.11) that the function $f(\alpha),\alpha\in(0,\infty)$ increases monotonically and it is easy to calculate that: $\lim_{\alpha\to\infty}f(\alpha)=\infty$, $\lim_{\alpha\to 0}f(\alpha)<0$ and so we conclude that in this case (3.7) has exactly one root $\alpha_1>0$.

Now Let $\delta_1=\varphi^{-1}(\alpha_1)$. then C_{δ_1} is in this case the unique cycle of maximal deviation of the family of equations $\mathcal{E}$. The function $\chi:\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$\chi(\delta)=\epsilon=(\psi\circ\varphi)^2(\delta) \tag{3.16}$$

is a Pointcare map for the cycle C_{δ_1} and simple calculations show that

$$\chi'(\delta_1)=g^2(\alpha_1)<1 \tag{3.17}$$

and so the cycle C_{δ_1} is a stable cycle. See figure 3.

Then from the invariance of the region bounded by C_{δ_1}, we have that all the solutions of $\mathcal{E}$ with initial condition in this closed bounded region are bounded. Now we will prove that in this case for all point x in the exterior of the cycle C_{δ_1} all the trajectories of $\mathcal{E}$ with initial condition in x are also bounded. In order to prove that we consider a maximum deviation trajectory with initial point $(A,0),A>0$

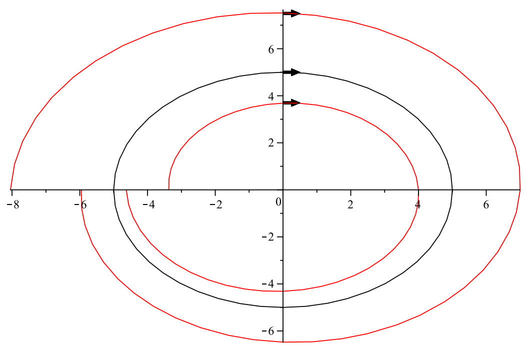

Figure 3: Case 1.

and end point $(B,0), B>0$ which takes a turn around the origin and we close this line with the vertical segment AB. Then the closed bounded region limited by this closed line is invariant for the family of equations $\mathcal{E}$, and for all point x in the exterior of C_{δ_1} choosing the number A sufficiently large we can assure that this point is in this bounded invariant region and then all trajectories of $\mathcal{E}$ with initial condition in x are bounded (See figure 4).

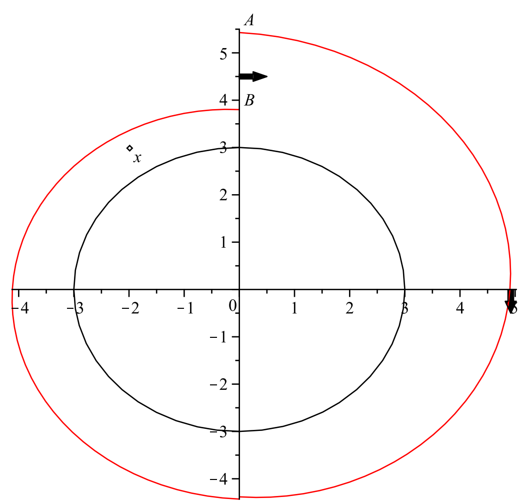

Figure 4. Boundedness of the trajectories in the exterior of the cycle.

We have proved that in this Case 1 all trajectories of $\mathcal{E}$ are bounded.

Case 2. $G(\underline{a},\underline{b},\overline{b})>1$. In this case we write the function $f(\alpha)$ in the form

$$f(\alpha)=\underline{b}e^{\underline{a}\tau(\alpha)}\sqrt{\overline{M}(\alpha)}\,h(\alpha)-c_0(\underline{b}+\overline{b}), \tag{3.18}$$

where

$$h(\alpha)=1-\frac{\overline{b}}{\underline{b}}e^{-\underline{a}(s(\alpha)+\tau(\alpha))}\sqrt{\frac{\underline{M}(\alpha)}{\overline{M}(\alpha)}}. \tag{3.19}$$

Now by differentiation we obtain

$$h'(\alpha)=(\overline{b}-\underline{b})c_0^2\alpha\ge 0 \tag{3.20}$$

for all $\alpha > 0$. Then

$$\max_{\alpha>0} h(\alpha) = \lim_{\alpha\to\infty} h(\alpha) = 1 - \lim_{\alpha\to\infty} g(\alpha) < 0, \tag{3.21}$$

and then from this and the expression (3.18) we conclude that $f(\alpha) < 0$ for all $\alpha > 0$, and so in this Case 2, the equation (3.7) does not have positive root and $\mathcal{E}$ doesn't have any cycle of maximum deviation. The negativity of the function $f(\alpha)$ means that for all positive δ we have $\epsilon(\delta) > \delta$ and so the trajectories of maximum deviation of the family $\mathcal{E}$ are all unbounded; see figure 5.

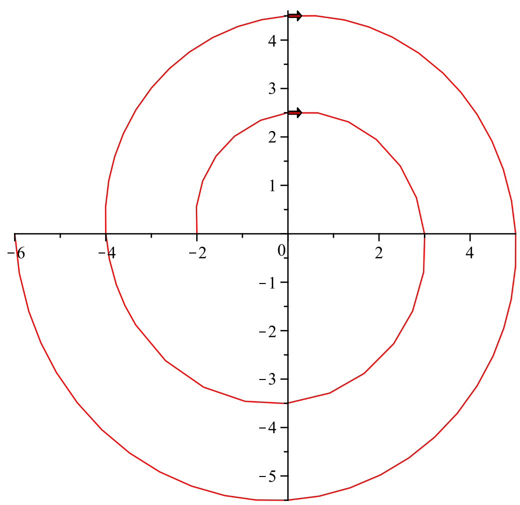

Figure 5: Case 2.

From the results for the two possible cases 1 and 2, we have the following result.

Theorem 3.1. *A necessary and sufficient condition for the boundedness of all the solutions of the family of equations $\mathcal{E}$ is the inequality*

$$G(\underline{a}, \underline{b}, \overline{b}) \le 1, \tag{3.22}$$

where the number $G(\underline{a}, \underline{b}, \overline{b})$, depending of the parameters $\underline{a}, \underline{b}, \overline{b}$ can be computed by the expressions (3.15)*) and* (3.14).

4. Examples

Example 4.1. Let $\underline{a} = 1$, $\underline{b} = 2$, $\overline{b} = 4$, and $\overline{a} \ge 1$, $c_0 > 0$ arbitrary real numbers. Then by direct computation using Maple we obtained

$$G(\underline{a}, \underline{b}, \overline{b}) = 0.192429 \le 1 \tag{4.1}$$

and so all the trajectories of the corresponding family of equations $\mathcal{E}$ are bounded.

Example 4.2. Let $\underline{a} = 1$, $\underline{b} = 2$, $\overline{b} = 40$, and $\overline{a} \ge 1$, $c_0 > 0$ arbitrary real numbers. Then by direct computation using Maple we obtained

$$G(\underline{a}, \underline{b}, \overline{b}) = 1.545757 > 1 \tag{4.2}$$

and so for all point x of the phase plane there is a non bounded trajectory of the corresponding family of equations $\mathcal{E}$ with initial point x, for example, the trajectory of maximum deviation with initial point x.

References

[1] B. V. Bulgakov; *About the accumulation of perturbations in oscillating linear systems with constant parameters*. Inform to the academic of Sciences URSS, 1946, T. 51, N^o 5.

[2] V. V. Alexandrov, V. N. Zhermolenko; *About the absolut stability of second order systems*. Viestnik of the Moscow University, N^o 5, 1972.

[3] V. N. Zhermolenko, H. Gonzalez M.; *Estabilidad absoluta de los sistemas lineales de segundo orden con perturbaciones paramétricas*. Revista Ciencias Matemáticas, Vol. IV, no. 2, 1983.

[4] M. Margaliot, D. Liberzon; *Lie-algebraic stability conditions for nonlinear switched systems and differential inclusions*. Systems And Control Letters, 55, 2006, pp. 8-16.

[5] V. N. Zhermolenko; *Limite cycles in the phase plane. In the Monograph: Bulgakov's Problem about maximum deviations and it's applications*. Moscow University Press, 1993.

[6] V. N. Zhermolenko, H. Gonzalez M.; *Sobre el ciclo límite en el problema de la desviación máxima de sistemas de segundo orden*. Investigación Operacional, N^o 2, 1979, pp. 56-69.

[7] L. S. Pontriaguin, V. G. Boltianski, P. V. Gamkrelidze, E. F. Mishenko; *Mathematical theory of optimal process*. M. Fizmatgiz, 1961.

[8] J. Macki, Strauss; *Introduction to Optimal Control Theory*. Springer-Verlag, New York, 1982.

Henry González

Faculty of Light Industry and Environmental Protection Engineering, Obuda University, 1034 Budapest, Bécsi út 96/B, Hungary

E-mail address: gonzalez.henry@rkk.uni-obuda.hu