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POSITIVE SOLUTIONS TO GENERALIZED SECOND-ORDER THREE-POINT INTEGRAL BOUNDARY-VALUE PROBLEMS

SAOWALUK CHASREECHAI, JESSADA TARIBOON

Abstract. In this article, by using Krasnoselskii's fixed point theorem, we obtain single and multiple positive solutions to the nonlinear second-order three-point integral boundary value problem

$$u''(t) + a(t)f(u(t)) = 0, \quad 0 < t < T,$$
$$u(0) = \beta \int_0^\eta u(s)ds, \quad \alpha \int_0^\eta u(s)ds = u(T),$$

where $0 < \eta < T$, $0 < \alpha < \frac{2T}{\eta^2}$, $0 < \beta < \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$ are given constants. As an application, we give some examples that illustrate our results.

1. Introduction

We are interested in obtaining positive solutions of the second-order three-point integral boundary-value problem (BVP)

$$u''(t) + a(t)f(u(t)) = 0, \quad t \in (0, T), \tag{1.1}$$
$$u(0) = \beta \int_0^\eta u(s)ds, \quad \alpha \int_0^\eta u(s)ds = u(T), \tag{1.2}$$

where $0 < \eta < T$, $0 < \alpha < \frac{2T}{\eta^2}$, $0 < \beta < \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$, $f \in C([0,\infty),[0,\infty))$, $a \in C([0,\infty),[0,\infty))$ and there exists a $t_0 \in (0,T)$, such that $a(t_0) > 0$. Set

$$f_0 = \lim_{u\to 0+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}.$$

The study of the existence of solutions of multi-point boundary-value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [5]. Then Gupta [3] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, the existence of positive solutions for nonlinear second order three-point boundary-value problems has been studied by many authors by using a nonlinear alternative of the Leray-Schauder approach, coincidence degree theory, the fixed point theorem for cones and so on. We refer the reader to [1, 2, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21] and the references therein. However, all of these papers are concerned with problems

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with three-point boundary conditions consisting of restrictions on the slope of the solutions and the solutions themselves, for example:

$$u(0)=0,\quad \alpha u(\eta)=u(1);$$
$$u(0)=\beta u(\eta),\quad \alpha u(\eta)=u(T);$$
$$u'(0)=0,\quad \alpha u(\eta)=u(1);$$
$$u(0)-\beta u'(0)=0,\quad \alpha u(\eta)=u(1);$$
$$\alpha u(0)-\beta u'(0)=0,\quad u'(\eta)+u'(1)=0;\ \text{etc.}$$

Recently, Tariboon [20] and the author proved the existence of positive solutions for the three-point boundary-value problem with integral condition

$$u''(t)+a(t)f(u(t))=0,\quad t\in(0,1), \tag{1.3}$$

$$u(0)=0,\quad \alpha\int_0^\eta u(s)ds=u(1), \tag{1.4}$$

where $0<\eta<1$ and $0<\alpha<2/\eta^2$. We note that the three-point integral boundary conditions (1.2) and (1.4) are related to the area under the curve of solutions $u(t)$ from $t=0$ to $t=\eta$.

The aim of this article is to establish some simple criteria for the existence of single positive solution for (1.1), (1.2) under $f_0=0$, $f_\infty=\infty$ (f is superlinear) or $f_0=\infty$, $f_\infty=0$ (f is sublinear). Moreover, we establish the existence conditions of two positive solutions for (1.1), (1.2) under $f_0=f_\infty=\infty$ or $f_0=f_\infty=0$. Finally, we give some examples to illustrate our results. The key tool in our approach is the Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1 ([6]). *Let E be a Banach space, and let $K\subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let*

$$A:K\cap(\overline{\Omega}_1\setminus\Omega_2)\to K$$

be a completely continuous operator such that

(i) $\|Au\|\leqslant\|u\|$, $u\in K\cap\partial\Omega_1$, *and* $\|Au\|\geqslant\|u\|$, $u\in K\cap\partial\Omega_2$; *or*
(ii) $\|Au\|\geqslant\|u\|$, $u\in K\cap\partial\Omega_1$, *and* $\|Au\|\geqslant\|u\|$, $u\in K\cap\partial\Omega_2$.

Then A has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

2. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Let $\beta\neq\frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$. Then for $y\in C[0,T]$, the problem*

$$u''+y(t)=0,\quad t\in(0,T), \tag{2.1}$$

$$u(0)=\beta\int_0^\eta u(s)ds,\quad \alpha\int_0^\eta u(s)ds=u(T), \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t)=&\frac{(\beta-\alpha)t-\beta T}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^\eta(\eta-s)^2y(s)ds\\ &+\frac{2(1-\beta\eta)t+\beta\eta^2}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T(T-s)y(s)ds-\int_0^t(t-s)y(s)ds. \end{aligned}$$

Proof. From (2.1), we have $u''(t) = -y(t)$. For $t \in [0, T)$, integrating from 0 to t, we obtain

$$u'(t) = u'(0) - \int_0^t y(s)ds.$$

For $t \in [0, T]$, integrating from 0 to t, we obtain

$$u(t) = u(0) + u'(0)t - \int_0^t \Big(\int_0^x y(s)ds \Big) dx;$$

i.e.,

$$u(t) = u(0) + u'(0)t - \int_0^t (t-s)y(s)ds := A + Bt - \int_0^t (t-s)y(s)ds. \tag{2.3}$$

Integrating (2.3) from 0 to η, where $\eta \in (0, T)$, we have

$$\begin{aligned}\int_0^\eta u(s)ds &= \eta A + \frac{\eta^2}{2}B - \int_0^\eta \Big(\int_0^x (x-s)y(s)ds \Big) dx \\ &= \eta A + \frac{\eta^2}{2}B - \frac{1}{2}\int_0^\eta (\eta - s)^2 y(s)ds.\end{aligned}$$

Since $u(0) = A$,

$$u(T) = A + BT - \int_0^T (T-s)y(s)ds.$$

By (2.2), from $u(0) = \beta \int_0^\eta u(s)ds$ we have

$$(1 - \beta\eta)A - \frac{\beta\eta^2}{2}B = -\frac{\beta}{2}\int_0^\eta (\eta - s)^2 y(s)ds,$$

and from $u(T) = \alpha \int_0^\eta u(s)ds$ we have

$$(1 - \alpha\eta)A + \big(T - \frac{\alpha\eta^2}{2}\big)B = \int_0^T (T-s)y(s)ds - \frac{\alpha}{2}\int_0^\eta (\eta - s)^2 y(s)ds.$$

Therefore,

$$\begin{aligned}A &= \frac{\beta\eta^2}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T (T-s)y(s)ds \\ &\quad - \frac{\beta T}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^\eta (\eta - s)^2 y(s)ds\end{aligned}$$

$$\begin{aligned}B &= \frac{2(1 - \beta\eta)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T (T-s)y(s)ds \\ &\quad + \frac{(\beta - \alpha)}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^\eta (\eta - s)^2 y(s)ds.\end{aligned}$$

Hence, (2.1)-(2.2) has a unique solution

$$\begin{aligned}u(t) &= \frac{(\beta - \alpha)t - \beta T}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^\eta (\eta - s)^2 y(s)ds \\ &\quad + \frac{2(1 - \beta\eta)t + \beta\eta^2}{(2T - \alpha\eta^2) - \beta\eta(2T - \eta)} \int_0^T (T-s)y(s)ds - \int_0^t (t-s)y(s)ds.\end{aligned}$$

□

Lemma 2.2. *Let $0 < \alpha < \frac{2T}{\eta^2}$, $0 < \beta < \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$. If $y \in C(0,T)$ and $y(t) \geqslant 0$ on $(0,T)$, then the unique solution u of (2.1)-(2.2) satisfies $u(t) \geqslant 0$ for $t \in [0,T]$.*

Proof. It is known that the graph of u is concave down on $[0,T]$ from $u''(t) = -y(t) \leqslant 0$, we obtain

$$\int_0^\eta u(s)ds \geqslant \frac{1}{2}\eta\big(u(0)+u(\eta)\big), \tag{2.4}$$

where $\frac{1}{2}\eta\big(u(0)+u(\eta)\big)$ is the area of the trapezoid under the curve $u(t)$ from $t=0$ to $t=\eta$ for $\eta \in (0,T)$. Combining (2.4) with (2.2), we can get

$$u(0) \geqslant \frac{\beta\eta}{2-\beta\eta}u(\eta), \tag{2.5}$$

$$u(T) \geqslant \frac{\alpha\eta}{2-\beta\eta}u(\eta), \tag{2.6}$$

such that

$$2-\beta\eta > 2 - \frac{2T-\alpha\eta^2}{2T-\eta} = \frac{2(T-\eta)+2\eta^2}{2T-\eta} > 0. \tag{2.7}$$

From the graph of u being concave down on $[0,T]$ again, we obtain

$$\frac{u(\eta)-u(0)}{\eta} \geqslant \frac{u(T)-u(0)}{T}. \tag{2.8}$$

Using (2.5), (2.6) and (2.8), we obtain

$$\frac{2-2\beta\eta}{\eta}u(\eta) \geqslant \frac{(\alpha-\beta)\eta}{T}u(\eta).$$

If $u(0) < 0$, then $u(\eta) < 0$. It implies $\frac{2T-\alpha\eta^2}{\eta(2T-\eta)} \leqslant \beta$, a contradiction to $\beta < \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$. If $u(T) < 0$, then $u(\eta) < 0$, and the same contradiction emerges. Thus, it is true that $u(0) \geqslant 0$, $u(T) \geqslant 0$, together with the concavity of u, we have $u(t) \geqslant 0$ for $t \in [0,T]$. This proof is complete. □

Lemma 2.3. *Let $\alpha\eta^2 \neq 2T$, $\beta > \max\big\{\frac{2T-\alpha\eta^2}{\eta(2T-\eta)}, 0\big\}$. If $y \in C(0,T)$ and $y(t) \geqslant 0$ for $t \in [0,T]$, then problem (2.1)-(2.2) has no positive solutions.*

Proof. Suppose that (2.1)-(2.2) has a positive solution u satisfying $u(t) \geqslant 0$, $t \in [0,T]$ and there is a $\tau_0 \in (0,T)$ such that $u(\tau_0) > 0$.

If $u(T) > 0$, then $\int_0^\eta u(s)ds > 0$. It implies

$$u(0) = \beta\int_0^\eta u(s)ds > \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}\int_0^\eta u(s)ds \geqslant \frac{\eta T(u(0)+u(\eta)) - \eta^2 u(T)}{\eta(2T-\eta)}; \tag{2.9}$$

that is

$$\frac{u(T)-u(0)}{T} > \frac{u(\eta)-u(0)}{\eta}, \tag{2.10}$$

which is a contradiction to the concavity of u.

If $u(T) = 0$, then $\int_0^\eta u(s)ds = 0$. When $\tau_0 \in (0,\eta)$, we obtain $u(\tau_0) > u(T) = 0 > u(\eta)$, which contradicts the concavity of u. When $\tau_0 \in (\eta,T)$, we obtain $u(\eta) \leqslant 0 = u(0) < u(\tau_0)$, which contradicts the concavity of u again. Therefore, no positive solutions exist. □

Let $E = C[0,T]$, then E is a Banach space with respect to the norm

$$\|u\| = \sup_{t\in[0,T]}|u(t)|.$$

Lemma 2.4. *Let $0 < \alpha < \frac{2T}{\eta^2}$, $0 < \beta < \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$. If $y \in C(0,T)$ and $y(t) \geqslant 0$ for $t \in [0,T]$, then the unique solution to problem (2.1)-(2.2) satisfies*

$$\min_{t\in[0,T]} u(t) \geqslant \gamma\|u\|, \tag{2.11}$$

where

$$\gamma := \min\Big\{\frac{\alpha\eta(T-\eta)}{T(2-\beta\eta)-\alpha\eta^2}, \frac{\alpha\eta^2}{(2-\beta\eta)T}, \frac{\beta\eta(T-\eta)}{(2-\beta\eta)T}, \frac{\beta\eta^2}{(2-\beta\eta)T}\Big\}. \tag{2.12}$$

Proof. From the fact that $u''(t) = -y(t) \leqslant 0$, we know that the graph of $u(t)$ is concave down on $[0,T]$. If $u(t)$ is maximum at $t = \tau_1$, then $\|u\| = u(\tau_1)$. We divide the proof into two cases.

Case (i) If $u(0) \geqslant u(T)$ and $\min_{t\in[0,T]} u(t) = u(T)$, then either $0 \leqslant \tau_1 \leqslant \eta < T$, or $0 < \eta < \tau_1 < T$. If $0 \leqslant \tau_1 \leqslant \eta < T$, then

$$\begin{aligned}
u(\tau_1) &\leqslant u(T) + \frac{u(T)-u(\eta)}{T-\eta}(\tau_1 - T)\\
&\leqslant u(T) + \frac{u(T)-u(\eta)}{T-\eta}(0 - T)\\
&= \frac{Tu(\eta)-\eta u(T)}{T-\eta}\\
&\leqslant \frac{T\Big[\frac{2-\beta\eta}{\alpha\eta}\Big]-\eta}{T-\eta}u(T) \quad \text{(by (2.6))}\\
&= \frac{T(2-\beta\eta)-\alpha\eta^2}{\alpha\eta(T-\eta)}u(T).
\end{aligned}$$

This implies

$$\min_{t\in[0,T]} u(t) \geqslant \frac{\alpha\eta(T-\eta)}{T(2-\beta\eta)-\alpha\eta^2}\|u\|.$$

If $0 < \eta < \tau_1 < T$, from

$$\frac{u(\eta)}{\eta} \geqslant \frac{u(\tau_1)}{\tau_1} \geqslant \frac{u(\tau_1)}{T},$$

together with (2.6), we have

$$u(T) \geqslant \frac{\alpha\eta^2}{(2-\beta\eta)T}\, u(\tau_1).$$

This implies

$$\min_{t\in[0,T]} u(t) \geqslant \frac{\alpha\eta^2}{(2-\beta\eta)T}\, \|u\|.$$

Case (ii) If $u(0) \leqslant u(T)$ and $\min_{t\in[0,T]} u(t) = u(0)$, then either $0 < \tau_1 < \eta < T$, or $0 < \eta \leqslant \tau_1 \leqslant T$. If $0 < \tau_1 < \eta < T$, from

$$\frac{u(\eta)}{T-\eta} \geqslant \frac{u(\tau_1)}{T-\tau_1} \geqslant \frac{u(\tau_1)}{T},$$

together with (2.5), we have

$$u(0) \geqslant \frac{\beta\eta(T-\eta)}{(2-\beta\eta)T}\, u(\tau_1).$$

Hence

$$\min_{t\in[0,T]} u(t) \geqslant \frac{\beta\eta(T-\eta)}{(2-\beta\eta)T}\,\|u\|.$$

If $0<\eta\leqslant\tau_1\leqslant T$, from

$$\frac{u(\tau_1)}{T}\leqslant\frac{u(\tau_1)}{\tau_1}\leqslant\frac{u(\eta)}{\eta},$$

together with (2.5), we have

$$u(0)\geqslant\frac{\beta\eta^2}{(2-\beta\eta)T}\,u(\tau_1).$$

This implies

$$\min_{t\in[0,T]} u(t) \geqslant \frac{\beta\eta^2}{(2-\beta\eta)T}\,\|u\|.$$

This completes the proof. □

In the rest of this article, we assume that $0<\alpha<2T/\eta^2$, $0<\beta<\frac{2T-\alpha\eta^2}{\eta(2T-\eta)}$. It is easy to see that (1.1)-(1.2) has a solution $u=u(t)$ if and only if u is a solution of the operator equation

$$\begin{aligned} u(t) &= \frac{(\beta-\alpha)t-\beta T}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^\eta(\eta-s)^2a(s)f(u(s))ds\\ &\quad+\frac{2(1-\beta\eta)t+\beta\eta^2}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T(T-s)a(s)f(u(s))ds\\ &\quad-\int_0^t(t-s)a(s)f(u(s))ds\triangleq Au(t). \end{aligned}$$

Denote

$$K=\big\{u\in E: u\geqslant 0,\ \min_{t\in[0,T]}u(t)\geqslant\gamma\|u\|\big\}, \tag{2.13}$$

where γ is defined in (2.12).

It is obvious that K is a cone in E. Moreover from Lemma 2.2 and Lemma 2.4, $A(K)\subset K$. It is also easy to check that $A:K\to K$ is completely continuous. In the following, for the sake of convenience, set

$$\begin{gathered} \Lambda_1=\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)ds,\\ \Lambda_2=\frac{\gamma(2-\beta\eta)(T-\eta)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T sa(s)ds. \end{gathered}$$

3. Main results

Now we are in the position to establish the main result.

Theorem 3.1. *Problem* (1.1)-(1.2) *has at least one positive solution under the assumptions:*

- (H1) $f_0=0$ *and* $f_\infty=\infty$ *(superlinear); or*
- (H2) $f_0=\infty$ *and* $f_\infty=0$ *(sublinear).*

Proof. At first, let (H1) hold. Since $f_0 = \lim_{u\to 0^+}(f(u)/u) = 0$ for any $\varepsilon \in (0, \Lambda_1^{-1}]$, there exists ρ_* such that

$$f(u) \leqslant \varepsilon u \quad \text{for } u \in [0, \rho_*]. \tag{3.1}$$

Let $\Omega_{\rho_*} = \{u \in E : \|u\| < \rho_*\}$ for any $u \in K \cap \partial\Omega_{\rho_*}$. From (3.1), we obtain

$$\begin{aligned}
Au(t) &\leqslant \frac{(\beta-\alpha)t - \beta T}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^\eta (\eta-s)^2 a(s) f(u(s))ds \\
&\quad + \frac{2(1-\beta\eta)t + \beta\eta^2}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\leqslant \frac{\beta t}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^\eta (\eta-s)^2 a(s) f(u(s))ds \\
&\quad + \frac{2t + \beta\eta^2}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\leqslant \frac{\beta T}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^\eta (\eta-s)^2 a(s) f(u(s))ds \\
&\quad + \frac{2T + \beta\eta^2}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\leqslant \frac{2T + \beta(T+\eta^2)}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T^2 - sT)a(s)f(u(s))ds \\
&\leqslant \varepsilon\rho_* \frac{2T + \beta(T+\eta^2)}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)ds \\
&= \varepsilon\Lambda_1\rho_* \leqslant \rho_* = \|u\|,
\end{aligned}$$

which yields

$$\|Au\| \leqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho_*}. \tag{3.2}$$

Further, since $f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty$, for any $M^* \in [\Lambda_2^{-1}, \infty)$, there exists $\rho^* > \rho_*$ such that

$$f(u) \geqslant M^* u \quad \text{for } u \geqslant \gamma\rho^*. \tag{3.3}$$

Set $\Omega_{\rho^*} = \{u \in E : \|u\| < \rho^*\}$ for $u \in K \cap \partial\Omega_{\rho^*}$. Since $u \in K$, $\min_{t\in[0,T]} u(t) \geqslant \gamma\|u\| = \gamma\rho^*$. Hence, for any $u \in K \cap \Omega_{\rho^*}$, from (3.3) and (2.7), we obtain

$$\begin{aligned}
Au(\eta) &= \frac{(\beta-\alpha)\eta - \beta T}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^\eta (\eta-s)^2 a(s) f(u(s))ds \\
&\quad + \frac{(2-\beta\eta)\eta}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\quad - \int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&= \frac{(2-\beta\eta)\eta}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\quad + \frac{1}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \\
&\quad \times \int_0^\eta (\eta-s)\Big[-(2-\beta\eta)T + \big(\beta(T-\eta) + \alpha\eta\big)s\Big] a(s)f(u(s))ds
\end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{(2-\beta\eta)\eta}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T (T-s)a(s)f(u(s))ds\\
&\quad + \frac{-T}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^\eta (\eta-s)(2-\beta\eta)a(s)f(u(s))ds\\
&\geqslant \frac{(2-\beta\eta)\eta}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T (T-s)a(s)f(u(s))ds\\
&\quad + \frac{-T}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T (\eta-s)(2-\beta\eta)a(s)f(u(s))ds\\
&= \frac{(2-\beta\eta)(T-\eta)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T sa(s)f(u(s))ds\\
&\geqslant \gamma\rho^* M^* \frac{(2-\beta\eta)(T-\eta)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T sa(s)ds\\
&= M^*\Lambda_2\rho^* \geqslant \rho^* = \|u\|,
\end{aligned}$$

which implies

$$\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho^*}. \tag{3.4}$$

Therefore, from (3.2), (3.4) and Theorem 1.1, it follows that A has a fixed point in $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$ such that $\rho_* \leqslant \|u\| \leqslant \rho^*$.

Next, let (H2) hold. In view of $f_0 = \lim_{u\to 0^+}(f(u)/u) = \infty$ for any $M_* \in [\Lambda_2^{-1}, \infty)$, there exists $r_* > 0$ such that

$$f(u) \geqslant M_* u \quad \text{for } 0 \leqslant u \leqslant r_*. \tag{3.5}$$

Set $\Omega_{r_*} = \{u \in E : \|u\| < r_*\}$ for $u \in K \cap \partial\Omega_{r_*}$. Since $u \in K$, it follows that $\min_{t\in[0,T]} u(t) \geqslant \gamma\|u\| = \gamma r_*$. Thus from (3.5) for any $u \in K \cap \partial\Omega_{r_*}$, we have

$$\begin{aligned}
Au(\eta) &= \frac{(\beta-\alpha)\eta-\beta T}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds\\
&\quad + \frac{(2-\beta\eta)\eta}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T (T-s)a(s)f(u(s))ds\\
&\quad - \int_0^\eta (\eta-s)a(s)f(u(s))ds\\
&\geqslant \gamma r_* M_* \frac{(2-\beta\eta)(T-\eta)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T sa(s)ds\\
&= M_*\Lambda_2 r_* \geqslant r_* = \|u\|,
\end{aligned}$$

which yields

$$\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r_*}. \tag{3.6}$$

Since $f_\infty = \lim_{u\to\infty}(f(u)/u) = 0$, for any $\varepsilon_1 \in (0, \Lambda_1^{-1}]$, there exists $r_0 > r_*$ such that

$$f(u) \leqslant \varepsilon_1 u \quad \text{for } u \in [r_0, \infty). \tag{3.7}$$

We have the next two cases:

Case (i): Suppose that $f(u)$ is unbounded, then from $f \in C([0,\infty),[0,\infty))$, we know that there is $r^* > r_0$ such that

$$f(u) \leqslant f(r^*) \quad \text{for } u \in [0, r^*]. \tag{3.8}$$

Since $r^* > r_0$, from (3.7) and (3.8), one has

$$f(u) \leqslant f(r^*) \leqslant \varepsilon_1 r^* \quad \text{for } u \in [0, r^*]. \tag{3.9}$$

For $u \in K$, $\|u\| = r^*$, from (3.9), we obtain

$$\begin{aligned} Au(t) &\leqslant \frac{2T + \beta(T+\eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)f(u(s))ds \\ &\leqslant \varepsilon_1 r^* \frac{2T + \beta(T+\eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)ds \\ &= \varepsilon_1 \Lambda_1 r^* \leqslant r^* = \|u\|. \end{aligned}$$

Case (ii) Suppose that $f(u)$ is bounded, say $f(u) \leqslant N$ for all $u \in [0, \infty)$. Taking $r^* \geqslant \max\{N/\varepsilon_1, r_*\}$, for $u \in K$, $\|u\| = r^*$, we have

$$\begin{aligned} Au(t) &\leqslant \frac{2T + \beta(T+\eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)f(u(s))ds \\ &\leqslant N \frac{2T + \beta(T+\eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)ds \\ &\leqslant \varepsilon_1 r^* \frac{2T + \beta(T+\eta^2)}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)ds \\ &= \varepsilon_1 \Lambda_1 r^* \leqslant r^* = \|u\|. \end{aligned}$$

Hence, in either case, we always may set $\Omega_{r^*} = \{u \in E : \|u\| < r^*\}$ such that

$$\|Au\| \leqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r^*}. \tag{3.10}$$

Hence, from (3.6), (3.10) and Theorem 1.1, it follows that A has a fixed point in $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$ such that $r_* \leqslant \|u\| \leqslant r^*$. The proof is complete. □

Theorem 3.2. *Suppose that the following assumptions are satisfied:*

(H3) $f_0 = f_\infty = \infty$,

(H4) *There exists a constant $\rho_1 > 0$, such that $f(u) \leqslant \Lambda_1^{-1}\rho_1$ for $u \in [0, \rho_1]$.*

Then (1.1), (1.2) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < \rho_1 < \|u_2\|.$$

Proof. At first, in view of $f_0 = \lim_{u\to 0^+}(f(u)/u) = \infty$, for any $M_* \in [\Lambda_2^{-1}, \infty)$, there exists $\rho_* \in (0, \rho_1)$ such that

$$f(u) \geqslant M_* u, \quad \text{for } 0 \leqslant u \leqslant \rho_*. \tag{3.11}$$

Set $\Omega_{\rho_*} = \{u \in E : \|u\| < \rho_*\}$ for $u \in K \cap \partial\Omega_{\rho_*}$. Since $u \in K$, then $\min_{t\in[0,T]} u(t) \geqslant \gamma\|u\| = \gamma\rho_*$. Thus from (3.11), for any $u \in K \cap \partial\Omega_{\rho_*}$, we obtain

$$\begin{aligned} Au(\eta) &= \frac{(\beta-\alpha)\eta - \beta T}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\ &\quad + \frac{(2-\beta\eta)\eta}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T-s)a(s)f(u(s))ds \\ &\quad - \int_0^\eta (\eta-s)a(s)f(u(s))ds \\ &\geqslant \gamma\rho_* M_* \frac{(2-\beta\eta)(T-\eta)}{(2T - \alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T sa(s)ds \end{aligned}$$

$$= M_*\Lambda_2\rho_* \geqslant \rho_* = \|u\|,$$

which implies

$$\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho_*}. \tag{3.12}$$

Next, since $f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty$, then for any $M^* \in [\Lambda_2^{-1}, \infty)$, there exists $\rho^* > \rho_1$ such that

$$f(u) \geqslant M^* u, \quad \text{for } u \geqslant \gamma\rho^*. \tag{3.13}$$

Set $\Omega_{\rho^*} = \{u \in E : \|u\| < \rho^*\}$ for $u \in K\cap\partial\Omega_{\rho^*}$. Since $u \in K$, then $\min_{t\in[0,T]} u(t) \geqslant \gamma\|u\| = \gamma\rho^*$. Thus from (3.13) for any $u \in K \cap \partial\Omega_{\rho^*}$, we have

$$\begin{aligned}
Au(\eta) &= \frac{(\beta-\alpha)\eta - \beta T}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^\eta (\eta-s)^2 a(s) f(u(s))ds \\
&\quad + \frac{(2-\beta\eta)\eta}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T (T-s)a(s)f(u(s))ds \\
&\quad - \int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&\geqslant \gamma\rho^* M^* \frac{(2-\beta\eta)(T-\eta)}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T sa(s)ds \\
&= M^*\Lambda_2\rho^* \geqslant \rho^* = \|u\|,
\end{aligned}$$

which implies

$$\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho^*}. \tag{3.14}$$

Finally, let $\Omega_{\rho_1} = \{u \in E : \|u\| < \rho_1\}$ for any $u \in K \cap \partial\Omega_{\rho_1}$. Then from (H4) we obtain

$$\begin{aligned}
Au(t) &\leqslant \frac{2T + \beta(T+\eta^2)}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)f(u(s))ds \\
&\leqslant \Lambda_1^{-1}\rho_1 \frac{2T + \beta(T+\eta^2)}{(2T-\alpha\eta^2) - \beta\eta(2T-\eta)} \int_0^T T(T-s)a(s)ds \\
&\leqslant \rho_1 = \|u\|,
\end{aligned}$$

which yields

$$\|Au\| \leqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho_*}. \tag{3.15}$$

Thus, from (3.12), (3.14) and (3.15), it follows from Theorem 1.1 that A has a fixed point u_1 in $K\cap(\overline{\Omega}_{\rho_1} \setminus \Omega_{\rho_*})$, and a fixed point u_2 in $K\cap(\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_1})$. Both are positive solutions of (1.1), (1.2) and $0 < \|u_1\| < \rho_1 < \|u_2\|$. The proof is complete. □

Theorem 3.3. *Suppose that the following assumptions are satisfied:*

- (H5) $f_0 = f_\infty = 0$,
- (H6) *There exists a constant* $\rho_2 > 0$*, such that*

$$f(u) \geqslant \Lambda_2^{-1}\rho_2 \quad \textit{for } u \in [\gamma\rho_2, \rho_2].$$

Then (1.1)*,* (1.2) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho_2 < \|u_2\|.$$

Proof. Firstly, since $f_0 = \lim_{u\to 0^+}(f(u)/u) = 0$, for any $\varepsilon \in (0, \Lambda_1^{-1}]$, there exists $\rho_* \in (0, \rho_2)$ such that

$$f(u) \leqslant \varepsilon u, \quad \text{for } u \in [0, \rho_*]. \tag{3.16}$$

Let $\Omega_{\rho_*}=\{u\in E:\|u\|<\rho_*\}$ for any $u\in K\cap\partial\Omega_{\rho_*}$. Then from (3.16), we obtain

$$\begin{aligned}Au(t)&\leqslant\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)f(u(s))ds\\&\leqslant\varepsilon\rho_*\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)ds\\&\leqslant\varepsilon\Lambda_1\rho_*\leqslant\rho_*=\|u\|,\end{aligned}$$

which implies

$$\|Au\|\leqslant\|u\|\quad\text{for }u\in K\cap\partial\Omega_{\rho_*}.\tag{3.17}$$

Secondly, in view of $f_\infty=\lim_{u\to\infty}(f(u)/u)=0$, for any $\varepsilon_1\in(0,\Lambda_1^{-1}]$ there exists $\rho_0>\rho_2$, such that

$$f(u)\leqslant\varepsilon_1u,\quad\text{for }u\in[\rho_0,\infty).\tag{3.18}$$

We consider the next two cases.

Case (i): Suppose that $f(u)$ is unbounded. Then from $f\in C([0,\infty),[0,\infty))$, there exists $\rho^*>\rho_0$ such that

$$f(u)\leqslant f(\rho^*),\quad\text{for }u\in[0,\rho^*].\tag{3.19}$$

Since $\rho^*>\rho_0$, from (3.18) and (3.18) one has

$$f(u)\leqslant f(\rho^*)\leqslant\varepsilon_1\rho^*,\quad\text{for }u\in[0,\rho^*].\tag{3.20}$$

For $u\in K$, and $\|u\|=\rho^*$, from (3.20), we obtain

$$\begin{aligned}Au(t)&\leqslant\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)f(u(s))ds\\&\leqslant\varepsilon_1\rho^*\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)ds\\&\leqslant\varepsilon_1\Lambda_1\rho^*\leqslant\rho^*=\|u\|.\end{aligned}$$

Case (ii): Suppose that $f(u)$ is bounded, say $f(u)\leqslant L$ for all $u\in[0,\infty)$. Taking $\rho^*\geqslant\max\{L/\varepsilon_1,\rho_0\}$, for $u\in K$ with $\|u\|=\rho^*$, we have

$$\begin{aligned}Au(t)&\leqslant\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)f(u(s))ds\\&\leqslant L\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)ds\\&\leqslant\varepsilon_1\rho^*\frac{2T+\beta(T+\eta^2)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T T(T-s)a(s)ds\\&\leqslant\varepsilon_1\Lambda_1\rho^*\leqslant\rho^*=\|u\|.\end{aligned}$$

Hence, in either case, we always may set $\Omega_{\rho^*}=\{u\in E:\|u\|<\rho^*\}$ such that

$$\|Au\|\leqslant\|u\|\quad\text{for }u\in K\cap\partial\Omega_{\rho^*}.\tag{3.21}$$

Finally, set $\Omega_{\rho_2}=\{u\in E:\|u\|<\rho_2\}$ for $u\in K\cap\partial\Omega_{\rho_2}$. Since $u\in K$, $\min_{t\in[0,T]}u(t)\geqslant\gamma\|u\|=\gamma\rho_2$. Hence, for any $u\in K\cap\partial\Omega_{\rho_2}$, and (H6), we have

$$\begin{aligned}Au(\eta)=&\frac{(\beta-\alpha)\eta-\beta T}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^\eta(\eta-s)^2a(s)f(u(s))ds\\&+\frac{(2-\beta\eta)\eta}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T(T-s)a(s)f(u(s))ds\end{aligned}$$

$$
\begin{aligned}
&- \int_0^\eta (\eta - s)a(s)f(u(s))ds \\
&\geqslant \gamma\rho_2\Lambda_2^{-1}\frac{(2-\beta\eta)(T-\eta)}{(2T-\alpha\eta^2)-\beta\eta(2T-\eta)}\int_0^T sa(s)ds \geqslant \rho_2 = \|u\|,
\end{aligned}
$$

which yields

$$
\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho_2}. \tag{3.22}
$$

Thus, since $\rho_* < \rho < \rho^*$ and from (3.17), (3.21) and (3.22), it follows from Theorem 1.1 that A has a fixed point u_1 in $K \cap (\overline{\Omega}_{\rho_2} \setminus \Omega_{\rho_*})$, and a fixed point u_2 in $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_2})$. Both are positive solutions of (1.1), (1.2) and $0 < \|u_1\| < \rho_2 < \|u_2\|$. The proof is complete. □

4. Some examples

In this section, to illustrate our results, we consider some examples.

Example 4.1. Consider the boundary-value problem

$$
u''(t) + t^2u^p = 0, \quad 0 < t < e^2, \tag{4.1}
$$

$$
u(0) = \frac{2}{9}\int_0^e u(s)ds, \quad u(e^2) = \frac{2}{3}\int_0^e u(s)ds. \tag{4.2}
$$

Set $\alpha = 2/3$, $\beta = 2/9$, $\eta = e$, $T = e^2$, $a(t) = t^2$, $f(u) = u^p$. We can show that

$$
0 < \alpha = \frac{2}{3} < 2 = \frac{2T}{\eta^2}, \quad 0 < \beta = \frac{2}{9} < \frac{4}{3(2e-1)} = \frac{2T-\alpha\eta^2}{\eta(2T-\eta)}.
$$

Case I: $p \in (1,\infty)$. In this case, $f_0 = 0$, $f_\infty = \infty$ and (H1) holds. Then (4.1), (4.2) has at least one positive solution.

Case II: $p \in (0,1)$ In this case, $f_0 = \infty$, $f_\infty = 0$ and (H2) holds. Then (4.1), (4.2) has at least one positive solution.

Example 4.2. Consider the boundary-value problem

$$
u''(t) + \frac{1}{8^3}(4-t)^{1/2}(u^{1/2}+u^2) = 0, \quad 0 < t < 4, \tag{4.3}
$$

$$
u(0) = \frac{1}{10}\int_0^1 u(s)ds, \quad u(4) = 2\int_0^1 u(s)ds. \tag{4.4}
$$

Set $\alpha = 2$, $\beta = 1/10$, $\eta = 1$, $T = 4$, $a(t) = \frac{1}{8^3}(4-t)^{1/2}$, $f(u) = u^{1/2} + u^2$. We can show that $0 < \alpha = 2 < 8 = 2T/\eta^2$, $0 < \beta = 1/10 < 6/7 = (2T - \alpha\eta^2)/(\eta(2T-\eta))$. Since $f_0 = f_\infty = \infty$, then (H_3) holds. Again $\Lambda_1^{-1} = ((2T-\alpha\eta^2)-\beta\eta(2T-\eta))/(2T+\beta(T+\eta^2))(\int_0^T T(T-s)a(s)ds)^{-1} = 530/85$, because $f(u)$ is monotone increasing function for $u \geqslant 0$, taking $\rho_1 = 4$, then when $u \in [0,\rho_1]$, we obtain

$$
f(u) \leqslant f(4) = 18 < \frac{530}{85}\rho_1 = \Lambda_1^{-1}\rho_1,
$$

which implies (H4) holds. Hence, by Theorem 3.2, BVP (4.3), (4.4) has at least two positive solutions u_1 and u_2 such that $0 < \|u_1\| < 4 < \|u_2\|$.

Example 4.3. Consider the boundary-value problem

$$
u''(t) + e^{32}u^2e^{-u} = 0, \quad 0 < t < \frac{4}{5}, \tag{4.5}
$$

$$u(0)=2\int_0^{1/5}u(s)ds,\quad u\Big(\frac{4}{5}\Big)=20\int_0^{1/5}u(s)ds. \tag{4.6}$$

Set $\alpha=20$, $\beta=2$, $\eta=1/5$, $T=4/5$, $a(t)\equiv e^{32}$, $f(u)=u^2e^{-u}$. We can show that $0<\alpha=20<40=2T/\eta^2$, $0<\beta=2<20/7=(2T-\alpha\eta^2)/(\eta(2T-\eta))$, $\gamma=\min\{\alpha\eta(T-\eta)/(T(2-\beta\eta)-\alpha\eta^2),\alpha\eta^2/((2-\beta\eta)T),\beta\eta(T-\eta)/((2-\beta\eta)T),\beta\eta^2/((2-\beta\eta)T)\}=\min\{5,5/8,3/16,1/16\}=1/16$. Since $f_0=f_\infty=0$, then (H5) holds. Again $\Lambda_2^{-1}=((2T-\alpha\eta^2)-\beta\eta(2T-\eta))/(\gamma(2-\beta\eta)(T-\eta))(\int_0^T sa(s)ds)^{-1}=\frac{25}{2}e^{-32}$, since $f(u)$ is monotone decreasing function for $u\geqslant 2$, taking $\rho_2=32$,when $u\in[\gamma\rho_2,\rho_2]$, we obtain

$$f(u)\geqslant f(32)=1024e^{-32}>400e^{-32}=\Lambda_2^{-1}\rho_2,$$

which implies (H6) holds. Hence, by Theorem 3.3, BVP (4.5), (4.6) has at least two positive solutions u_1 and u_2 such that $0<\|u_1\|<32<\|u_2\|$.

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Saowaluk Chasreechai
Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
E-mail address: slc@kmutnb.ac.th

Jessada Tariboon
Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.
Centre of Excellence in Mathematics, CHE, Sri Ayutthaya Road, Bangkok 10400, Thailand
E-mail address: jessadat@kmutnb.ac.th