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EXISTENCE OF THREE POSITIVE SOLUTIONS FOR A SYSTEM OF NONLINEAR THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this work, we use the Leggett-Williams fixed point theorem, we prove the existence of at least three positive solutions of a boundary-value problem for system of third-order ordinary differential equations.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics. In recent years, boundary-value problems (BVPs for short) have included, as special cases, multi-point BVPs considered by many authors (see [1, 2, 4, 9, 11, 12] and references therein). Naturally, further study in this specific field is on BVPs for systems of ordinary differential equations. However, to our knowledge, various results for systems of second and third order differential equations have been established (see [5, 8, 10] and references therein). Guo et al. [4] obtained some existence results for positive solutions for the BVP

$$
\begin{gathered}
u'''(t) + a(t)f(u(t)) = 0 \quad 0 < t < 1 \\
u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta),
\end{gathered}
$$

by using the well-know Guo-Krasnoselshkii and Leggett-Williiams fixed point theorems [3, 6, 7] when f is superlinear or sublinear. Hu et al. [5] established some results on the existence and multiplicity of positive solution for the BVP

$$
\begin{gathered}
-u''(t) = f(x, v) \\
-v''(t) = g(x, u), \\
\alpha u(0) - \beta u'(0) = 0, \gamma u(1) + \sigma u'(1) = 0, \\
\alpha v(0) - \beta v'(0) = 0, \quad \gamma v(1) + \sigma v'(1) = 0.
\end{gathered}
$$

Li et al. [8], considered the existence of positive solutions for the boundary-value problem

$$
\begin{gathered}
-u'''(t) = a(t)f(t, v(t)) \\
-v'''(t) = b(t)h(t, u(t)),
\end{gathered}
$$

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$$
\begin{gathered}
u(0)=u'(0)=0, \quad u'(1)=\alpha u'(\eta), \\
v(0)=v'(0)=0, \quad v'(1)=\alpha v'(\eta)=0.
\end{gathered}
$$

Motivated by the above-mentioned works, in this article, we prove the existence of at least three positive solutions for the boundary-value problem

$$
\begin{gathered}
u'''(t)+a(t)f(t,u(t),v(t))=0, \quad 0<t<1, \\
v'''(t)+b(t)h(t,u(t),v(t)), \quad 0<t<1, \\
u(0)=u'(0)=0, \quad u'(1)=\beta u'(\eta), \\
v(0)=v'(0)=0, \quad v'(1)=\beta v'(\eta),
\end{gathered} \tag{1.1}
$$

where $f,h:[0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty)$ are continuous and $0<\eta<1$, $1<\beta<1/\eta$, $a(t),b(t)\in C([0,1],[0,\infty))$ and are not identically zero on $[\eta/\beta,\eta]$.

A pair of functions $(u,v)\in C^3((0,1),\mathbb{R}^+)\times C^3((0,1),\mathbb{R}^+)$ is said to be a positive solution of (1.1) if (u,v) satisfies (1.1) and $u(t)\geq 0$, $v(t)>0$, or $u(t)>0$, $v(t)\geq 0$, for all $t\in(0,1)$.

For the convenience of the reader, we present here the Leggett-Williams fixed point theorem.

Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp. convex) functional on K provided that $\alpha:K\to[0,+\infty)$ is continuous and

$$
\begin{gathered}
\alpha(tx+(1-t)y)\geq t\alpha(x)+(1-t)\alpha(y), \\
(\text{resp. } \alpha(tx+(1-t)y)\leq t\alpha(x)+(1-t)\alpha(y)),
\end{gathered}
$$

for all $x,y\in K$ and $t\in[0,1]$. Let $0<a<b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha,a,b)$ by

$$
P_r=\{x\in K|\|x\|<r\}, \quad P(\alpha,a,b)=\{x\in K|a\leq\alpha(x),\|x\|\leq b\}.
$$

Theorem 1.1 (Leggett-Williams fixed point theorem)**.** *Let $A:\overline{P_c}\to\overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x)\leq\|x\|$ for all $x\in\overline{P_c}$. Suppose there exist $0<a<b<d\leq c$ such that*

- (A1) *$\{x\in P(\alpha,b,d):\alpha(x)>b\}\neq\emptyset$, and $\alpha(Ax)>b$ for $x\in P(\alpha,b,d)$;*
- (A2) *$\|Ax\|<a$ for $\|x\|\leq a$; and*
- (A3) *$\alpha(Ax)>b$ for $x\in P(\alpha,b,c)$ with $\|Ax\|>d$.*

Then A has at least three fixed points x_1, x_2 and x_3 and such that $\|x_1\|<a$, $b<\alpha(x_2)$ and $\|x_3\|>a$, with $\alpha(x_3)<b$.

Inspired and motivated by the works mentioned above, in this work we consider the existence of positive solutions to (1.1). We shall first give a new form of the solution, and then determine the properties of the Green's function for associated linear boundary-value problems; finally, by employing the Leggett-Williams fixed point theorem, some sufficient conditions guaranteeing the existence of a positive solution. The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Section 3. The main results and proofs will be given in Section 3. Finally, in Section 4, we shall give an example to illustrate our main result.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proof of the main result. Obviously, $(u(t), v(t)) \in C^3([0,1],(0,+\infty)) \times C^3([0,1],(0,+\infty))$ is a solution of (1.1) if and only if $(u(t), v(t))$ is a solution of the system of integral equations

$$u(t) = \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds, \tag{2.1}$$

$$v(t) = \int_0^1 G(t,s)b(s)h(s,u(s),v(s))ds, \tag{2.2}$$

where $G(t,s)$ is the Green's function [3] defined as follows:

$$G(t,s) = \frac{1}{2(1-\beta\eta)} \begin{cases} (2ts - s^2)(1-\beta\eta) + t^2 s(\beta - 1), & s \le \min\{\eta, t\} \\ t^2(1-\beta\eta) + t^2 s(\beta-1), & t \le s \le \eta \\ (2ts - s^2)(1-\beta\eta) + t^2(\beta\eta - s), & \eta \le s \le t, \\ t^2(1-s), & \max\{\eta, t\} \le s\,. \end{cases}$$

We need some properties of function $G(t,s)$ in order to discuss the existence of positive solutions. For convenience, we define

$$g(s) = \frac{1+\alpha}{1-\beta\eta} s(1-s) \quad s \in [0,1]. \tag{2.3}$$

For the Green's function $G(t,s)$, we have the following two lemmas [3].

Lemma 2.1. *Let $0 < \eta < 1$ and $1 < \beta < 1/\eta$. Then for any $(t,s) \in [0,1] \times [0,1]$, we have $0 \le G(t,s) \le g(s)$.*

Lemma 2.2. *Let $0 < \eta < 1$ and $1 < \beta < 1/\eta$. Then for any $(t,s) \in [\eta/\beta, \eta] \times [0,1]$, we have*

$$\lambda g(s) \le G(t,s),$$

where $0 < \lambda = \eta^2 \min\{\beta - 1, 1\}/(2\beta^2(1+\beta)) < 1$.

In this article, we assume that the following conditions are satisfied

$$0 < \int_0^1 g(s)a(s)ds < +\infty, \quad 0 < \int_0^1 g(s)b(s)ds < +\infty. \tag{2.4}$$

Also, we use the following notation

$$M_1 = \max_{0 \le t \le 1} \int_0^1 G(t,s)a(s)ds, \quad M_2 = \max_{0 \le t \le 1} \int_0^1 G(t,s)b(s)ds,$$

$$m_1 = \min_{\eta/\beta \le t \le \eta} \int_{\eta/\beta}^{\eta} G(t,s)a(s)ds, \quad m_2 = \min_{\eta/\beta \le t \le \eta} \int_{\eta/\beta}^{\eta} G(t,s)b(s)ds.$$

Clearly, we see that $0 < m_i < M_i$; for $i = 1,2$.

Let $E = C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ endowed with the norm $\|(u,v)\| := \|u\| + \|v\|$, where $\|u\| = \max_{0 \le t \le 1} |u(t)|$, $\|v\| = \max_{0 \le t \le 1} |v(t)|$, and define

$$K = \{(u,v) \in E; u(t) \ge 0, v(t) \ge 0, t \in [0,1], \min_{\eta/\beta \le t \le \eta} (u(t) + v(t)) \ge \gamma \|(u,v)\|\}.$$

It is obvious that E is a Banach space and K is a cone in E. Define operator $T : E \to E$ as

$$T(u,v)(t) = (A(u,v)(t), B(u,v)(t)), \quad \forall t \in (0,1), \tag{2.5}$$

where

$$A(u,v)(t) = \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds, \tag{2.6}$$

$$B(u,v)(t) = \int_0^1 G(t,s)b(s)f(s,u(s),v(s))ds. \tag{2.7}$$

Lemma 2.3. *For T and K as above, $T(K) \subset K$.*

Proof. For any $(u,v) \in K$, from properties of $G(t,s)$, $A(u,v)(t) \geq 0, B(u,v)(t) \geq 0, t \in [0,1]$, and it follows from (2.6), (2.7) and Lemma 2.1 that

$$\begin{aligned} \|A(u,v)\| &\leq \int_0^1 g(s)a(s)f(s,u(s),v(s))ds, \\ \|B(u,v)\| &\leq \int_0^1 g(s)b(s)f(s,u(s),v(s))ds, \end{aligned} \tag{2.8}$$

Thus, for any $(u,v) \in K$, by Lemma 2.2 and the above inequality,

$$\begin{aligned} \min_{\eta/\beta \leq t \leq \eta} A(u,v)(t) &= \min_{\eta/\beta \leq t \leq \eta} \int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds, \\ &\geq \lambda \int_0^1 g(s)a(s)f(s,u(s),v(s))ds \\ &\geq \lambda \|A(u,v)\|. \end{aligned}$$

In the same way, for any $(u,v) \in K$, we have

$$\min_{\eta/\beta \leq t \leq \eta} B(u,v)(t) \geq \lambda \|B(u,v)\|.$$

Therefore,

$$\begin{aligned} &\min_{\eta/\beta \leq t \leq \eta} (A(u,v)(t) + B(u,v)(t) \geq \lambda \|A(u,v)\| + \lambda \|B(u,v)\|) \\ &= \lambda \|(A(u,v), B(u,v))\|. \end{aligned}$$

From the above, we conclude that $T(u,v) = (A(u,v), B(u,v)) \in K$, that is, $T(K) \subset K$. The proof is complete. $\square$

It is clear that the existence of a positive solution for (1.1) is equivalent to the existence of a nontrivial fixed point of T in K.

3. Main results

In this section, we discuss the existence of a positive solution (1.1). We define the nonnegative continuous concave functional on K by

$$\alpha(u,v) = \min_{\eta/\beta \leq t \leq \eta} (u(t), v(t)).$$

It is obvious that, for each $(u,v) \in K, \alpha(u,v) \leq \|(u,v)\|$. In this section, we assume that $p_i, i = 1,2$, are two positive numbers satisfying $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$. Also, we use the following assumptions: There exist nonnegative numbers a, b, c such that $0 < a < b \leq \min\{\lambda, \frac{m_1}{p_1 M_1}, \frac{m_2}{p_2 M_2}\}c$, and $f(t,u,v)$, $h(t,u,v)$ satisfy the following conditions:

- (H1) $f(t,u,v) < \frac{1}{p_1} \cdot \frac{c}{M_1}$, $h(t,u,v) < \frac{1}{p_2} \cdot \frac{c}{M_2}$, for all $t \in [0,1]$, $u + v \in [0,c]$;
- (H2) $f(t,u,v) < \frac{1}{p_1} \cdot \frac{a}{M_1}$, $h(t,u,v) < \frac{1}{p_2} \cdot \frac{a}{M_2}$, for all $t \in [0,1]$, $u + v \in [0,a]$;

(H3) $f(t,u,v) > b/m_1$ or $h(t,u,v) > b/m_2$, for all $t \in [0,1]$, $u+v \in [b, b/\lambda]$.

Theorem 3.1. *Assume* (2.4) *and* (H1)–(H3)*. Then* (1.1) *has at least three positive solutions* $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ *such that* $\|(u_1,v_1)\| < a$, $b < \min_{[\eta/\beta,\eta]}(u_2(t)+v_2(t))$, *and* $\|(u_3,v_3)\| > a$, *with* $\min_{\eta/\beta\le t\le\eta}(u_3(t)+v_3(t)) < b$.

Proof. First, we show that $T : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator. If $(u,v) \in \overline{P_c}$, by condition (H1), we have

$$
\begin{aligned}
\|T(u,y)\| &= \max_{0\le t\le 1}|A(u,v)(t)| + \max_{0\le t\le 1}|B(u,v)(t)| \\
&= \max_{0\le t\le 1}\int_0^1 G(t,s)a(s)f(s,u(s),v(s))ds \\
&\quad + \max_{0\le t\le 1}\int_0^1 G(t,s)b(s)h(s,u(s),v(s))ds \\
&\le \frac{1}{p_1}\cdot\frac{c}{M_1}\max_{0\le t\le 1}\int_0^1 G(t,s)a(s)ds + \frac{1}{p_2}\cdot\frac{c}{M_2}\max_{0\le t\le 1}\int_0^1 G(t,s)b(s)ds \\
&\le \frac{1}{p_1}\cdot\frac{c}{M_1}\cdot M_1 + \frac{1}{p_2}\cdot\frac{c}{M_2}\cdot M_2 \le c.
\end{aligned}
$$

Therefore, $\|T(u,y)\| \le c$, that is, $T : \overline{P_c} \to \overline{P_c}$. The operator T is completely continuous by an application of the Ascoli-Arzela theorem.

In the same way, the condition (H2) implies that the condition (A2) of Theorem 1.1 is satisfied. We now show that condition (A1) of Theorem 1.1 is satisfied. Clearly, $\{(u,v) \in P(\alpha,b,b/\lambda)|\alpha(u,v) > b\} \ne \emptyset$. If $(u,v) \in P(\alpha,b,b/\lambda)$, then $b \le u(s)+v(s) \le \frac{b}{\lambda}, s \in [\eta/\beta,\eta]$.

By condition (H3), we obtain

$$
\begin{aligned}
\alpha(T(u,v)(t)) &= \min_{\eta/\beta\le t\le\eta}(A(u,v)(t)+B(u,v)(t)) \\
&\ge \min_{\eta/\beta\le t\le\eta}\int_{\eta/\beta}^{\eta} G(t,s)a(s)f(s,u(s),v(s))ds \\
&\quad + \min_{\eta/\beta\le t\le\eta}\int_{\eta/\beta}^{\eta} G(t,s)b(s)h(s,u(s),v(s))ds \\
&\ge \frac{b}{m_1}\min_{\eta/\beta\le t\le\eta}\int_{\eta/\beta}^{\eta} G(t,s)a(s)ds = \frac{b}{m_1}\cdot m_1 = b.
\end{aligned}
$$

Therefore, condition (A3) of Theorem 1.1 is satisfied.

Finally, we show that the condition (A3) of Theorem 1.1 is satisfied. If $(u,v) \in P(\alpha,b,c)$, and $\|T(u,v)\| > \frac{b}{\lambda}$, then

$$
\alpha(T(u,v)(t)) = \min_{\eta/\beta\le t\le\eta}(A(u,v)(t)+B(u,v)(t)) \ge \lambda\|T(u,v)\| > b.
$$

Therefore, the condition (A3) of Theorem 1.1 is also satisfied. By Theorem 1.1, there exist three positive solutions $(u_1,v_1),(u_2,v_2),(u_3,v_3)$ such that $\|(u_1,v_1)\| < a, b < \min_{\eta/\beta\le t\le\eta}(u_2(t)+v_2(t))$, and $\|(u_3,v_3)\| > a$, with $\min_{\eta/\beta\le t\le\eta}(u_3(t)+v_3(t)) < b$. $\square$

4. Application

Consider the system of nonlinear third-order ordinary differential equations

$$
\begin{gathered}
u'''(t) + f(t, u(t), v(t)) = 00 < t < 1, \\
v'''(t) + h(t, u(t), v(t)), 0 < t < 1, \\
u(0) = u'(0) = 0, \quad u'(1) = \frac{3}{2}u'(\frac{1}{2}), \\
v(0) = v'(0) = 0, \quad v'(1) = \frac{3}{2}v'(\frac{1}{2}),
\end{gathered} \tag{4.1}
$$

where

$$
\begin{aligned}
& f(t,u,v) = h(t,u,v) \\
& = \begin{cases}
\frac{t}{100} + \frac{1}{200}(u+v)^2, & t \in [0,1],\ 0 \le u+v \le 1, \\
\frac{t}{100} + 240[(u+v)^2 - (u+v)] + \frac{1}{200}, & t \in [0,1],\ 1 < u+v < 2, \\
\frac{t}{100} + 30[10\log_2(u+v) + 3(u+v)] + \frac{1}{200}, & t \in [0,1],\ 2 \le u+v \le 4 \\
\frac{t}{100} + \frac{\sqrt{u+v}}{2} + \frac{191801}{200}, & t \in [0,1],\ 4 < u+v < +\infty.
\end{cases}
\end{aligned}
$$

It is easy to check that $g(s) = 10s(1-s)$, for all $s \in [0,1]$, $0 < \int_0^1 g(s)a(s)ds < +\infty, 0 < \int_0^1 g(s)b(s)ds < +\infty$ hold. Choose $p_1 = p_2 = 2$. Then by direct calculations, we can obtain that $M_1 = M_2 = 5/3$, $m_1 = m_2 = 13/2916$. So we choose $a = 1$, $b = 2$, $c = 3500$. It is easy to check that f, h satisfy the conditions (H1)–(H3). So system (4.1) has at least three positive solutions $(u_1, v_1), (u_2, v_2), (u_3, v_3)$ such that $\|(u_1, v_1)\| < 1, 2 < \min_{\frac{1}{3} \le t \le \frac{1}{2}}(u_2(t) + v_2(t))$, and $\|(u_3, v_3)\| > 1$, with $\min_{\frac{1}{3} \le t \le \frac{1}{2}}(u_3(t) + v_3(t)) < 2$.

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References

[1] D. R. Anderson; *Green's function for a third- order generalized right focal problem*, Math. Anal. Appl. 288 (2003) 1-14.

[2] D. R. Anderson, J. M. Davis; *Multiple solutions and eigenvalues for third-order right focal boundary-value problems*, J. Math. Anal. Appl. 267 (2002) 135-157.

[3] D. Guo, V. Lakshmikantham; *Nonlinear problem in Abstract Cones*, Academic Press, New York, 1988.

[4] L. J. Guo, J. P. Sun, Ya H. Zhao; *Existence of positive solutions for nonlinear third-order three-point boundary-value problems*, Nonlinear Anal. 68 (2008) 3151-3158.

[5] L. Hu, L. L. Wang; *Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations*, J. Math. Anal. Appl. 335 (2007) 1052-1060.

[6] M. A. Krasnoselskii; *Positive solutions of operator equations*, Noordhoff, Groningen, Netherlands, 1964.

[7] R. W. Leggett, L. R. Williams; *Multiple positive fixed point of nonlinear operators on orderd Banach space*, Indiana Univ. Math. J. 28 (1979) 673-688.

[8] Y. Li, Y. Guo, G. Li; *Existence of positive solutions for systems of nonlinear third-order differential equations*, Commun Nonlinear Sci Numer Simulat 14 (2009) 3792-3797.

[9] Y. Sun; *Positive solutions of singular third-order three-point boundary value problem*, J. Math. Anal. Appl. 306 (2005) 589-603.

[10] Z. Wei; *Positive solution of singular Dirichlet boundary value problems for second order ordinary differential equation system*, J. Math. Anal. Appl. 328 (2007) 1255-1267.

[11] Q. Yao; *The existence and multiplicity of positive solutions of a third-order three-point boundary value problem*, Acta Math. Appl. Sin. 19 (2003) 117-122.

[12] H. Yu, H. L, Y. Liu; *Multiple positive solutions to third-order three-point singular semipositone boundary value problem*, Proc. Indian Sci. 114 (2004) 409-422.

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