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ASYMPTOTICALLY LINEAR FOURTH-ORDER ELLIPTIC PROBLEMS WHOSE NONLINEARITY CROSSES SEVERAL EIGENVALUES

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Abstract. In this article we prove the existence of multiple solutions for the fourth-order elliptic problem

$$\begin{gathered}\Delta^2 u + c\Delta u = g(x,u) \quad \text{in } \Omega\\ u = \Delta u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(x,0) = 0$ and it is asymptotically linear at infinity. We study the cases when the parameter c is less than the first eigenvalue, and between two consecutive eigenvalues of the Laplacian. To obtain solutions we use the Saddle Point Theorem, the Linking Theorem, and Critical Groups Theory.

1. Introduction

Let us consider the problem

$$\begin{gathered}\Delta^2 u + c\Delta u = g(x,u) \quad \text{in } \Omega\\ u = \Delta u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(x,0) = 0$. Assume that

$$g_0 := \lim_{t\to 0} \frac{g(x,t)}{t}, \quad \text{uniformly in } \Omega, \tag{1.2}$$

$$g_\infty := \lim_{|t|\to\infty} \frac{g(x,t)}{t}, \quad \text{uniformly in} \Omega, \tag{1.3}$$

where g_0 and g_∞ are constants.

Denote by $0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_j \le \dots$ the eigenvalues of $(-\Delta, H_0^1)$ and $\mu_k(c) = \lambda_k(\lambda_k - c)$ the eigenvalues of $(\Delta^2 + c\Delta, H_0^1 \cap H^2)$. We also denote by φ_j the eigenfunction associated with λ_j and consequently with μ_j.

This fourth-order problem with g asymptotically linear has been studied by Qian and Li [6], where the authors considered the case $c < \lambda_1$ and $g_0 < \mu_1 < \mu_k < g_\infty < \mu_{k+1}$ and they obtained three nontrivial solutions. Tarantelo [8] found a negative

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solution of (1.1) with the nonlinearity of the form $g(x,u) = b[(u+1)^+ - 1]$, where b is a constant. With the same type of nonlinearity Micheletti and Pistoia [4] showed that there exist two solutions when $b > \lambda_1(\lambda_1 - c)$ and three solutions when b is close to $\lambda_k(\lambda_k - c)$. Micheletti and Pistoia [5] showed the existence of two solutions for problem (1.1) with linear growth at infinity by the classical Mountain Pass Theorem and a variation of the Linking Theorem. In [7] the authors considered the superlinear case and showed the existence of two nontrivial solutions. Zhang [9] and Zhang and Li [10] proved the existence of solutions when $f(x,u)$ is sublinear at ∞.

In our work we suppose that $c < \lambda_1$ and $\mu_{k-1} \le g_0 < \mu_k \le \mu_m < g_\infty \le \mu_{m+1}$, and we prove the existence of two nontrivial solutions of (1.1). We also obtain results for the case when $\lambda_\nu < c < \lambda_{\nu+1}$. The case $\mu_{k-1} < g_\infty < \mu_k \le \mu_m < g_0 < \mu_{m+1}$ is also considered.

The classical solutions of problem (1.1) correspond to critical points of the functional F defined on $V = H_0^1(\Omega) \cap H^2(\Omega)$, by

$$F(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx - \int_\Omega G(x,u)dx, \quad u \in V, \tag{1.4}$$

where $G(x,t) = \int_0^t g(x,s)ds$. Notice that V is a Hilbert space with the usual inner product $\int_\Omega (|\Delta v|^2 + |\nabla v|^2)dx$. Let $\|\cdot\|$ be norm induced by this inner product. Under the above assumptions F is a functional of class C^2.

For the convenience of the reader, we recall some notation of Morse Theory. Let H be a Hilbert space and $F : H \to \mathbb{R}$ be a functional of class C^1. We assume that the set of critical points of F, denoted by K, is finite. Let $y \in H$ be a critical point of F with $c = F(y)$. The group

$$C_p(F,y) = H_p(F^c, F^c \setminus \{y\}), p = 0,1,2,\dots,$$

is called the p^{th} critical group of F at y, where $F^c = \{x \in H : F(x) \le c\}$ and $H_p(\cdot,\cdot)$ is the singular relative homology group with integer coefficients.

2. Case $c < \lambda_1$

We denote $\frac{d}{dt}g(x,t)$ by $g'(x,t)$. We start with following result.

Theorem 2.1. *Assume that $g'(x,t) \ge g(x,t)/t$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Suppose that there exists $k \ge 2$, $m \ge k+1$ such that $\mu_{k-1} \le g_0 < \mu_k$, $\mu_{k-1} < g(x,t)/t$ and $\mu_m < g_\infty < \mu_{m+1}$. Then problem (1.1) has at least two nontrivial solutions.*

First we will prove that the associated functional satisfies the Palais-Smale condition. We remind that V is a Hilbert space with the inner product

$$(u,v)_0 = \int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx.$$

Indeed, $\|u\|_0 = \sqrt{(u,u)_0}$ is equivalent to norm $\|u\|$, provided $c < \lambda_1$.

Lemma 2.2. *If there exists $m \ge 1$ such that $\mu_m < g_\infty < \mu_{m+1}$ then the functional F defined in* (1.4) *satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset V$ be a Palais-Smale sequence; that is, a sequence such that $F(u_n) \to C$ and $F'(u_n) \to 0$. Since g is a sublinear function, it is sufficient to prove that $(\|u_n\|_0)_{n\in\mathbb{N}}$ is bounded. By contradiction we suppose that $\lim_{n\to\infty} \|u_n\|_0 = \infty$.

Up to a subsequence we can assume that $v_n = u_n/\|u_n\|_0$ converge to v weakly in V, strongly in $L^2(\Omega)$ and pointwise in Ω.

Let $\phi \in V$. Then

$$F'(u_n)\phi = \int_\Omega (\Delta u_n \Delta\phi - c\nabla u_n \nabla\phi)dx - \int_\Omega g(x,u_n)\phi dx,$$

thus

$$\frac{F'(u_n)}{\|u_n\|_0}\phi = \int_\Omega (\Delta v_n \Delta\phi - c\nabla v_n \nabla\phi)dx - \int_\Omega \frac{g(x,u_n)}{u_n} v_n \phi dx.$$

Taking the limit in the last expression and using the above convergence, we obtain

$$\Delta^2 v + c\Delta v = g_\infty v. \tag{2.1}$$

in the weak sense.

In fact, define $A_+ = \{x \in \Omega; v(x) > 0 \text{ a. e. }\}$ and $A_- = \{x \in \Omega; v(x) < 0 \text{ a. e. }\}$ then $u_n(x) \to \infty$ a.e. if $x \in A_+$ and $u_n(x) \to -\infty$ a.e. if $x \in A_-$. Using (g_∞) and the fact that over $A_0 = \{x \in \Omega; v(x) = 0 \text{ a.e. }\}$, we obtain $\frac{g(x,u_n)}{u_n}$ is bounded.

Now we will prove that $v \neq 0$. Note that

$$\frac{F(u_n)}{\|u_n\|_0^2} = \frac{1}{2} - \int_\Omega \frac{G(x,u_n)}{\|u_n\|_0^2}dx = \frac{1}{2} - \int_\Omega \frac{G(x,u_n)}{u_n^2}v_n^2 dx.$$

Taking the limit in this expression and using the fact $F(u_n) \to C$ as $n \to \infty$, we obtain

$$\int_\Omega g_\infty v^2 dx = \frac{1}{2},$$

which proves that $v \neq 0$. Thus, we conclude that g_∞ is an eigenvalue of $(\Delta^2 + c\Delta, V)$, contradiction. Therefore, $(\|u_n\|_0)_{n\in\mathbb{N}}$ is bounded. The proof is complete. $\square$

For the next lemma, we split the space V in the following way: $V = H \oplus H_3$, where $H = \operatorname{span}\{\varphi_1, \dots, \varphi_m\}$ and $H_3 = H^\perp$.

Lemma 2.3. *Suppose there exists $m \geq 1$ such that $\mu_m < g_\infty < \mu_{m+1}$. Then:*

1. *$F(u) \to -\infty$ as $\|u\|_0 \to \infty$ for $u \in H$.*
2. *There exists $C_1 > 0$ such that $F(u) \geq -C_1$ for all $u \in H_3$.*

Proof. Because $\mu_m < g_\infty$, there exist $\epsilon, C > 0$ such that

$$G(x,t) \geq \frac{t^2}{2}(\mu_m + \epsilon) - C. \tag{2.2}$$

Thus

$$\begin{aligned} F(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx - \int_\Omega G(x,u)dx \\ &\leq \frac{1}{2}\|u\|_0^2 - \int_\Omega \frac{u^2}{2}(\mu_m + \epsilon)dx + C\int_\Omega |u|dx \\ &\leq \frac{1}{2}\|u\|_0^2(1 - \frac{\mu_m + \epsilon}{\mu_m}) + C|\Omega|, \end{aligned}$$

which proves (1).

Using the fact $g_\infty < \mu_{m+1}$ and a similar argument as in the proof of (1), we obtain (2). $\square$

Now, we split the space H as follows

$$H = H_1 \oplus H_2,$$

where $H_1 = \operatorname{span}\{\varphi_1 \dots, \varphi_{k-1}\}$ and $H_2 = \operatorname{span}\{\varphi_k, \dots, \varphi_m\}$. Thus $V = H_1 \oplus H_2 \oplus H_3$.

Lemma 2.4. *Suppose that there are $\alpha, \delta > 0$ such that $\mu_{k-1} \leq g(x,t)/t \leq \alpha < \mu_k$, for $|t| < \delta$, $k \geq 2$, and $g'(x,t) \geq \mu_{k-1}$. Moreover, assume that there exists $m \geq k+1$ such that $\mu_m < g_\infty < \mu_{m+1}$. The following statements hold:*

1. *There are $r > 0$ and $A > 0$ such that $F(u) \geq A$ for all $u \in H_2 \oplus H_3$ with $\|u\|_0 = r$.*
2. *$F(u) \to -\infty$, as $\|u\|_0 \to \infty$ for all $u \in H_1 \oplus H_2$.*
3. *$F(u) \leq 0$ for all $u \in H_1$.*

Proof. Let $H^{m+1} = \ker(\Delta^2 + c\Delta - \mu_{m+1} I)$. Then $H_2 \oplus H_3 = U \oplus W$, where $U = H_2 \oplus H^{m+1}$. For $v \in V$ put $v = u + w$, $u \in U$ and $w \in W$. Since $\dim U < +\infty$ then U is generated by eigenfunctions which are $L^\infty(\Omega)$, then there exists $r > 0$ such that

$$\sup_{x \in \Omega} |u(x)| \leq \frac{\gamma - \mu_k}{\gamma - \alpha} \delta \quad \text{if } \|u\|_0 \leq r,$$

where $\gamma > \mu_k$ and $\int_\Omega (|\Delta w|^2 - c|\nabla w|^2) dx \geq \gamma \int_\Omega |w|^2 dx$, for all $w \in W$.

Suppose that $\|u\|_0 \leq r$. If $|u(x) + w(x)| \leq \delta$, then

$$\begin{aligned}
&\frac{1}{2}\mu_2 |u|^2 + \frac{1}{4}\gamma |w|^2 - G(x, u+w) \\
&\geq \frac{1}{2}\mu_2 |u|^2 + \frac{1}{4}\gamma |w|^2 - \frac{1}{2}\alpha (u+w)^2 \\
&= -\frac{1}{4}\alpha |w|^2 + \frac{1}{4}(\gamma - \alpha)|w|^2 + \frac{1}{2}(\mu_2 - \alpha) u^2 - \alpha u w \\
&\geq -\frac{1}{4}\mu_2 |w|^2 + \frac{1}{2}(\mu_2 - \alpha) u^2 - \alpha u w
\end{aligned}$$

If $|u(x) + w(x)| > \delta$, then

$$|G(x, u+w)| \leq \frac{1}{2}\mu_k (u+w)^2 - \frac{1}{2}(\mu_k - \alpha)\delta^2.$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\mu_2 |u|^2 + \frac{1}{4}\gamma |w|^2 - G(x, u+w) \\
&\geq \frac{1}{2}\mu_2 |u|^2 + \frac{1}{4}\gamma |w|^2 - \frac{1}{2}\mu_k (u+w)^2 + \frac{1}{2}(\mu_k - \alpha)\delta^2 \\
&= -\frac{1}{4}\mu_k |w|^2 + \frac{1}{2}(\mu_2 - \alpha)|u|^2 - \alpha u w + \frac{1}{4}(\gamma - \mu_k)|w|^2 + (\alpha - \mu_k) u w \\
&\quad + \frac{1}{2}(\alpha - \mu_k)|u|^2 + \frac{1}{2}(\mu_k - \alpha)\delta^2 \\
&\geq -\frac{1}{4}\mu_k |w|^2 + \frac{1}{2}(\mu_2 - \alpha)|u|^2 - \alpha u w,
\end{aligned}$$

where the last inequality follows from the fact that the quadrat form below is positive (see [3, p. 235]).

$$\frac{1}{4}(\gamma - \mu_k)|w|^2 + (\alpha - \mu_k) u w + \frac{1}{2}(\alpha - \mu_k)|u|^2 + \frac{1}{2}(\mu_k - \alpha)\delta^2.$$

Therefore,

$$\begin{aligned} F(v) &= \frac{1}{2}\|u+w\|_0^2 - \int_\Omega G(x,u+w)dx \\ &\geq \frac{1}{4}\|w\|_0^2 - \frac{1}{4}\mu_k \int_\Omega |w|^2 dx + \frac{1}{2}(\mu_2-\alpha)\int_\Omega |u|^2 dx \\ &\geq \min\big\{\frac{1}{4}\big(1-\frac{\mu_k}{\gamma}\big), \frac{\mu_2-\alpha}{2\mu_k}\big\}\|v\|_0^2, \end{aligned}$$

which proves assertion (1).

The proof of (2) follows by the same argument as in the proof of (1) of Lemma 2.2. For (3), observe that $g'(x,s) \geq \mu_{k-1}$ and so $G(x,t) \geq \mu_{k-1}t^2/2$. Thus, if $u \in H_1$ then $u = \sum_{i=1}^{k-1} m_i\varphi_i$ for some constant $m \in \mathbb{R}$. Hence

$$\begin{aligned} F(u) &\leq \sum_{i=1}^{k-1} \frac{1}{2}\int_\Omega (|\Delta\varphi_i|^2 - c|\nabla\varphi_i|^2)dx - \sum_{i=1}^{k-1}\mu_{k-1}\int_\Omega \frac{\varphi_i^2}{2}dx \\ &\leq \sum_{i=1}^{k-1}\frac{m_i^2}{2}\Big(\|\varphi_i\|_0^2 - \mu_i\int_\Omega \varphi_i^2\Big) = 0. \end{aligned}$$

which proves (3). The proof of lemma is complete. □

Conclusion of de proof Theorem 2.1. By Lemmas 2.2 and 2.3, we have that the functional F satisfies the (PS) condition and has the geometry of Saddle Point Theorem. Therefore there exists u_1, a critical point of F, such that

$$C_m(F,u_1) \neq 0. \tag{2.3}$$

Moreover, by conditions $\mu_{k-1} \leq g_0 < \mu_k$ and $g'(x,t) \geq g(x,t)/t$ for all $x \in \Omega$ and $t \in \mathbb{R}$, we verifies the hypotheses of Lemma 2.4. It follows that the functional F satisfies the geometry of Linking Theorem. Thus, there is a critical point u_2 of F satisfying

$$C_k(F,u_2) \neq 0.$$

Since $\mu_{k-1} \leq g_0 < \mu_k$, then $m(0)+n(0) \leq k-1$, and by a corollary of Shifting Theorem [2, Corollary 5.1, Chapter 1], we have $C_p(F,0) = 0$ for all $p > k-1$. Therefore u_1 and u_2 are nontrivial critical points of F. The theorem follows from the next claim.

Claim: $C_p(F,u_2) = \delta_{pk}G$.

From (2.3)) and the Shifting Theorem we have that $m(u_2) \leq k$. We will show that $m(u_2) = k$. Indeed, by $g(x,t)/t > \mu_{k-1}$ we have that $\beta_i(g(x,u_2)/u_2) < \beta_i(\mu_{k-1}) \leq 1$ for all $i \leq k-1$. Now, we have that

$$\Delta^2 u_2 + c\Delta u_2 = \frac{g(x,u_2)}{u_2}u_2.$$

This implies that $\beta_k(g(x,u_2)/u_2) \leq 1$. Then, it follows from $g'(x,t) \geq g(x,t)/t$, that $\beta_k(g'(x,u_2)) < 1$. This implies that $m(u_2) \geq k$, then $m(u_2) = k$. Again, the Shifting Theorem and (2.3)) imply the Claim.

Theorem 2.5. *Assume that $\mu_{k-1} \leq g'(x,t) < \mu_{m+1}$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Suppose that there exists $k \geq 2$, $m \geq k+1$ such that $\mu_{k-1} < g_0 < \mu_k$ and $\mu_m < g_\infty < \mu_{m+1}$. Then problem (1.1)) has at least two nontrivial solutions.*

Proof. By hypotheses $\mu_{k-1} < g_0 < \mu_k$ and $\mu_{k-1} \leq g'(x,t) < \mu_{m+1}$ for all $x \in \Omega$ and $t \in \mathbb{R}$, we verifies the Lemma 2.4. Thus, as in the proof of the previous theorem there exists critical points u_1 and u_2 such that

$$C_m(F, u_1) \neq 0 \quad \text{and} \quad C_k(F, u_2) \neq 0,$$

moreover, we can conclude that u_1 and u_2 are nontrivial solutions, provided $\mu_{k-1} \leq g_0 < \mu_k$.

We will show that $u_1 \neq u_2$. Since $g'(x,t) < \mu_{m+1}$ we obtain

$$\begin{aligned} F''(u_1)(v,v) &= \int_\Omega (|\Delta v|^2 - c|\nabla v|^2)dx - \int_\Omega g'(x,u_1)v^2 dx \\ &> \int_\Omega (|\Delta v|^2 - c|\nabla v|^2)dx - \mu_{m+1}\int_\Omega |v|^2 dx \geq 0, \end{aligned}$$

for all $v \in \operatorname{span}\{\varphi_{m+1}, \dots\}$. Hence $m(u_1) + n(u_1) \leq m$. On the other hand, $C_m(F, u_1) \neq 0$. Thus, by a corollary of Shifting Theorem $C_p(F, u_1) = \delta_{pm}\mathbb{Z}$. Therefore $u_1 \neq u_2$, which completes the proof. □

Theorem 2.6. *Assume that $\mu_1 < g_\infty < \mu_2$ and there exists $m \geq 2$ such that $\mu_m < g_0 < \mu_{m+1}$. Then (1.1) has at least two nontrivial solutions.*

Proof. By Lemmas 2.2 and 2.3, we can apply the Saddle Point Theorem to obtain a solution $u_1 \neq 0$ such that $C_1(F, u_1) \neq 0$.

Claim: $C_p(F, u_1) = \delta_{p1}\mathbb{Z}$.

Actually, we have that $m(u_1) \leq 1$. If $m(u_1) = 1$ the claim is proved. If $m(u_1) = 0$, then we have that the first eigenvalue β_1 of the problem

$$\begin{gathered} \Delta^2 v + c\Delta v = \beta g'(x,u_1)v \quad \text{in } \Omega \\ v = \Delta v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.4}$$

satisfies $\beta_1 = 1$ and is simple. It follows that $n(u_1) = 1$, and so the claim follows by Shifting Theorem.

We also have that $C_p(F, 0) = \delta_{pm}\mathbb{Z}$, provided $\mu_m < g_0 < \mu_{m+1}$. Now, suppose by contradiction that u_1 and 0 are the unique critical points of F. Thus the Morse Inequality reads as

$$(-1) = (-1) + (-1)^m.$$

This is a contradiction. So there is at least one more nontrivial solution. □

3. The case $\lambda_1 < c < \lambda_2$

Since $\lambda_1 < c < \lambda_2$ the first eigenvalue of the problem

$$\begin{aligned} \Delta^2 u + c\Delta u = \mu u \quad &\text{in} \quad \Omega \\ u = \Delta u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{3.1}$$

is negative. Thus, $\int_\Omega (|\Delta v|^2 - c|\nabla v|^2)dx$ is not an inner product in V. In this case, consider the following norm: for all $\phi \in V$

$$\begin{aligned} \|\phi\|_1^2 &= \alpha_1^2 \int_\Omega (|\Delta\varphi_1|^2 + |\nabla\varphi_1|^2)dx + \int_\Omega (|\Delta\overline{\phi}|^2 - c|\nabla\overline{\phi}|^2)dx \\ &= \alpha_1^2(\lambda_1^2 + \lambda_1) + \int_\Omega (|\Delta\overline{\phi}|^2 - c|\nabla\overline{\phi}|^2)dx \\ &= \alpha_1^2(\lambda_1^2 + \lambda_1) + \|\overline{\phi}\|_0^2 \end{aligned}$$

where $\phi = \alpha_1\varphi_1 + \overline{\phi}$ with $\overline{\phi} \in \operatorname{span}\{\varphi_1\}^\perp$ and $\|\cdot\|_0$ was defined in the previous section. Notice that $\|\cdot\|_0$ is a norm in $\operatorname{span}\{\varphi_1\}^\perp$.

Clearly, the norm $\|\cdot\|_1$ is equivalent to usual norm $\|\cdot\|$.

Next lemma will prove that the functional (1.4) with the above conditions satisfies the *Palais-Smale Condition*, (PS)-Condition.

Lemma 3.1. *Suppose that g_∞ is not eigenvalue from* (3.1)*. Then the functional* (1.4) *satisfies the (PS)-Condition.*

Proof. Let $(u_n) \subset V$ be a Palais-Smale sequence, that is, a sequence such that $F(u_n) \to C$ and $F'(u_n) \to 0$. This lemma is proved with the same arguments used in Lemma 2.2. By contradiction, suppose that $\lim_{n\to\infty} \|u_n\|_1 = \infty$. Up to a subsequence we can assume that $v_n = u_n/\|u_n\|_1$ converge to v weakly in V strongly in $L^2(\Omega)$ and pointwise in Ω. Therefore

$$\Delta^2 v + c\Delta v = g_\infty v.$$

As in the proof of Lemma 2.2 we have to show that $v \neq 0$. In fact, let $u_n = t_1^n\varphi + \overline{\phi}_n$,

$$\begin{aligned} F(u_n) &= \frac{1}{2}\int_\Omega (|\Delta u_n|^2 - c|\nabla u_n|^2)dx - \int_\Omega G(x,u_n)dx \\ &= \frac{1}{2}\|u_n\|_1^2 - \frac{1}{2}(t_1^n)^2(\lambda_1 + c\lambda_1) - \int_\Omega G(x,u_n)dx. \end{aligned} \tag{3.2}$$

Since $v_n \to v$ in $L^2(\Omega)$ as $n \to \infty$ then $\int v_n\varphi_1 \to \int v\varphi_1 = t_1$ as $n \to \infty$. Taking limit in the expression

$$\frac{F(u_n)}{\|u_n\|_1^2} = \frac{1}{2} - \frac{1}{2}\frac{(t_1^n)^2}{\|u_n\|_1^2}(\lambda_1 + c\lambda_1) - \int_\Omega \frac{G(\,u_n)}{u_n^2}v_n^2dx, \tag{3.3}$$

we obtain

$$0 = \frac{1}{2} - \frac{1}{2}(t_1)^2(\lambda_1 + c\lambda_1) - \int_\Omega g_\infty v^2dx, \tag{3.4}$$

this implies $v \neq 0$. Thus, Lemma 3.1 is proved. □

In the next result we obtain the functional geometry to establish existence of two nontrivial solutions from (1.1)).

Lemma 3.2. *Suppose that $\mu_1 < g_\infty < \mu_2$. Then*

(i) *$F(t\varphi_1) \to -\infty$, as $t \to \infty$.*
(ii) *There exists $C_1 > 0$ such that $F(u) \geq -C_1$ for all $u \in \operatorname{span}\{\varphi_1\}^\perp$.*

Proof. (i). Hence $\mu_1 < g_\infty < \mu_2$ there exists $\epsilon > 0$ and $B > 0$ such that

$$G(x,s) \geq \frac{\mu_1+\epsilon}{2}s^2 - B.$$

So,

$$F(t\varphi_1) \leq \frac{1}{2}t^2(\lambda_1^2 - c\lambda_1) - \frac{\mu_1+\epsilon}{2}t^2\int_\Omega \varphi_1^2dx + B|\Omega| = -\frac{1}{2}t^2\epsilon + B|\Omega|.$$

this implies $F(t\varphi_1) \to -\infty$ as $t \to \infty$.

The proof of (ii) is analogous of (ii) of Lemma 2.3. □

The next lemma is analogous to Lemma 2.4.

Lemma 3.3. *Suppose that there are $\alpha, \delta > 0$ such that $\mu_{k-1} \le g(x,t)/t \le \alpha < \mu_k$, for $|t| < \delta$, $k \ge 2$, and $g'(x,t) \ge \mu_{k-1}$. Moreover, assume that there exists $m \ge k+1$ such that $\mu_m < g_\infty < \mu_{m+1}$. The following statements hold:*

- (i) *There exists $r > 0$ and $A > 0$ such that $F(u) \ge A$ for all $u \in H_2 \oplus H_3$ with $\|u\|_1 = r$.*
- (ii) *$F(u) \to -\infty$, as $\|u\|_1 \to \infty$ for all $u \in H_1 \oplus H_2$.*
- (iii) *$F(u) \le 0$ for all $u \in H_1$.*

Proof. The proof of (i) is analogous to proof of (i), Lemma 2.4.

Proof of (ii). Let $u \in H_1 \oplus H_2$. Then $u = t\varphi_1 + w$, where $w \in \operatorname{span}\{\varphi_1\}^\perp$. By $\mu_m < g_\infty$ there exists $\epsilon, C > 0$ such that $G(x,s) \ge ((\mu_m + \epsilon)/2)s^2 - C$. Thus,

$$
\begin{aligned}
F(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)dx - \int_\Omega G(x,u)dx \\
&\le \frac{1}{2}\|w\|_0^2 + \frac{1}{2}t^2\lambda_1(\lambda_1 - c) - \frac{\mu_m + \epsilon}{2}\int_\Omega (t^2\varphi_1^2 + w^2)dx + C|\Omega| \\
&\le \frac{1}{2}\|w\|_0^2\big(1 - \frac{\mu_m + \epsilon}{\mu_m}\big) + \frac{1}{2}t^2(\lambda_1^2 - c\lambda_1) - t^2\frac{\mu_m + \epsilon}{2} + C|\Omega|
\end{aligned}
$$

this implies $F(u) \to -\infty$ as $\|u\|_1 \to \infty$.

Proof of (iii). Since $g'(x,s) \ge \mu_1$ we obtain $G(x,s) \ge \mu_1 t^2/2$ and

$$
\begin{aligned}
F(t\varphi_1) &= \frac{1}{2}t^2\int_\Omega (|\Delta\varphi_1|^2 - c|\nabla\varphi_1|^2)dx - \int_\Omega G(x,t\varphi_1)dx \\
&\le \frac{t^2}{2}(\mu_1 - \int_\Omega \mu_1\varphi_1^2 dx) = 0.
\end{aligned}
$$

The proof is complete. □

From Lemmas 3.2 and 3.3, we find analogous geometries as in Lemmas 2.3 and 2.4 for functional (1.4). Furthermore, we have the Palais-Smale Condition by Lemma 3.1. Thus, with the same proofs of Theorems 2.1, 2.5 and 2.6, we obtain the following results.

Theorem 3.4. *Assume that $g'(x,t) \ge g(x,t)/t$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Suppose that there exists $k \ge 2$, $m \ge k+1$ such that $\mu_{k-1} \le g_0 < \mu_k$ and $\mu_m < g_\infty < \mu_{m+1}$ and $\mu_{k-1} < g(x,t)/t$. Then (1.1) has at least two nontrivial solutions.*

Theorem 3.5. *Assume that $\mu_{k-1} \le g'(x,t) < \mu_{m+1}$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Suppose that there exists $k \ge 2$, $m \ge k+1$ such that $\mu_{k-1} \le g_0 < \mu_k$ and $\mu_m < g_\infty < \mu_{m+1}$. Then (1.1) has at least two nontrivial solutions.*

Theorem 3.6. *Assume that $\mu_1 < g_\infty < \mu_2$. Suppose there exists $m \ge 2$ such that $\mu_m < g_0 < \mu_{m+1}$. Then (1.1) has at least two nontrivial solutions.*

4. The case $\lambda_\nu < c < \lambda_{\nu+1}$, $\nu \ge 2$

In this section we consider $\lambda_\nu < c < \lambda_{\nu+1}$. Thus, the problem

$$
\begin{gathered}
\Delta^2 u + c\Delta u = \mu u \quad \text{in } \Omega \\
u = \Delta u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{4.1}
$$

has ν first negative eigenvalues. Therefore, we will define the following norm in V:

$$\begin{aligned}\|\phi\|_\nu^2 &= \sum_{i=1}^{\nu}\alpha_i^2\int_\Omega(|\Delta\varphi_i|^2+|\nabla\varphi_i|^2)dx+\int_\Omega(|\Delta\overline{\phi}|^2-c|\nabla\overline{\phi}|^2)dx\\ &=\sum_{i=1}^{\nu}\alpha_i^2(\lambda_i^2+\lambda_i)+\int_\Omega(|\Delta\overline{\phi}|^2-c|\nabla\overline{\phi}|^2)dx\\ &=\sum_{i=1}^{\nu}\alpha_i^2(\lambda_i^2+\lambda_i)+\|\overline{\phi}\|_0^2,\quad \text{for all } \phi\in V,\end{aligned}$$

where $\phi=\alpha_1\varphi_1+\dots+\alpha_\nu\varphi_\nu+\overline{\phi}$ with $\overline{\phi}\in\operatorname{span}\{\varphi_1,\dots,\varphi_\nu\}^\perp$.

In this section, results will be obtained with the same arguments used in previous section. The Palais-Smale Condition is proved as Lemma 3.1 with equation (3.2) changed by

$$\begin{aligned}F(u_n)&=\frac{1}{2}\int_\Omega(|\Delta u_n|^2-c|\nabla u_n|^2)dx-\int_\Omega G(x,u_n)dx\\ &=\frac{1}{2}\|u_n\|_\nu^2-\sum_{i=1}^{\nu}\frac{1}{2}(t_i^n)^2(\lambda_i+c\lambda_i)-\int_\Omega G(x,u_n)dx.\end{aligned}$$

and the equation (3.4) changed by

$$0=\frac{1}{2}-\frac{1}{2}\sum_{i=1}^{\nu}(t_i)^2(\lambda_i+c\lambda_i)-\int_\Omega g_\infty v^2dx.$$

Suppose V as before and $\mu_m<g_\infty<\mu_{m+1}$. We can split $V=H\oplus W$ where $H=\operatorname{span}\{\varphi_1,\dots,\varphi_m\}$ and $W=H^\perp$.

Next lemma is analogous to Lemma 3.2.

Lemma 4.1. *Assume that $\mu_m<g_\infty<\mu_{m+1}$ and $\nu\le m$. Then*

(i) *$F(u)\to-\infty$, as $\|u\|_\nu\to\infty$, for $u\in H$.*

(ii) *There exists $C_1>0$ such that $F(w)\ge -C_1$ for all $w\in W$.*

Proof. The proof of (ii) is similar to the proof of Lemma 3.2, (ii).

The proof of (i) follows from $g_\infty>\mu_m$. In fact, let $u\in H$. Since $\nu\le m$, we have $u=\sum_{i=1}^{\nu}t_i\varphi_i+w$. Thus, we have two cases to consider:

Case 1: $\nu<m$. Then there exists $\epsilon,B>0$ such that

$$\begin{aligned}F(u)&=\frac{1}{2}\int_\Omega(|\Delta u|^2-c|\nabla u|^2)dx-\int_\Omega G(x,u)dx\\ &\le\frac{1}{2}\|w\|_0^2+\frac{1}{2}\sum_{i=1}^{\nu}t_i^2(\lambda_i^2-c\lambda_i)-\frac{\mu_m+\epsilon}{2}\Big(\sum_{i=1}^{\nu}t_i^2+\int_\Omega|w|^2dx\Big)+B|\Omega|\\ &\le\frac{1}{2}\|w\|_0^2(1-\frac{\mu_m+\epsilon}{\mu_m})+\frac{1}{2}\sum_{i=1}^{\nu}t_i^2(\lambda_i^2-c\lambda_i-(\mu_m+\epsilon))+B|\Omega|.\end{aligned}$$

Case 2: $\nu=m$. Then

$$\begin{aligned}F(u)&=\frac{1}{2}\int_\Omega(|\Delta u|^2-c|\nabla u|^2)dx-\int_\Omega G(x,u)dx\\ &\le\frac{1}{2}\sum_{i=1}^{\nu}t_i^2(\lambda_i^2-c\lambda_i-(\mu_\nu+\epsilon))+B|\Omega|.\end{aligned}$$

In both cases $F(u) \to -\infty$ as $\|u\|_\nu \to \infty$, which completes the proof. □

From the Palais-Smale Condition and Lemma 4.1, we obtain the following result.

Theorem 4.2. *Assume that $\mu_m < g_\infty < \mu_{m+1}$ and $\nu \le m$. Suppose, there exists $s \ge m+1$ such that $\mu_s < g_0 < \mu_{s+1}$. Then (1.1) has at least one nontrivial solution.*

To study multiplicity of solutions we have an analogous lemma to Lemma 3.3.

Lemma 4.3. *Assume that $\nu \le k$. Suppose that there are $\alpha, \delta > 0$ such that $\mu_{k-1} \le g(x,t)/t \le \alpha < \mu_k$, for $|t| < \delta$, $k \ge 2$, and $g'(x,t) \ge \mu_{k-1}$. Moreover, assume that there exists $m \ge k+1$ such that $\mu_m < g_\infty < \mu_{m+1}$. The following statements hold:*

- (i) *There exists $r > 0$ and $A > 0$ such that $F(u) \ge A$ for all $u \in H_2 \oplus H_3$ with $\|u\|_\nu = r$.*
- (ii) *$F(u) \to -\infty$, as $\|u\|_\nu \to \infty$ for $u \in H_1 \oplus H_2$.*
- (iii) *$F(u) \le 0$ for all $u \in H_1$.*

Thus we obtain the main theorem of this section.

Theorem 4.4. *Suppose there exist $k \in \mathbb{N}$, $m \ge k+1$ such that $\mu_{k-1} < g_0 < \mu_k$, $\mu_m < g_\infty < \mu_{m+1}$ and $\nu \le m$. Assume that $\mu_{k-1} \le g'(x,t) \le \mu_{m+1}$, for all $x \in \Omega$ and $t \in \mathbb{R}$. If $\nu \le k$ problem 1.1 has at least two nontrivial solutions; If $k+1 \le \nu$ problem 1.1 has at least one nontrivial solution.*

Proof. Since $\nu \le k+1$ then, by Lemma 4.1 and the Palais-Smale Condition, we conclude that functional F has the geometry of Saddle Point Theorem. Then there exists u_1, a critical point of F, such that

$$C_m(F, u_1) \neq 0. \tag{4.2}$$

On the other hand, from Lemma 4.3 there exists u_2 a critical point of F, such that

$$C_k(F, u_2) \neq 0. \tag{4.3}$$

The proof is completed with the same arguments as Theorem 2.5.

If $k \le \nu$ is immediate from Lemma 4.1 and $\mu_{k-1} \le g_0 < \mu_k$ that there exists nontrivial solution u_1. □

To finish, with the same arguments as in Theorem 2.1 we obtain the following result.

Theorem 4.5. *Suppose there exist $k \in \mathbb{N}$, $m \ge k+1$ such that $\mu_{k-1} \le g_0 < \mu_k$, $\mu_m < g_\infty < \mu_{m+1}$ and $\nu \le m$. Assume that $g'(x,t) \ge g(x,t)/t$ for all $x \in \Omega$ and $t \in \mathbb{R}$; and $\mu_{k-1} \le g'(x,t)$. Then: if $\nu \le k+1$ problem 1.1 has at least two nontrivial solutions; if $k \le \nu$ problem 1.1 has at least one nontrivial solution.*

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