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KRASNOSEL'SKII FIXED POINT THEOREM FOR DISSIPATIVE OPERATORS

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ABSTRACT. In this note, a sufficient condition guaranteing the existence of fixed points in a nonempty, closed convex K for $T+S$ is given, where $T : K \subset E \to E$ is dissipative and $S : K \to E$ is condensing. This may indicate a new direction of the Krasnoselskii type fixed point theorem.

1. INTRODUCTION

In 1955, Krasnoselskii [8] proved a fixed point theorem for the sum of two operators. The theorem was motivated by an observation that the inversion of a perturbed differential operator may yield the sum of a compact and contraction operator. Krasnoselskii's theorem actually combines the Banach contraction mapping principle and the Schauder [12] fixed point theorem. It asserts that the sum $T + S$ has at least one fixed point in a nonempty, closed convex subset K of a Banach space E, where S and T verify:

(i) T is a contraction with constant $\alpha < 1$;
(ii) S is continuous and $S(K)$ resides in a compact subset of E;
(iii) any $x, y \in K$ imply $Tx + Sy \in K$.

The theorem is useful in establishing the existence theorems for perturbed operator equations. The importance of this kind of results relies on, among other things, its many applications in nonlinear analysis. For instance, it has a wide range of applications to nonlinear integral equations of mixed type for proving the existence of solutions. Thus the existence of fixed points for the sum of two operators has been focus of tremendously interest and their applications are frequent in nonlinear analysis. For example, O'regan [10] proved a kind of such results with applications to boundary-value problems of second-order with nonlinearities. Several improvements of Krasnosel'skii Theorem have been established in the literature in the course of time by modifying assumption (i), (ii) or (iii). See [4, 6, 13, 14]. It was mentioned in [4] that the condition (iii) is too stringent and can be replaced by a mild one, in which Burton proposed the following improvement for (*iii*): if $x = Tx + Sy$ with $y \in K$, then $x \in K$. Subsequently, Dhage [6] replaced (i) by the following requirement: T is a bounded linear operator on E, and T^p is a nonlinear contraction for some $p \in \mathbb{N}$. In [13], the authors firstly replaced the contraction

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map by an expansion and then replaced the compactness of the operator S by a $k-$set contractive one, and obtained some new fixed point results, which extend and develop some previous related fixed point results. More recently, a pretty universal compact-type and noncompact-type Krasnoselskii fixed point theorem were established in [9] and [14], respectively.

So far, for the sum of two operators, many kinds of generalizations and variants of Krasnoselskii's fixed point theorem have been obtained, see for example [4, 5, 6, 9, 10, 13, 14] and the references therein.

The fixed point results obtained in [14] facilitate the application of Krasnosel'skii Theorem to another kind of operator, namely, dissipative operator. We thus consider in this note the fixed point of the sum $T+S$ in a nonempty, closed convex K, where $T : K \subset E \to E$ is dissipative and $S : K \to E$ is condensing. Since dissipative operators may not be continuous, the assumption on T lessens the usual continuity assumption on T. Thus, this may indicate a new direction of the Krasnoselskii type fixed point theorem. In a special case with $K = B_\rho$, under mild condition $S(B_\rho) \subset \mathrm{R}(I-T)$, it is shown that $T+S$ has at least one fixed point in B_ρ, where B_ρ is the closed ball with radius $\rho > 0$ and center at origin.

2. A Fixed point theorem for the sum of dissipative and condensing operators

Throughout this note, we denote by $(E, \|\cdot\|)$ a Banach space. Define the duality set of $x \in E$, a subset of the dual space E^* of E, by

$$J(x) = \big\{x^* \in E^* : \|x^*\|_{E^*}^2 = \|x\|_E^2 = \langle x^*, x\rangle\big\}.$$

Let $T : D(T) \subset E \to E$ be a (possibly) nonlinear operator. Then T is said to be dissipative if for each $x, y \in D(T)$ there exists $f \in J(x-y)$ such that

$$\mathrm{Re}\langle f, Tx - Ty\rangle \le 0. \tag{2.1}$$

This notion is a nonlinear version of linear dissipative operators, introduced in [3] and [7] independently. Dissipative or accretive operators play a significant role in the study of nonlinear semigroups, differential equations in Banach spaces, and fully nonlinear partial differential equations.

For a Hilbert space H, it is clear that (2.1) is equivalent to $(x-y, Tx-Ty) \le 0$ for all $x, y \in D(T)$. Using this equivalent characterization, one can easily show that the Laplacian operator, Δ, defined on the dense subspace of compactly supported smooth functions on the domain $\Omega \subset \mathbb{R}^n$, is a dissipative operator. The following proposition is useful in our further purpose, we may provide all the details for completeness.

Proposition 2.1. *Assume that $T : D(T) \subset E \to E$ is a dissipative operator. Then*

(i) *$(I-T)$ is one-to-one;*
(ii) *T is closed if and only if $R(\lambda I - T) = (\lambda I - T)(D(T))$ is closed for any $\lambda > 0$.*

Proof. (i). Since $T : D(T) \subset E \to E$ is a dissipative operator, we obtain (cf. [7, Lemma 1.1])

$$\|x-y\| \le \|x-y-\lambda(Tx-Ty)\| \tag{2.2}$$

for all $\lambda > 0$ and $x, y \in D(T)$. Setting $\lambda = 1$ in (2.2), we see that $(I-T)$ is injective.

(ii) "if part": Take $y_n = (\lambda I - T)x_n \in R(\lambda I - T)$ with $y_n \to y$. One can readily deduce from (2.2) that

$$\|x_n - x_m\| \le \frac{1}{\lambda}\|y_n - y_m\|,$$

which illuminates that $\{x_n\}$ is a Cauchy sequence, and therefore converges to some x in E. This in turn gives $Tx_n = \lambda x_n - y_n \to \lambda x - y$. The closedness of T then implies that $x \in D(T)$ and $\lambda x - y = Tx$; i.e., $y = (\lambda I - T)x \in R(\lambda I - T)$.

"only if part": Since T is dissipative, it follows from (2.2) that

$$(\lambda I - T)^{-1} : \mathrm{R}(\lambda I - T) \to D(T)$$

is Lipschitz continuous, and

$$\|(\lambda I - T)^{-1}x - (\lambda I - T)^{-1}y\| \le \frac{1}{\lambda}\|x - y\| \tag{2.3}$$

for all $\lambda > 0$ and all $x, y \in R(\lambda I - T)$.

Let now $x_n \in D(T)$ with $x_n \to x$ and $Tx_n \to y$. Set $y_n = (\lambda I - T)x_n$ then $y_n \to \lambda x - y$. Since $\mathrm{R}(\lambda I - T)$ is closed, it follows that $\lambda x - y \in \mathrm{R}(\lambda I - T)$. Consequently, the continuity of $(\lambda I - T)^{-1}$ gives $x_n = (\lambda I - T)^{-1}y_n \to (\lambda I - T)^{-1}(\lambda x - y)$. The uniqueness of limit yields that $x \in D(T)$ and $y = Tx$. □

Theorem 2.2. *Let K be a nonempty, closed convex subset E and let $T : K \to E$ be a dissipative operator and let $S : K \to E$ be condensing. Assume that there exists $R > 0$ such that $\|Sx + T0\| \le R$ whenever $x \in K$ and $\|x\| = R$. Assume additionally that either one of the following holds.*

(i) *T is closed and linear, and there exists $\lambda_0 > 0$ such that $K \cup S(K) \subset \mathrm{R}(\lambda_0 I - T)$.*
(ii) *T is nonlinear and $K \cup S(K) \subset R(I - T)$.*

Then $S + T$ has at least one fixed point in K.

Proof. It is sufficient to give the proof for the linear case. Since T is a linear and dissipative, we know from (2.3) that $(\lambda I - T)^{-1} : \mathrm{R}(\lambda I - T) \to D(T)$ is bounded and $\|(\lambda I - T)^{-1}\| \le 1/\lambda$ for each $\lambda > 0$.

We next show that $\mathrm{R}(\lambda I - T) = \mathrm{R}(\lambda_0 I - T)$ whenever $|\lambda - \lambda_0| < \|(\lambda_0 I - T)^{-1}\|$, which makes sense by (2.3). Observe that in such case, we have

$$\lambda I - T = \lambda_0 I - T + \lambda I - \lambda_0 I = [I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}](\lambda_0 I - T). \tag{2.4}$$

Note that T is linear. It then follows easily from $K \subset \mathrm{R}(\lambda_0 I - T)$ that $I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}$ transforms $\mathrm{R}(\lambda_0 I - T)$ into itself. Now for any $y \in \mathrm{R}(\lambda_0 I - T)$, define $T_y : \mathrm{R}(\lambda_0 I - T) \to \mathrm{R}(\lambda_0 I - T)$ by

$$T_y x = (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}x + y.$$

In light of (ii) of proposition 2.1, $R(\lambda_0 I - T)$ is closed. Note now that $|\lambda - \lambda_0| < \|(\lambda_0 I - T)^{-1}\|^{-1}$. Then an application of Banach contraction mapping principle shows that

$$[I - (\lambda_0 - \lambda)(\lambda_0 I - T)^{-1}]\, \mathrm{R}(\lambda_0 I - T) = \mathrm{R}(\lambda_0 I - T). \tag{2.5}$$

Joining (2.4) and (2.5) we confirm the said assertion.

Proceeding in this manner, we finally deduce that $\mathrm{R}(\lambda_0 I - T) = \mathrm{R}(\lambda I - T)$ for all $\lambda > 0$. Consequently, $S(K) \subset \mathrm{R}(\lambda_0 I - T) = \mathrm{R}(I - T)$. We now define the mapping of $K \to K$ by

$$x \to (I - T)^{-1}Sx.$$

We claim that $\psi((I-T)^{-1}S(A)) < \psi(A)$ for all $A \subset K$ with $\psi(A) > 0$, where ψ denotes the Kuratowskii measure of non-compactness or the Hausdorff measure of non-compactness. Indeed, take any bounded $A \subset K$ with $\psi(A) > 0$, it follows from (2.3), the basic properties of measure of non-compactness [1, 2] and the assumption that S is condensing that

$$\psi((I-T)^{-1}S(A)) \leq \psi(S(A)) < \psi(A).$$

Now, if K is bounded, then the well-known Sadovskii fixed point theorem finishes the proof. If K is unbounded, we need to verify that $(I-T)^{-1}S : K \to K$ satisfies the Leray-Schauder condition. As a matter of fact, let $R > 0$ be as given in our assumption and suppose that there exist $\lambda > 1$ and $x \in K$ with $\|x\| = R$ such that $(I-T)^{-1}Sx = \lambda x$. Then $(I-T)(\lambda x) = Sx$. Since T is dissipative, we obtain from (2.2) and our hypothesis that

$$\lambda R = \|\lambda x - 0\| \leq \|\lambda x - (T(\lambda x) - T0)\| = \|Sx + T0\| \leq R,$$

which is a contradiction since $\lambda > 1$.

Now, according to the results in [11], $(I-T)^{-1}S$ admits at least one fixed point in K. $\square$

Rechecking the proof, it can be seen that the crucial point is to ensure that $\mathrm{R}(\lambda_0 I - T) = \mathrm{R}(\lambda I - T)$ for all $\lambda > 0$ if $\lambda_0 \neq 1$. But, if λ_0 happens to be one, then it suffices to require $S(K) \subset \mathrm{R}(\lambda_0 I - T)$ to make the theorem true. Thus we have the following corollary.

Corollary 2.3. *Let $T : B_\rho \to E$ be dissipative and let $S : B_\rho \to E$ be condensing for some $\rho > 0$. Assume $S(B_\rho) \subset \mathrm{R}(I-T)$. Then $S + T$ has at least one fixed point in B_ρ.*

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