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NONEXISTENCE OF RADIAL POSITIVE SOLUTIONS FOR A NONPOSITONE SYSTEM IN AN ANNULUS

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Abstract. In this article we study the nonexistence of radial positive solutions for a nonpositone system in an annulus by using energy analysis and comparison methods.

1. Introduction

We study the nonexistence of radial positive solutions for the system

$$\begin{aligned} -\Delta u(x) &= \lambda f(v(x)), \quad x \in \Omega \\ -\Delta v(x) &= \mu g(u(x)), \quad x \in \Omega \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where λ, $\mu \geq \varepsilon_0 > 0$, Ω is an annulus in $\mathbb{R}^N$: $\Omega = C(0, R, \widehat{R}) = \{x \in \mathbb{R}^N : R < |x| < \widehat{R}\}$, $(0 < R < \widehat{R},\ N \geq 2)$, f and g are smooth functions that grow at least linearly at infinity. When Ω is a ball, problem (1.1) has been studied by Hai, Oruganti and Shivaji [7].

The nonexistence of radial positive solutions of (1.1) is equivalent of the nonexistence of positive solutions of the system

$$\begin{aligned} -(r^{N-1}u')' &= \lambda r^{N-1} f(v), \quad R < r < \widehat{R} \\ -(r^{N-1}v')' &= \mu r^{N-1} g(u), \quad R < r < \widehat{R} \\ u(R) = u(\widehat{R}) &= 0 = v(R) = v(\widehat{R}). \end{aligned} \tag{1.2}$$

The purpose of this paper is to prove that the nonexistence of radial positive solutions of (1.1) remains valid when Ω is an annulus and f and g satisfy the following hypotheses

- (C1) f, $g : [0, +\infty) \to \mathbb{R}$ are continuous, increasing and $f(0) < 0$ and $g(0) < 0$.
- (C2) There exist two positive real numbers a_i and b_i, $i = 1, 2$ such that

$$f(z) \geq a_1 z - b_1, \quad g(z) \geq a_2 z - b_2,$$

  for all $z \geq 0$.

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2. Main result

Our main result is the following theorem.

Theorem 2.1. *Assume that* (C1)–(C2) *are satisfied. Then there exists a positive real number σ such that* (1.1) *has no radial positive solution for $\lambda\mu > \sigma$.*

Remark. Existence result for positive solutions with superlinearities satisfying (C1), $\lambda = \mu$ and λ small can be found in [5, 6]. Existence results, for the single equation case can be found in [1, 3, 8], and non-existence results in [1, 2, 9].

To prove Theorem 2.1, we need the next three lemmas. Here we use ideas adapted from Hai, Oruganti and Shivaji [7].

Lemma 2.2. *There exists a positive constant C such that for $\lambda\mu$ large,*

$$u(R_0) + v(R_0) \le C,$$

where $R_0 = (R + \widehat{R})/2$.

Proof. Multiplying the first equation in (1.2) by a positive eigenfunction, say ϕ corresponding to λ_1, and using (C1) we obtain

$$-\int_R^{\widehat{R}} (r^{N-1}u')'\phi dr \ge \int_R^{\widehat{R}} \lambda(a_1 v - b_1)\phi r^{N-1} dr;$$

that is,

$$\int_R^{\widehat{R}} \lambda_1 u r^{N-1}\phi dr \ge \int_R^{\widehat{R}} \lambda(a_1 v - b_1)\phi r^{N-1} dr. \tag{2.1}$$

Similarly, using the second equation in (1.2) and (C2), we obtain

$$\int_R^{\widehat{R}} \lambda_1 v r^{N-1}\phi dr \ge \int_R^{\widehat{R}} \mu(a_2 u - b_2)\phi r^{N-1} dr. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$\int_R^{\widehat{R}} [\lambda_1 - \lambda\mu\frac{a_1a_2}{\lambda_1}] v\Phi r^{N-1} dr \ge \int_R^{\widehat{R}} \mu[-\lambda\frac{a_2b_1}{\lambda_1} - b_2]\Phi r^{N-1} dr.$$

Now, if $\lambda\mu a_1a_2/2 \ge \lambda_1^2$, then

$$\int_R^{\widehat{R}} \mu[-\lambda a_2b_1 - b_2\lambda_1]\Phi r^{N-1} dr \le \int_R^{\widehat{R}} -\frac{\lambda\mu}{2} a_1a_2 v\Phi r^{N-1} dr;$$

that is,

$$\int_R^{\widehat{R}} \frac{a_1a_2}{2} v\Phi r^{N-1} dr \le \int_R^{\widehat{R}} [a_2b_1 + \frac{b_2\lambda_1}{\varepsilon_0}]\Phi r^{N-1} dr, \tag{2.3}$$

(because $\lambda \ge \varepsilon_0$). Similarly

$$\int_R^{\widehat{R}} \frac{a_1a_2}{2} u\Phi r^{N-1} dr \le \int_R^{\widehat{R}} [a_1b_2 + \frac{b_1\lambda_1}{\varepsilon_0}]\Phi r^{N-1} dr. \tag{2.4}$$

Adding (2.3) and (2.4), we obtain the inequality

$$\int_R^{\widehat{R}} (u+v)\Phi r^{N-1} dr \le \frac{2}{a_1a_2}\int_R^{\widehat{R}} [a_1b_2 + \frac{b_1\lambda_1}{\varepsilon_0} + a_2b_1 + \frac{b_2\lambda_1}{\varepsilon_0}]\Phi r^{N-1} dr.$$

Then

$$\begin{aligned}(u+v)(R_0)\int_{\bar t}^{R_0}\Phi r^{N-1}dr &\le \int_{\bar t}^{R_0}(u+v)\Phi r^{N-1}dr\\ &\le \int_R^{\widehat R}(u+v)\Phi r^{N-1}dr\\ &\le \frac{2}{a_1a_2}\int_R^{\widehat R}[a_1b_2+\frac{b_1\lambda_1}{\varepsilon_0}+a_2b_1+\frac{b_2\lambda_1}{\varepsilon_0}]\Phi r^{N-1}dr,\end{aligned}$$

where $\bar t=\max(\bar t_1,\bar t_2)$ with $\bar t_1$ and $\bar t_2$ are such that

$$\bar t_1=\max\{r\in(R,\widehat R): u'(r)=0\},\quad \bar t_2=\max\{r\in(R,\widehat R): v'(r)=0\}.$$

The proof is complete. □

We remark that $\bar t_i\le R_0$, for $i=1,2$, was shown in [4]. Now, assume that there exists $z\ge 0$ ($z\not\equiv 0$) on $\overline{I}$ where $I=(\alpha,\beta)$, and a constant γ such that

$$-(r^{N-1}z')'\ge \gamma r^{N-1}z\,,\quad r\in I. \tag{2.5}$$

Let $\lambda_1=\lambda_1(I)>0$ denote the principal eigenvalue of

$$\begin{gathered}-(r^{N-1}\Psi')'=\lambda r^{N-1}\Psi,\quad r\in(\alpha,\beta)\\ \Psi(\alpha)=0=\Psi(\beta),\end{gathered}\tag{2.6}$$

where $0<\alpha<\beta\le 1$.

Lemma 2.3. *Let* (2.5) *hold. Then* $\gamma\le\lambda_1(I)$.

Proof. Multiplying (2.5) by Ψ ($\Phi>0$), an eigenfunction corresponding to the principal eigenvalue $\lambda_1(I)$, and integrating by parts (twice) we obtain

$$\int_\alpha^\beta[\gamma-\lambda_1(I)]r^{N-1}z\Psi dr\le \beta^{N-1}\Psi'(\beta)z(\beta)-\alpha^{N-1}\Psi'(\alpha)z(\alpha). \tag{2.7}$$

However, $\Psi'(\beta)<0$ and $\Psi'(\alpha)>0$; hence the right-hand side of (2.7) is less than or equal to zero. Then $\gamma\le\lambda_1(I)$, and the proof is complete. □

Now, we define

$$R_1=R_0+\frac{\widehat R-R_0}{3},\quad R_2=R_0+\frac{2(\widehat R-R_0)}{3}.$$

Lemma 2.4. *For $\lambda\mu$ sufficiently large, $u(R_2)\le\beta_2$ or $v(R_2)\le\beta_1$, where β_1 and β_2 are the unique positive zeros of f and g respectively.*

Proof. We argue by contradiction. Suppose that $u(R_2)>\beta_2$ and $v(R_2)>\beta_1$.
Case 1: $u(R_1)>\rho_2$ or $v(R_1)>\rho_1$, where $\rho_1=\frac{\beta_1+\theta_1}{2}$ and $\rho_2=\frac{\beta_2+\theta_2}{2}$ (θ_1 and θ_2 are the unique zeros of F and G respectively where $F(x)=\int_0^x f(t)dt$ and $G(x)=\int_0^x g(t)dt$). If $u(R_1)>\rho_2$ then

$$-(r^{N-1}v')'=\mu r^{N-1}g(u)\ge\varepsilon_0 r^{N-1}g(\rho_2)\quad\text{in } J=(R_0,R_1)$$

and $v(r)\ge\beta_1$ on $\bar J$.

Let ω be the unique solution of

$$\begin{gathered}-(r^{N-1}\omega')'=\varepsilon_0 r^{N-1}g(\rho_2)\quad\text{in } J\\ \omega=\beta_1\quad\text{in }\partial J.\end{gathered}$$

Then by comparison arguments, $v(r) \geq \omega(r) = \varepsilon_0 g(\rho_2)\omega_0(r) + \beta_1$ on $\bar{J}$, where ω_0 is the unique (positive) solution of

$$\begin{gathered} -(r^{N-1}\omega_0')' = r^{N-1} \quad \text{in } J \\ \omega_0 = 0 \quad \text{on } \partial J. \end{gathered}$$

In particular, there exists $\overline{\beta}_1 > \beta_1$ (we choose $\overline{\beta}_1$ such that $f(\overline{\beta}_1) \neq 0$) such that

$$v(R_0 + \frac{2(R_1 - R_0)}{3}) \geq \omega(R_0 + \frac{2(R_1 - R_0)}{3}) \geq \overline{\beta}_1$$

in $J^* = (R_0 + \frac{R_1 - R_0}{3}, R_0 + \frac{2(R_1-R_0)}{3})$. Then

$$\begin{aligned} -(r^{N-1}(u - \beta_2)')' &= \lambda r^{N-1} f(v) \\ &\geq \lambda r^{N-1} f(\overline{\beta}_1) \\ &\geq (\frac{\lambda f(\overline{\beta}_1)}{C}) r^{N-1}(u - \beta_2) \quad \text{on } J^*, \end{aligned}$$

(where C is as in Lemma 2.2). Since $u - \beta_2 > 0$ on $\bar{J}^*$, it follows that

$$\frac{\lambda f(\overline{\beta}_1)}{C} \leq \lambda_1(J^*), \tag{2.8}$$

where $\lambda_1(J^*)$ is the principal value of (2.6) (with $(\alpha, \beta) = J^*$).

Next we consider

$$\begin{aligned} (r^{N-1}(v - \beta_1)')' &= \mu r^{N-1} g(u) \\ &\geq \mu r^{N-1} g(\rho_2) \\ &\geq (\frac{\mu g(\rho_2)}{C}) r^{N-1}(v - \beta_1) \quad \text{on } J. \end{aligned}$$

Since $v - \beta_1 > 0$ on $\bar{J}$, it follows that

$$\frac{\mu g(\rho_2)}{C} \leq \lambda_1(J), \tag{2.9}$$

where $\lambda_1(J)$ is the principal value of (2.6) (with $(\alpha, \beta) = J$). Combining (2.8) and (2.9), we obtain

$$\frac{\lambda \mu f(\overline{\beta}_1) g(\rho_2)}{C^2} \leq \lambda_1(J^*)\lambda_1(J),$$

But $f(\overline{\beta}_1)$, $g(\rho_2)$ and C are fixed positive constants. This is a contradiction for $\lambda\mu$ large. A similar contradiction can be reached for the case $v(R_1) > \rho_1$.

Case 2: $u(R_1) \leq \rho_2$ and $v(R_1) \leq \rho_1$. Then $\beta_2 < u \leq \rho_2$ and $\beta_1 < v \leq \rho_1$ on $J_1 = [R_1, R_2]$. Then by the mean value theorem, there exist $c_1, c_2 \in (R_1, R_2)$ such that

$$|u'(c_2)| \leq \frac{\rho_2}{R_2 - R_1}, \quad |v'(c_1)| \leq \frac{\rho_1}{R_2 - R_1}.$$

Since $-(r^{N-1}u')' \geq 0$ on $[R_1, R_2)$, we have

$$-r^{N-1}u'(r) \leq -c_2^{N-1}u'(c_2) \quad \text{on } J_2 = [R_1, c_2);$$

thus

$$|u'(r)| \leq \frac{c_2^{N-1}}{r^{N-1}} u'(c_2) \leq (\frac{R_2}{R_1})^{N-1} \frac{\rho_2}{R_2 - R_1} \quad \text{in } J_2.$$

Similarly, we obtain

$$|v'(r)| \le (\frac{R_2}{R_1})^{N-1}\frac{\rho_1}{R_2-R_1} \quad \text{in } J_3=[R_1,c_1).$$

Hence there exists $r_0 \in (R_1,R_2)$ such that

$$|u'(r_0)| \le \widetilde{c}, \quad v'(r_0)| \le \widetilde{c},$$

where

$$\widetilde{c} = \frac{1}{R_2-R_1}(\frac{R_2}{R_1})^{N-1}\max(\rho_2,\rho_1).$$

Now, we define the energy function

$$E(r) = u'(r)v'(r) + \lambda F(v(r)) + \mu G(u(r)).$$

Then

$$E'(r) = -\frac{2(N-1)}{r}u'(r)v'(r) \le 0,$$

and hence $E \ge 0$ on $[R,\widehat{R}]$, (because $u'(\widehat{R})v'(\widehat{R}) \ge 0$). However,

$$E(r_0) \le \widetilde{c}^2 + \lambda F(\rho_1) + \mu G(\rho_2), \tag{2.10}$$

and $F(\rho_1) < 0$ and $G(\rho_2) < 0$. Hence $E(r_0) < 0$ for $\lambda\mu$ large which is a contradiction. The proof is complete. □

Proof of Theorem 2.1. Assume $\lambda\mu$ is large enough so that both lemmas 2.2, 2.4 hold. We take the case when $u(R_2) \le \beta_2$. Then

$$\begin{gathered} -(r^{N-1}v')' = \mu r^{N-1}g(u) \le 0 \quad \text{on } J_3=(R_2,\widehat{R}) \\ v(R_2) \le C, \quad v(\widehat{R}) = 0, \end{gathered}$$

hence, by a comparison argument, $v(r) \le \widetilde{\omega}(r)$, where $\widetilde{\omega}$ is the solution of

$$\begin{gathered} -(r^{N-1}\widetilde{\omega}')' = 0 \quad \text{on } J_3 \\ \widetilde{\omega}(R_2) = C, \quad \widetilde{\omega}(\widehat{R}) = 0. \end{gathered}$$

However, $\widetilde{\omega}(r) = C\int_r^{\widehat{R}} s^{1-N}ds/\int_{R_2}^{\widehat{R}} s^{1-N}ds$ decreases from C to 0 on $[R_2,\widehat{R}]$, hence there exists $r_1 \in (R_2,\widehat{R})$ (independent of $\lambda\mu$) such that $\widetilde{\omega}(r_1) = \beta_1/2$.

Remark. Here, we assume that $\beta_1/2 < C$, unless we can choose N_0 such that $\beta_1/N_0 < C$.

Hence $v(r_1) \le \beta_1/2$, and

$$\begin{aligned} -(r^{N-1}(\beta_2-u)')' &= -\lambda r^{N-1}f(v) \\ &\ge -\lambda r^{N-1}f(\frac{\beta_1}{2}) \\ &\ge \lambda\big(-f(\frac{\beta_1}{2})\big)r^{N-1}\frac{\beta_2-u}{\beta_2} \quad \text{on } J_4=(r_1,\widehat{R}). \end{aligned}$$

Since $\beta_2 - u > 0$ on $\bar{J}_4$, we have

$$\frac{\lambda\widetilde{K}_1}{\beta_2} \le \lambda_1(J_4), \tag{2.11}$$

where $\widetilde{K}_1 = -f(\beta_1/2)$ and $\lambda_1(J_4)$ is the principal eigenvalue of (2.6) (with $(\alpha, \beta) = J_4$). Similarly, there exists $r_2 \in (r_1, \widehat{R})$ (independent of $\lambda\mu$) such that

$$v(r_2) < \frac{\beta_1}{2}.$$

Hence

$$\begin{gathered} -(r^{N-1}u')' = \mu r^{N-1} f(v) \leq 0 \quad \text{on } J_5 = (r_2, \widehat{R}) \\ u(r_2) \leq C, \quad u(\widehat{R}) = 0, \end{gathered}$$

then, by a comparison argument we obtain

$$u(r) \leq \omega_1(r) = \frac{C}{\int_{r_2}^{\widehat{R}} s^{1-N} ds} \int_r^{\widehat{R}} s^{1-N} ds;$$

thus

$$\begin{gathered} -(r^{N-1}\omega_1')' = 0, \quad \text{on } J_5, \\ \omega_1(r_2) = C, \quad \omega_1(\widehat{R}) = 0. \end{gathered}$$

Arguing as before there exists $r_3 \in (r_2, \widehat{R})$ (independent of $\lambda\mu$) such that

$$u(r_3) \leq \omega_1(r_3) \leq \frac{\beta_2}{2} < C.$$

Hence

$$\begin{aligned} -(r^{N-1}(\beta_1 - v)')' &= -\mu r^{N-1} g(v) \\ &\geq -\mu r^{N-1} g(\frac{\beta_2}{2}) \\ &\geq \mu\big(-g(\frac{\beta_2}{2})\big) r^{N-1} \frac{\beta_1 - v}{\beta_1} \quad \text{on } J_6 = (r_3, \widehat{R}). \end{aligned}$$

Since $\beta_1 - v > 0$ on $\bar{J}_6$, it follows that

$$\frac{\mu \widetilde{K}_2}{\beta_1} \leq \lambda_1(J_6), \tag{2.12}$$

where $\widetilde{K}_2 = -g(\beta_1/2)$ and $\lambda_1(J_6)$ is the principal eigenvalue of (2.6) (with $(\alpha, \beta) = J_6$). Combining (2.11) and (2.12), we obtain

$$\frac{\lambda\mu \widetilde{K}_1 \widetilde{K}_2}{\beta_1\beta_2} \leq \lambda_1(J_4)\lambda_1(J_6),$$

which is a contradiction to $\lambda\mu$ being large.

A similar contradiction can be reached for the case $v(R_2) \leq \beta_1$. Hence Theorem 2.1 is proven. □

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