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GROWTH OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS OF [P,Q]-ORDER IN THE UNIT DISC

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Abstract. In this article, we study the growth of solutions to complex higher-order linear differential equations in which the coefficients are analytic functions of $[p,q]$-order in the unit disc.

1. Introduction and statement of results

For $k \geq 2$ we consider the linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = F(z), \tag{1.2}$$

where $A_0(z), \dots, A_{k-1}(z)$, $F(z) \not\equiv 0$ are analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. It is well-known that all solutions of (1.1) and (1.2) are analytic functions in Δ and that there are exactly k linearly independent solutions of (1.1) (see [11]). Juneja, Kapoor and Bajpai [14, 15] have investigated properties of entire functions of $[p,q]$-order and obtained some results. Liu, Tu and Shi [20], by using the concept of $[p,q]$-order have considered equations (1.1), (1.2) with entire coefficients and obtained different results concerning the growth of its solutions. Recently, there has been an increasing interest in studying the growth of analytic solutions of linear differential equations in the unit disc by making use of Nevanlinna theory (see [2, 3, 5, 6, 7, 9, 11, 12, 19]). In this article, we continue to consider this subject and investigate the complex linear differential equations (1.1) and (1.2) when the coefficients $A_0, A_1, \dots, A_{k-1}$, F are analytic functions of $[p,q]$-order in Δ.

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna's theory in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ (see [10, 11, 18, 21]).

Before, we state our results we need to give some definitions and discussions. Firstly, let us give definition about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many definitions of small growth order of functions in Δ; see [7, 8].

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Definition 1.1. For a meromorphic function f in Δ let

$$D(f) := \limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. If $D(f)<\infty$, we say that f is of finite degree $D(f)$ (or is non-admissible); if $D(f)=\infty$, we say that f is of infinite degree (or is admissible). If f is an analytic function in Δ, and

$$D_M(f) := \limsup_{r\to 1^-} \frac{\log^+ M(r,f)}{\log \frac{1}{1-r}}$$

in which $M(r,f)=\max_{|z|=r}|f(z)|$ is the maximum modulus function, then we say that f is a function of finite degree $D_M(f)$ if $D_M(f)<\infty$; otherwise, f is of infinite degree.

Now, we give the definitions of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$; see [4, 16, 17]. Let us define inductively, for $r\in[0,1)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p\in\mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p\in\mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.2 ([5, 6, 18])**.** Let f be a meromorphic function in Δ. Then the iterated p-order of f is defined by

$$\rho_p(f) = \limsup_{r\to 1^-} \frac{\log_p^+ T(r,f)}{\log \frac{1}{1-r}} \quad (p \text{ is an integer, } p\ge 1),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. For $p=1$, this notation is called order and for $p=2$ hyper-order [11, 19]. If f is analytic in Δ, then the iterated p-order of f is defined by

$$\rho_{M,p}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log \frac{1}{1-r}} \quad (p \text{ is an integer, } p\ge 1).$$

Remark 1.3. It follows by Tsuji [21, p. 205] that if f is an analytic function in Δ, then we have the inequalities

$$\rho_1(f)\le \rho_{M,1}(f)\le \rho_1(f)+1$$

which are the best possible in the sense that there are analytic functions g and h such that $\rho_{M,1}(g)=\rho_1(g)$ and $\rho_{M,1}(h)=\rho_1(h)+1$, see [8]. However, it follows by [17, Proposition 2.2.2] that $\rho_{M,p}(f)=\rho_p(f)$ for $p\ge 2$.

Definition 1.4 ([5])**.** The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible},\\ \min\{j\in\mathbb{N} : \rho_j(f)<\infty\} & \text{if } f \text{ is admissible},\\ \infty, & \text{if } \rho_j(f)=\infty \text{ for all } j\in\mathbb{N}. \end{cases}$$

For an analytic function f in Δ, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible},\\ \min\{j\in\mathbb{N} : \rho_{M,j}(f)<\infty\} & \text{if } f \text{ is admissible},\\ \infty, & \text{if } \rho_{M,j}(f)=\infty \text{ for all } j\in\mathbb{N}. \end{cases}$$

Remark 1.5. If $\rho_p(f) < \infty$ or $i(f) \le p$, then we say that f is of finite iterated p-order; if $\rho_p(f) = \infty$ or $i(f) > p$, then we say that f is of infinite iterated p-order. In particular, we say that f is of finite order if $\rho_1(f) < \infty$ or $i(f) \le 1$; f is of infinite order if $\rho_1(f) = \infty$ or $i(f) > 1$.

Now, we introduce the concept of $[p, q]$-order for meromorphic and analytic functions in the unit disc.

Definition 1.6. Let $p \ge q \ge 1$ be integers. Let f be meromorphic function in Δ, the $[p, q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^+ T(r, f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function f in Δ, we also define

$$\rho_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^+ M(r, f)}{\log_q \frac{1}{1-r}}.$$

Remark 1.7. It is easy to see that $0 \le \rho_{[p,q]}(f) \le \infty$. If $f(z)$ is non-admissible, then $\rho_{[p,q]}(f) = 0$ for any $p \ge q \ge 1$. By Definition 1.6, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$, $\rho_{[2,1]}(f) = \rho_2(f)$ and $\rho_{[p+1,1]}(f) = \rho_{p+1}(f)$.

Proposition 1.8. *Let $p \ge q \ge 1$ be integers, and let f be analytic function in Δ of $[p, q]$-order. The following two statements hold:*

(i) *If $p = q$, then*

$$\rho_{[p,q]}(f) \le \rho_{M,[p,q]}(f) \le \rho_{[p,q]}(f) + 1.$$

(ii) *If $p > q$, then $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f)$.*

Proof. By the standard inequalities [17, p. 26]

$$T(r, f) \le \log^+ M(r, f) \le \frac{1 + 3r}{1 - r} T(\frac{1 + r}{2}, f),$$

we easily deduce that (i) and (ii) hold. □

The present article may be understood as an extension and improvement of the recent article of the author [3]. We obtain the following results.

Theorem 1.9. *Let $p \ge q \ge 1$ be integers, and let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in the unit disc Δ. Suppose that there exists a sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ with $|z_n| = r_n \to 1^-$, $n \to \infty$ such that for real constants α, β where $0 \le \beta < \alpha$, we have*

$$T(r_n, A_0) \ge \exp_p\{\alpha \log_q(\frac{1}{1 - r_n})\} \tag{1.3}$$

as $n \to \infty$, and

$$T(r, A_j) \le \exp_p\{\beta \log_q(\frac{1}{1 - r})\} \quad (j = 1, \dots, k - 1) \tag{1.4}$$

holds for all $r \in [0, 1)$. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \ge \alpha$.

Theorem 1.10. *Let $p \geq q \geq 1$ be integers, and let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in the unit disc Δ satisfying $\max\{\rho_{[p,q]}(A_j) : j = 1, \dots, k-1\} \leq \rho_{[p,q]}(A_0) = \rho$. Suppose that there exist a sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ with $|z_n| = r_n \to 1^-$, $n \to \infty$ and a real number μ satisfying $0 \leq \mu < \rho$ such that for any given ε $(0 < \varepsilon < \rho - \mu)$ sufficiently small, we have*

$$T(r_n, A_0) \geq \exp_p\{(\rho - \varepsilon)\log_q(\frac{1}{1-r_n})\} \tag{1.5}$$

as $n \to \infty$, and

$$T(r, A_j) \leq \exp_p\{\mu \log_q(\frac{1}{1-r})\} \quad (j = 1, \dots, k-1) \tag{1.6}$$

holds for all $r \in [0, 1)$. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and

$$\rho_{[p,q]}(A_0) \leq \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1\}.$$

Furthermore, if $p > q$, then

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

Theorem 1.11. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), \dots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ be analytic functions in the unit disc Δ such that for some integer s, $1 \leq s \leq k-1$ satisfying $\max\{\rho_{[p,q]}(A_j)\ (j \neq s),\ \rho_{[p,q]}(F)\} < \rho_{[p,q]}(A_s)$. Then every admissible solution f of (1.2) with $\rho_{[p,q]}(f) < \infty$ satisfies $\rho_{[p,q]}(f) \geq \rho_{[p,q]}(A_s)$.*

Theorem 1.12. *Let $p \geq q \geq 1$ be integers. Let $A_0(z), \dots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ be analytic functions in the unit disc Δ such that for some integer s, $0 \leq s \leq k-1$, we have $\rho_{[p,q]}(A_s) = \infty$ and $\max\{\rho_{[p,q]}(A_j)\ (j \neq s), \rho_{[p,q]}(F)\} < \infty$. Then every solution f of (1.2) satisfies $\rho_{[p,q]}(f) = \infty$.*

2. Preliminaries

In this section we give some lemmas which are used in the proofs of our theorems.

Lemma 2.1 ([11]). *Let f be a meromorphic function in the unit disc Δ, and let $k \geq 1$ be an integer. Then*

$$m\big(r, \frac{f^{(k)}}{f}\big) = S(r, f), \tag{2.1}$$

where $S(r, f) = O\big(\log^+ T(r, f) + \log(\frac{1}{1-r})\big)$, possibly outside a set $E_1 \subset [0, 1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$.

Next we give the generalized logarithmic derivative lemma.

Lemma 2.2. *Let $p \geq q \geq 1$ be integers. Let f be a meromorphic function in the unit disc Δ such that $\rho_{[p,q]}(f) = \rho < \infty$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m\big(r, \frac{f^{(k)}}{f}\big) = O\big(\exp_{p-1}\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}\big) \tag{2.2}$$

holds for all r outside a set $E_2 \subset [0, 1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$.

Proof. First for $k = 1$. Since $\rho_{[p,q]}(f) = \rho < \infty$, for all $r \to 1^-$ we have

$$T(r, f) \le \exp_p\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}. \tag{2.3}$$

By Lemma 2.1, we have

$$m\big(r, \frac{f'}{f}\big) = O\big(\ln^+ T(r, f) + \ln(\frac{1}{1-r})\big) \tag{2.4}$$

holds for all r outside a set $E_2 \subset [0, 1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. Hence, we obtain

$$m(r, \frac{f'}{f}) = O(\exp_{p-1}\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}), r \notin E_2. \tag{2.5}$$

Next, we assume that we have

$$m\big(r, \frac{f^{(k)}}{f}\big) = O\big(\exp_{p-1}\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}\big), \quad r \notin E_2 \tag{2.6}$$

for some an integer $k \ge 1$. Since $N(r, f^{(k)}) \le (k + 1)N(r, f)$, it holds that

$$\begin{aligned} T(r, f^{(k)}) &= m(r, f^{(k)}) + N(r, f^{(k)}) \\ &\le m\big(r, \frac{f^{(k)}}{f}\big) + m(r, f) + (k + 1)N(r, f) \\ &\le m\big(r, \frac{f^{(k)}}{f}\big) + (k + 1)T(r, f) = O\big(\exp_{p-1}\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}\big) \\ &\quad + (k + 1)T(r, f) = O\big(\exp_p\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}\big). \end{aligned} \tag{2.7}$$

By (2.4) and (2.7), we obtain

$$m\big(r, \frac{f^{(k+1)}}{f^{(k)}}\big) = O\big(\exp_{p-1}\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}\big), \quad r \notin E_2 \tag{2.8}$$

and hence,

$$\begin{aligned} m\big(r, \frac{f^{(k+1)}}{f}\big) &\le m\big(r, \frac{f^{(k+1)}}{f^{(k)}}\big) + m\big(r, \frac{f^{(k)}}{f}\big) \\ &= O\big(\exp_{p-1}\{(\rho + \varepsilon)\log_q(\frac{1}{1-r})\}\big), \quad r \notin E_2. \end{aligned} \tag{2.9}$$

□

Lemma 2.3 ([1]). *Let $g : (0, 1) \to \mathbb{R}$ and $h : (0, 1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \le h(r)$ holds outside of an exceptional set $E_3 \subset [0, 1)$ for which $\int_{E_3} \frac{dr}{1-r} < \infty$. Then there exists a constant $d \in (0, 1)$ such that if $s(r) = 1 - d(1 - r)$, then $g(r) \le h(s(r))$ for all $r \in [0, 1)$.*

Lemma 2.4 ([13]). *Let f be a solution of equation (1.1), where the coefficients $A_j(z)$ $(j = 0, \dots, k-1)$ are analytic functions in the disc $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$, $0 < R \le \infty$. Let $n_c \in \{1, \dots, k\}$ be the number of nonzero coefficients $A_j(z)$ $(j = 0, \dots, k - 1)$, and let $\theta \in [0, 2\pi]$ and $\varepsilon > 0$. If $z_\theta = \nu e^{i\theta} \in \Delta_R$ is such that $A_j(z_\theta) \ne 0$ for some $j = 0, \dots, k - 1$, then for all $\nu < r < R$,*

$$|f(re^{i\theta})| \le C \exp\Big(n_c \int_\nu^r \max_{j=0,\dots,k-1} |A_j(te^{i\theta})|^{1/(k-j)} dt\Big), \tag{2.10}$$

where $C > 0$ is a constant satisfying

$$C \leq (1+\varepsilon) \max_{j=0,\dots,k-1} \Big(\frac{|f^{(j)}(z_\theta)|}{(n_c)^j \max\limits_{n=0,\dots,k-1} |A_n(z_\theta)|^{j/(k-n)}} \Big). \tag{2.11}$$

Lemma 2.5. *Let $p \geq q \geq 1$ be integers. If $A_0(z), \dots, A_{k-1}(z)$ are analytic functions of $[p,q]$-order in the unit disc Δ, then every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1\}. \tag{2.12}$$

Proof. Set $\sigma = \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1\}$. Let $f \not\equiv 0$ be a solution of (1.1). Let $\theta_0 \in [0, 2\pi)$ be such that $|f(re^{i\theta_0})| = M(r,f)$. By Lemma 2.4, we have

$$\begin{aligned} M(r,f) &\leq C \exp\Big(n_c \int_\nu^r \max_{j=0,\dots,k-1} |A_j(te^{i\theta})|^{1/(k-j)} dt\Big) \\ &\leq C \exp\big(n_c \int_\nu^r \max_{j=0,\dots,k-1} (M(r,A_j))^{1/(k-j)} dt\big) \\ &\leq C \exp(n_c(r-\nu) \max_{j=0,\dots,k-1} \{M(r,A_j)\}). \end{aligned} \tag{2.13}$$

By Definition 1.6,

$$M(r,A_j) \leq \exp_{p+1}\{(\sigma+\varepsilon)\log_q(\frac{1}{1-r})\} \quad (j = 0, \dots, k-1) \tag{2.14}$$

holds for any $\varepsilon > 0$. Hence from (2.13) and (2.14) we obtain

$$\rho_{M,[p+1,q]}(f) \leq \sigma + \varepsilon. \tag{2.15}$$

Since $\varepsilon > 0$ is arbitrary, we have by Proposition 1.8 (ii)

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \sigma = \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1\}.$$

□

3. Proof of Theorem 1.9

Suppose that $f \not\equiv 0$ is a solution of (1.1). By (1.1), we can write

$$A_0(z) = -\Big(\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \dots + A_1(z)\frac{f'}{f}\Big). \tag{3.1}$$

From the condition (1.4), by using (3.1) and Lemma 2.1 we obtain

$$\begin{aligned} m(r,A_0) &\leq \sum_{j=1}^{k-1} m(r,A_j) + \sum_{j=1}^{k} m\big(r, \frac{f^{(j)}}{f}\big) + O(1) \\ &\leq (k-1)\exp_p\{\beta \log_q(\frac{1}{1-r})\} + S(r,f) \end{aligned} \tag{3.2}$$

holds for all r outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$. By Lemma 2.3 and (3.2), we have

$$\begin{aligned} m(r,A_0) &\leq (k-1)\exp_p\{\beta \log_q(\frac{1}{1-s(r)})\} \\ &\quad + O\Big(\log^+ T(s(r),f) + \log(\frac{1}{1-s(r)})\Big) \end{aligned} \tag{3.3}$$

holds for all $r \in [0,1)$. The assumption (1.3) gives us

$$
\begin{aligned}
m(r_n, A_0) = T(r_n, A_0) &\geq \exp_p\{\alpha \log_q \big(\frac{d}{1-s(r_n)}\big)\} \\
&\geq \exp_p\{\gamma \log_q \big(\frac{1}{1-s(r_n)}\big)\},
\end{aligned} \tag{3.4}
$$

where γ is an arbitrary number satisfying $\beta < \gamma < \alpha$ and n is sufficiently large. By combining (3.3) and (3.4) for $r = r_n$, for n sufficiently large we obtain

$$
\begin{aligned}
\exp_p\{\gamma \log_q \big(\frac{1}{1-s(r_n)}\big)\} &\leq (k-1)\exp_p\{\beta \log_q \big(\frac{1}{1-s(r_n)}\big)\} \\
&\quad + O(\log^+ T(s(r_n), f) + \log(\frac{1}{1-s(r_n)})).
\end{aligned} \tag{3.5}
$$

Noting that $\gamma > \beta \geq 0$, it follows from (3.5) that

$$
(1-o(1))\exp_p\{\gamma \log_q \big(\frac{1}{1-s(r_n)}\big)\} \leq O\Big(\log^+ T(s(r_n), f) + \log(\frac{1}{1-s(r_n)})\Big) \tag{3.6}
$$

holds as $r_n \to 1^-$. Hence, by (3.6) we obtain $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and

$$
\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \limsup_{s(r_n)\to 1^-} \frac{\log_{p+1}^+ T(s(r_n), f)}{\log_q \frac{1}{1-s(r_n)}} \geq \gamma.
$$

Since γ is an arbitrary number less than α, we obtain $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \geq \alpha$.

4. Proof of Theorem 1.10

Suppose that $f \not\equiv 0$ is a solution of (1.1). Then for any given $\varepsilon > 0$, by the results of Theorem 1.9, we have $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and

$$
\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \geq \rho - \varepsilon. \tag{4.1}
$$

Since $\varepsilon > 0$ is arbitrary, from (4.1) we obtain $\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \geq \rho = \rho_{[p,q]}(A_0)$. On the other hand, by Lemma 2.5, we have

$$
\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1\}. \tag{4.2}
$$

It yields

$$
\rho_{[p,q]}(A_0) \leq \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1\}.
$$

If $p > q$, then

$$
\max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1\} = \rho_{[p,q]}(A_0).
$$

Therefore,

$$
\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A_0).
$$

5. Proof of Theorem 1.11

Set $\max\{\rho_{[p,q]}(A_j)(j \neq s), \rho_{[p,q]}(F)\} = \beta < \rho_{[p,q]}(A_s) = \alpha$. Suppose that f is an admissible solution of (1.2) with $\rho = \rho_{[p,q]}(f) < \infty$. It follows from (1.2) that

$$\begin{aligned} A_s(z) = &\frac{F(z)}{f^{(s)}} - \frac{f^{(k)}}{f^{(s)}} - A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} - \dots - A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}} \\ &- A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} - \dots - A_1(z)\frac{f'}{f^{(s)}} - A_0(z)\frac{f}{f^{(s)}}. \end{aligned} \tag{5.1}$$

Applying Lemma 2.2, we have

$$m\big(r, \frac{f^{(j+1)}}{f}\big) = O\Big(\exp_{p-1}\{(\rho+\varepsilon)\log_q(\frac{1}{1-r})\}\Big) \quad (j = 0, \dots, k-1) \tag{5.2}$$

holds for all r outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. Since $N(r, f^{(j+1)}) = 0$, it holds for $j = 0, \dots, k-1$ that

$$\begin{aligned} T(r, f^{(j+1)}) = m(r, f^{(j+1)}) &\le m\big(r, \frac{f^{(j+1)}}{f}\big) + m(r, f) \\ &\le T(r,f) + m\big(r, \frac{f^{(j+1)}}{f}\big). \end{aligned} \tag{5.3}$$

By (5.3), from (5.1) and (5.2) we obtain

$$\begin{aligned} T(r, A_s) \le T(r,F) + cT(r,f) + \sum_{j \neq s} T(r, A_j) \\ + O\Big(\exp_{p-1}\{(\rho+\varepsilon)\log_q(\frac{1}{1-r})\}\Big) \quad (r \notin E_2), \end{aligned} \tag{5.4}$$

where $c > 0$ is a constant. Since $\rho_{[p,q]}(A_s) = \alpha$, there exists a sequence $\{r_n'\}$ ($r_n' \to 1^-$) such that

$$\lim_{r_n' \mapsto 1^-} \frac{\log_p^+ T(r_n', A_s)}{\log_q \frac{1}{1-r_n'}} = \alpha. \tag{5.5}$$

Set $\int_{E_2} \frac{dr}{1-r} := \log \gamma < \infty$. Since $\int_{r_n'}^{1-\frac{1-r_n'}{\gamma+1}} \frac{dr}{1-r} = \log(\gamma+1)$, there exists a point $r_n \in [r_n', 1 - \frac{1-r_n'}{\gamma+1}] - E_2 \subset [0,1)$. From

$$\frac{\log_p^+ T(r_n, A_s)}{\log_q \frac{1}{1-r_n}} \ge \frac{\log_p^+ T(r_n', A_s)}{\log_q(\frac{\gamma+1}{1-r_n'})} = \frac{\log_p^+ T(r_n', A_s)}{\log_q \frac{1}{1-r_n'} + \log\big(\frac{\log_{q-1}(\frac{\gamma+1}{1-r_n'})}{\log_{q-1}\frac{1}{1-r_n'}}\big)}, \tag{5.6}$$

it follows that

$$\lim_{r_n \to 1^-} \frac{\log_p^+ T(r_n, A_s)}{\log_q \frac{1}{1-r_n}} = \alpha. \tag{5.7}$$

So, for any given ε $(0 < 2\varepsilon < \alpha - \beta)$, we have

$$T(r_n, A_s) > \exp_p\{(\alpha-\varepsilon)\log_q(\frac{1}{1-r_n})\} \tag{5.8}$$

and for $j \neq s$,

$$T(r_n, A_j) \le \exp_p\{(\beta+\varepsilon)\log_q(\frac{1}{1-r_n})\}, \tag{5.9}$$

$$T(r_n, F) \le \exp_p\{(\beta+\varepsilon)\log_q(\frac{1}{1-r_n})\} \tag{5.10}$$

hold as $r_n \to 1^-$. By (5.4), (5.8), (5.9) and (5.10), for $r_n \to 1^-$ we can obtain

$$\begin{aligned} \exp_p\{(\alpha-\varepsilon)\log_q(\frac{1}{1-r_n})\} &\le k\exp_p\{(\beta+\varepsilon)\log_q(\frac{1}{1-r_n})\} + cT(r_n, f) \\ &\quad + O\Big(\exp_{p-1}\{(\rho+\varepsilon)\log_q(\frac{1}{1-r_n})\}\Big)). \end{aligned} \tag{5.11}$$

Noting that $\alpha-\varepsilon > \beta+\varepsilon$, it follows from (5.11) that for $r_n \to 1^-$,

$$\begin{aligned} &(1-o(1))\exp_p\{(\alpha-\varepsilon)\log_q(\frac{1}{1-r_n})\} \\ &\le cT(r_n, f) + O\Big(\exp_{p-1}\{(\rho+\varepsilon)\log_q(\frac{1}{1-r_n})\}\Big). \end{aligned} \tag{5.12}$$

Therefore, by (5.12) we obtain

$$\limsup_{r_n\mapsto 1^-} \frac{\log_p^+ T(r_n, f)}{\log_q \frac{1}{1-r_n}} \ge \alpha - \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, we obtain $\rho_{[p,q]}(f) \ge \rho_{[p,q]}(A_s) = \alpha$. This completes the proof.

6. Proof of Theorem 1.12

Setting $\max\{\rho_{[p,q]}(A_j)(j \ne s), \rho_{[p,q]}(F)\} = \beta$, for a given $\varepsilon > 0$, we have

$$T(r, A_j) \le \exp_p\{(\beta+\varepsilon)\log_q(\frac{1}{1-r})\}\ (j \ne s), \tag{6.1}$$

$$T(r, F) \le \exp_p\{(\beta+\varepsilon)\log_q(\frac{1}{1-r})\} \tag{6.2}$$

as $r \to 1^-$. Now from (1.2) we can write

$$\begin{aligned} A_s(z) = \frac{F(z)}{f^{(s)}} - \frac{f^{(k)}}{f^{(s)}} - A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} - \dots - A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}} \\ - A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} - \dots - A_1(z)\frac{f'}{f^{(s)}} - A_0(z)\frac{f}{f^{(s)}}. \end{aligned} \tag{6.3}$$

Hence by (5.3) and (6.3) we obtain

$$T(r, A_s) \le T(r, F) + cT(r, f) + \sum_{j=0}^{k-1} m\big(r, \frac{f^{(j+1)}}{f}\big) + \sum_{j\ne s} T(r, A_j), \tag{6.4}$$

where $c > 0$ is a constant. If $\rho = \rho_{[p,q]}(f) < \infty$, then by Lemma 2.2

$$m\big(r, \frac{f^{(j+1)}}{f}\big) = O\Big(\exp_{p-1}\{(\rho+\varepsilon)\log_q(\frac{1}{1-r})\}\Big) \quad (j = 0, \dots, k-1) \tag{6.5}$$

holds for all r outside a set $E_2 \subset [0, 1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. For $r \to 1^-$, we have

$$T(r, f) \le \exp_p\{(\rho+\varepsilon)\log_q(\frac{1}{1-r})\}. \tag{6.6}$$

Thus, by (6.1), (6.2), (6.4), (6.5) and (6.6), we obtain

$$\begin{aligned} T(r,A_s) \le k\exp_p\{(\beta+\varepsilon)\log_q(\frac{1}{1-r})\} + c\exp_p\{(\rho+\varepsilon)\log_q(\frac{1}{1-r})\} \\ + O(\exp_{p-1}\{(\rho+\varepsilon)\log_q(\frac{1}{1-r})\}) \end{aligned} \tag{6.7}$$

for $r \notin E_2$ and $r \to 1^-$. By Lemma 2.3, for any $d \in [0,1)$, we have

$$\begin{aligned} T(r,A_s) &\le k\exp_p\{(\beta+\varepsilon)\log_q(\frac{1}{d(1-r)})\} \\ &\quad + c\exp_p\{(\rho+\varepsilon)\log_q\big(\frac{1}{d(1-r)}\big)\} \\ &\quad + O\Big(\exp_{p-1}\{(\rho+\varepsilon)\log_q(\frac{1}{d(1-r)})\}\Big) \end{aligned} \tag{6.8}$$

as $r \to 1^-$. Therefore,

$$\rho_{[p,q]}(A_s) \le \max\{\beta+\varepsilon, \rho+\varepsilon\} < \infty.$$

This contradicts that $\rho_{[p,q]}(A_s) = \infty$. This completes the proof

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