Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 157, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS HAVING THE SAME ORDER AND TYPE

NACERA BERRIGHI, SAADA HAMOUDA

Abstract. In this article, we study the growth of solutions to the differential equation

$$\begin{aligned} &f^k + (A_{k-1}(z)e^{P_{k-1}(z)}e^{\lambda z^m} + B_{k-1}(z))f^{k-1} + \dots \\ &+ (A_0(z)e^{P_0(z)}e^{\lambda z^m} + B_0(z))f = 0, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $m \geq 2$ is an integer and $\max_{j=0,\dots,k-1}\{\deg P_j(z)\} < m$, A_j, B_j $(j = 0, \dots, k-1)$ are entire functions of orders less than m.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory (see [8]). In addition, we use the notation $\sigma_2(f)$ to denote the hyper-order of nonconstant entire function f; that is,

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic of f and $M(r,f) = \max_{|z|=r}|f(z)|$ (see [11]).

We define the linear measure of a set $E \subset [0, 2\pi)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $F \subset [1, +\infty)$ by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where $\chi_H(t)$ is the characteristic function of a set H.

Several authors have studied the particular differential equations

$$f'' + e^{-z}f' + Q(z)f = 0, \tag{1.1}$$

(see [1, 4, 6, 9]). Gundersen [6] proved that if $\deg Q(z) \neq 1$, then every nonconstant solution of (1.1) is of infinite order. Chen considered the case $Q(z) = h(z)e^{bz}$, where $h(z)$ is nonzero polynomial and $b \neq -1$, see [2]; more precisely, he proved that every nontrivial solution f of (1.1) satisfies $\sigma_2(f) = 1$. The same paper contains a discussion about more general equations of the type

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0, \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34M10, 30D35.
Key words and phrases. Linear differential equations; growth of solutions; hyper-order.

Submitted October 20, 2011. Published November 21, 2011.

where $A_1 \not\equiv 0, A_0 \not\equiv 0$, are entire functions of order less than 1, and a, b are complex constants. He proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$, then every solution $f(z) \not\equiv 0$ of (1.2) is of infinite order. He also proved the following result.

Theorem 1.1 ([2]). *Let $A_j(z)$ $(\not\equiv 0)$, $D_j(z)$ $(j = 0, 1)$ be entire functions with $\sigma(A_j) < 1$, $\sigma(B_j) < 1$ $(j = 0, 1)$, a, b be complex constants such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every solution $f(z) \not\equiv 0$ of*

$$f'' + (A_1(z)e^{az} + D_1)f' + (A_0(z)e^{bz} + D_0)f = 0,$$

is of infinite order.

In another paper, Chen and Shon [3] Proved the following result.

Theorem 1.2. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, \dots, n)$ are complex numbers, $a_n \neq 0$, $b_n \neq 0$. Let $A_1(z) \not\equiv 0$ and $A_0(z) \not\equiv 0$ be entire functions. Suppose that either (i) or (ii) below, holds:*

(i) $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$ $\sigma(A_j) < n$ $(j = 0, 1)$
(ii) $a_n = cb_n$ $(c > 1)$ *and* $\deg(P - cQ) = m \geq 1$, $\sigma(A_j) < m$ $(j = 0, 1)$.

Then every solution $f(z) \not\equiv 0$ of the differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = 0,$$

is of infinite order with $\sigma_2(f) = n$.

Hamouda and Belaidi [7] investigated the linear differential equation

$$w^{(n)} + e^{az^m}w' + Q(z)w = 0$$

and some related extensions.

In this paper, we investigate the differential equation

$$\begin{aligned} &f^k + (A_{k-1}(z)e^{P_{k-1}(z)}e^{\lambda z^m} + B_{k-1}(z))f^{k-1} + \dots \\ &+ (A_0(z)e^{P_0(z)}e^{\lambda z^m} + B_0(z))f = 0, \end{aligned} \tag{1.3}$$

where $\lambda \in \mathbb{C}^*$, $m \geq 2$ is an integer and $\max_{j=0,\dots,k-1}\{\deg P_j(z)\} < m$. We obtain the following results.

Theorem 1.3. *Let $\lambda \in \mathbb{C}^*$, $m \geq 2$ is an integer and $P_0(z), \dots, P_{k-1}(z)$ be non constant polynomials such that $\max_{j=0,\dots,k-1}\{\deg P_j(z)\} < m$; $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(j = 0, \dots, k-1)$ be entire functions such that $\sigma(A_j) < \deg P_j(z)$, $\sigma(B_j) < m$ $(j = 0, \dots, k-1)$. Suppose that there exist $\theta_1 < \theta_2$ such that $\delta(\lambda z^m, \theta) > 0$, $\delta(P_0, \theta) > 0$ and $\delta(P_j, \theta) < 0$ $(j = 1, \dots, k-1)$ for all $\theta \in (\theta_1, \theta_2)$. Then every non trivial solution f of (1.3) is of infinite order with $n \leq \sigma_2(f) \leq m$, where $n = \deg P_0$.*

Corollary 1.4. *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \dots, k-1)$ be non constant polynomials where $a_{i,j}$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, \dots, k-1)$, $\arg a_{n,j} = \arg a_{n,1}$ $(j = 2, \dots, k-1)$ and $\arg a_{n,1} \neq \arg a_{n,0}$; $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$, $(j = 0, \dots, k-1)$ be entire functions such that $\sigma(A_j) < n$, $\sigma(B_j) < m$ $(j = 0, \dots, k-1)$. Then every non trivial solution f of (1.3) is of infinite order with $n \leq \sigma_2(f) \leq m$.*

Now we give examples for Theorem 1.3 for cases other than Corollary 1.4.

Example 1.5. From Theorem 1.3, every non trivial solution f of the differential equation

$$\begin{aligned} &f''' + (A_2(z)e^{z^3}e^{z^4} + B_2(z))f'' + (A_1(z)e^{z^2}e^{z^4} + B_1(z))f' \\ &+ (A_0(z)e^{z}e^{z^4} + B_0(z))f = 0, \end{aligned}$$

is of infinite order with $1 \le \sigma_2(f) \le 4$. We cane take $(\theta_1, \theta_2) \subset (\pi/3, \pi/2) \cup (3\pi/2, 5\pi/3)$

Example 1.6. Every non trivial solution f of the differential equation

$$\begin{aligned} &f''' + (A_2(z)e^{z}e^{z^3} + B_2(z))f'' + (A_1(z)e^{(i+1)z^2}e^{z^3} + B_1(z))f' \\ &+ (A_0(z)e^{z^2}e^{z^3} + B_0(z))f = 0, \end{aligned}$$

is of infinite order with $2 \le \sigma_2(f) \le 3$. Here we can take $(\theta_1, \theta_2) \subset (3\pi/4, 5\pi/6)$.

Theorem 1.7. *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j}z^i$ $(j = 0, \dots, k-1)$ be non constant polynomials where $a_{i,j}$ are complex numbers such that $a_{n,0} \neq 0$ and $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, \dots, k-1)$; $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$, $(j = 0, \dots, k-1)$ be entire functions such that $\sigma(A_j) < n$, $\sigma(B_j) < m$ $(j = 0, \dots, k-1)$. Then every non trivial solution f of (1.3) is of infinite order with $n \le \sigma_2(f) \le m$.*

By combining Corollary 1.4 and Theorem 1.7, we can obtain the following corollary.

Corollary 1.8. *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j}z^i$ $(j = 0, \dots, k-1)$ be non constant polynomials where $a_{i,j}$ are complex numbers such that $a_{n,0} \neq 0$. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $\arg a_{n,s} \neq \arg a_{n,0}$ and for all $j \neq 0, s, a_{n,j}$ satisfies either $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ or $\arg a_{n,j} = \arg a_{n,s}$. Then every non trivial solution f of (1.3) is of infinite order with $n \le \sigma_2(f) \le m$.*

Now we investigate cases when $a_{n,j}$ have distinct arguments or $a_{n,j} = c_j a_{n,0}$ $(c_j > 1)$ and obtain the following results.

Theorem 1.9. *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j}z^i$ $(j = 0, \dots, k-1)$ be non constant polynomials where $a_{i,j}$ are complex numbers such that $a_{n,0} \neq 0$. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that*

$$\arg(a_{n,j} - a_{n,s}) = \varphi \neq \arg(a_{n,0} - a_{n,s}) \quad \textit{for all } j \neq 0, s. \tag{1.4}$$

Then every non trivial solution f of (1.3) is of infinite order with $n \le \sigma_2(f) \le m$.

Theorem 1.10. *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j}z^i$ $(j = 0, \dots, k-1)$ be non constant polynomials where $a_{i,j}$ are complex numbers such that $a_{n,0} \neq 0$. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that*

$$a_{n,j} - a_{n,s} = c_j(a_{n,0} - a_{n,s}) \ (0 < c_j < 1) \tag{1.5}$$

for all $j \neq 0, s$. Then every non trivial solution f of (1.3) is of infinite order with $n \le \sigma_2(f) \le m$.

By combining Theorem 1.9 and Theorem 1.10, we obtain the following corollary.

Corollary 1.11. *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j}z^i$ $(j = 0, \dots, k-1)$ be non constant polynomials where $a_{i,j}$ are complex numbers such that $a_{n,0} \neq 0$. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $\arg(a_{n,j} - a_{n,s}) \neq \arg(a_{n,0} - a_{n,s})$ and for $j \neq 0, s$ we have either (1.4) or (1.5). Then every non trivial solution f of (1.3) is of infinite order with $n \le \sigma_2(f) \le m$.*

Theorem 1.12. *If $P_j(z)$ and $A_j(z)$ $(j=0,\dots,k-1)$ satisfy the conditions of one of our previous theorems or corollaries, then every non trivial solution f of*

$$f^k + A_{n-1}(z)e^{P_{k-1}(z)}e^{\lambda z^m}f^{k-1} + \dots + A_0(z)e^{P_0(z)}e^{\lambda z^m}f = 0, \tag{1.6}$$

is of infinite order with $\sigma_2(f)=n$ or $\sigma_2(f)=m$.

2. Preliminary lemmas

We need the following lemmas for our proofs.

Lemma 2.1 ([5]). *Let $f(z)$ be a transcendental meromorphic function and $\alpha>1$. There exist a set $E\subset[0,2\pi)$ that has linear measure zero and a constant $M>0$ that depends only on α such that for any $\theta\in[0,2\pi)\backslash E$ there exists a constant $R_0=R_0(\theta)>1$ such that for all z satisfying $\arg z=\theta$ and $|z|=r>R_0$, we have*

$$|\frac{f^{(k)}(z)}{f(z)}| \le M\big[T(\alpha r,f)\frac{(\log^\alpha r)}{r}\log T(\alpha r,f)\big]^k, \quad k\in\mathbb{N}.$$

Lemma 2.2 ([5]). *Let $f(z)$ be a transcendental meromorphic function of finite order σ, and let $\varepsilon>0$ be a given constant. Then there exists a set $E\subset[0,2\pi)$ of linear measure zero such that for all $z=re^{i\theta}$ with $|z|$ sufficiently large and $\theta\in[0,2\pi)\backslash E$, and for all k,j, $0\le j\le k$, we have*

$$|\frac{f^{(k)}(z)}{f^{(j)}(z)}| \le |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Using the Wiman-Valiron theory, we can easily prove the following lemma (see [2]).

Lemma 2.3. *Let A,B be entire functions of finite order. If f is a solution of the differential equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1f' + A_0f = 0,$$

then $\sigma_2(f)\le \max_{j=0,\dots,k-1}\{\sigma(Aj)\}$.

Lemma 2.4 ([2]). *Let $P(z)=a_nz^n+\dots$, $(a_n=\alpha+i\beta\ne0)$ be a polynomial with degree $n\ge1$ and $A(z)$ $(\not\equiv0)$ be entire function with $\sigma(A)<n$. Set $f(z)=A(z)e^{P(z)}$, $z=re^{i\theta}$, $\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$. Then for any given $\varepsilon>0$, there exists a set $H\subset[0,2\pi)$ that has linear measure zero, such that for any $\theta\in[0,2\pi)\backslash H$, where $H=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ is a finite set, there is $R>0$ such that for $|z|=r>R$, we have*

(i) *if $\delta(P,\theta)>0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} \le |f(z)| \le \exp\{(1+\varepsilon)\delta(P,\theta)r^n\},$$

(ii) *if $\delta(P,\theta)<0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} \le |f(z)| \le \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}.$$

Lemma 2.5 ([3]). *Let $f(z)$ be a entire function with $\sigma(f)=+\infty$ and $\sigma_2(f)=\alpha<+\infty$, let a set $E_2\subset[1,+\infty)$ has finite logarithmic measure. Then there exists a sequence $\{z_p=r_pe^{i\theta_p}\}$ such that $f(z_p)=M(r_p,f)$, $\theta_p\in[0,2\pi)$, $\lim_{p\to\infty}\theta_p=\theta_0\in[0,2\pi)$, $r_p\notin E_2$, and for any given $\epsilon>0$, as $r_p\to\infty$, we have*

$$\exp\{r_p^{\alpha-\epsilon}\} \le \nu(r_p) \le \exp\{r_p^{\alpha+\epsilon}\},$$

where $\nu(r)$ is the central index of f.

3. Proofs of theorems

Proof of theorem 1.3. From (1.3), we obtain

$$
\begin{aligned}
&|A_0(z)e^{P_0(z)}+B_0(z)e^{-\lambda z^m}| \\
&\le |e^{-\lambda z^m}||\frac{f^{(k)}}{f}|+\sum_{j=1}^{k-1}|A_j(z)e^{P_j(z)}+B_j(z)e^{-\lambda z^m}||\frac{f^{(j)}}{f}|.
\end{aligned}
\tag{3.1}
$$

Since $\delta(\lambda z^m,\ \theta)>0$, $\delta(P_0,\theta)>0$ and $\delta(P_j,\theta)<0$ ($j=1,\dots,k-1$) for all $\theta\in(\theta_1,\theta_2)$, by Lemma 2.4, for any $\theta\in(\theta_1,\theta_2)$ there is $R_0(\theta)>0$ such that for $|z|=r>R_0$, we have

$$\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\}\le|A_0(z)e^{P_0(z)}+B_0(z)e^{-\lambda z^m}|, \tag{3.2}$$

$$|A_j(z)e^{P_j(z)}+B_j(z)e^{-\lambda z^m}|\le C_1 \quad (j=1,\dots,k-1). \tag{3.3}$$

From Lemma 2.1, there exist a set $E\subset[0,2\pi)$ that has linear measure zero and a constant $M>0$ such that for any $\theta\in[0,2\pi)\backslash E$ there exists a constant $R_1=R_1(\theta)>1$ such that for all z satisfying $\arg z=\theta$ and $|z|=r>R_1$, we have

$$|\frac{f^{(j)}}{f}|\le C_2[T(2r,f)]^{2k} \quad (j=1,\dots,k). \tag{3.4}$$

By using (3.2)-(3.4) in (3.1), for $r>\max\{R_0,R_1\}$, we obtain

$$\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\}\le C_3[T(2r,f)]^{2k},$$

which implies that $\sigma_2(f)\ge n$. By Lemma 2.3, we obtain $n\le\sigma_2(f)\le m$. □

Proof of Corollary 1.4. In these conditions, there exist θ_1,θ_2 such that $\theta_1<\theta_2$, $\delta(\lambda z^m,\theta)>0$, $\delta(P_0,\theta)>0$ and $\delta(P_j,\theta)<0$ ($j=1,\dots,k-1$) for all $\theta\in(\theta_1,\theta_2)$. □

Proof of Theorem 1.7. Since $m>n$ and $a_{n,j}=c_ja_{n,0}$ ($0<c_j<1$) ($j=1,\dots,k-1$), there exist $\theta_1<\theta_2$ such that $\delta(\lambda z^m,\theta)>0$ and $\delta(P_j,\theta)>0$ ($j=0,\dots,k-1$) for all $\theta\in(\theta_1,\theta_2)$. In this case from Lemma 2.4, for sufficiently large r, we have

$$\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\}\le|A_0(z)e^{P_0(z)}+B_0(z)e^{-\lambda z^m}|, \tag{3.5}$$

$$|A_j(z)e^{P_j(z)}+B_j(z)e^{-\lambda z^m}|\le\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\}, \tag{3.6}$$

where $c=\max\{c_j\}$. Using (3.5), (3.6) and (3.4) in (3.1), for r large enough,

$$\exp\{(1-\varepsilon)\delta(P_0,\theta)r^n\}\le C_4\exp\{(1+\varepsilon)c\delta(P_0,\theta)r^n\}[T(2r,f)]^{2k}, \tag{3.7}$$

and thus

$$\exp\{(1-\varepsilon-(1+\varepsilon)c)\delta(P_0,\theta)r^n\}\le C_4[T(2r,f)]^{2k}. \tag{3.8}$$

Taking $0<\varepsilon<\frac{1-c}{1+c}$, we obtain, from (3.8) and Lemma 2.3, the desired estimate $n\le\sigma_2(f)\le m$. □

Proof of Corollary 1.8. In this case also there exist $\theta_1<\theta_2$ such that $\delta(\lambda z^m,\theta)>0$, $\delta(P_0,\theta)>0$ and $\delta(P_s,\theta)<0$ for all $\theta\in(\theta_1,\theta_2)$. We have

$$|A_s(z)e^{P_s(z)}+B_s(z)e^{-\lambda z^m}|\le C_5 \tag{3.9}$$

and also for j such that $\arg a_{n,j}=\arg a_{n,s}$

$$|A_j(z)e^{P_j(z)}+B_j(z)e^{-\lambda z^m}|\le C_6. \tag{3.10}$$

Now for j such that $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$, we have (3.6). By combining (3.6), (3.9), (3.10) with (3.1) we obtain $n \leq \sigma_2(f)$, and by Lemma 2.3 we obtain $n \leq \sigma_2(f) \leq m$. □

Proof of Theorem 1.9. There exist $\theta_1 < \theta_2$ such that $\delta(\lambda z^m, \theta) > 0$, $\delta(P_0 - P_s, \theta) > 0$ and $\delta(P_j - P_s, \theta) > 0$ for every $\theta \in (\theta_1, \theta_2)$. Thus for r sufficiently large, we have

$$\exp\{(1-\varepsilon)\delta(P_0 - P_s, \theta)r^n\} \leq |A_0(z)e^{P_0(z)-P_s(z)} + B_0(z)e^{-\lambda z^m - P_s(z)}|, \quad (3.11)$$

$$|A_s(z) + B_s(z)e^{-\lambda z^m - P_s(z)}| \leq \exp\{r^{\sigma(A_s)+\varepsilon}\}, \quad (3.12)$$

$$|A_0(z)e^{P_j(z)-P_s(z)} + B_0(z)e^{-\lambda z^m - P_s(z)}| \leq C_7. \quad (3.13)$$

From (1.3), we obtain

$$\begin{aligned} &|A_0(z)e^{P_0(z)-P_s(z)} + B_0(z)e^{-\lambda z^m - P_s(z)}| \\ &\leq |e^{-\lambda z^m - P_s(z)}||\frac{f^{(k)}}{f}| + \sum_{j=1}^{k-1} |A_j(z)e^{P_j(z)-P_s(z)} + B_j(z)e^{-\lambda z^m - P_s(z)}||\frac{f^{(j)}}{f}|. \end{aligned} \quad (3.14)$$

Substituting (3.11)-(3.13) and (3.4) in (3.14), we obtain

$$\exp\{(1-\varepsilon)\delta(Q - P, \theta)r^n\} \leq C_8 \exp\{r^{\sigma(A_s)+\varepsilon}\}[T(2r, f)]^{2k}.$$

Which implies $n \leq \sigma_2(f)$, and by Lemma 2.3, we obtain $n \leq \sigma_2(f) \leq m$. □

Proof of Theorem 1.10. There exist $\theta_1 < \theta_2$ such that $\delta(\lambda z^m, \theta) > 0$ and $\delta(P_j, \theta) > 0$ $(j = 0, \dots, k-1)$ for all $\theta \in (\theta_1, \theta_2)$. In this case from Lemma 2.4, for sufficiently large r, we have

$$\exp\{(1-\varepsilon)\delta(P_0 - P_s, \theta)r^n\} \leq |A_0(z)e^{P_0(z)-P_s(z)} + B_0(z)e^{-\lambda z^m - P_s(z)}|, \quad (3.15)$$

$$|A_s(z) + B_s(z)e^{-\lambda z^m - P_s(z)}| \leq \exp\{r^{\sigma(A_s)+\varepsilon}\} \quad (3.16)$$

and for $j \neq 0, s$

$$|A_j(z)e^{P_j(z)-P_s(z)} + B_j(z)e^{-\lambda z^m - P_s(z)}| \leq \exp\{(1+\varepsilon)c\delta(P_0 - P_s, \theta)r^n\}, \quad (3.17)$$

where $c = \max\{c_j\}$. Using (3.15)-(3.17) and (3.4) in (3.14), for r large enough, we obtain

$$\exp\{(1-\varepsilon)\delta(P_0, \theta)r^n\} \leq C_9 \exp\{r^{\sigma(A_s)+\varepsilon}\} \exp\{(1+\varepsilon)c\delta(P_0, \theta)r^n\}[T(2r, f)]^{2k},$$

and thus

$$\exp\{(1 - \varepsilon - (1+\varepsilon)c)\delta(P_0, \theta)r^n\} \leq C_9 \exp\{r^{\sigma(A_s)+\varepsilon}\}[T(2r, f)]^{2k} \quad (3.18)$$

By taking $0 < \varepsilon < \frac{1-c}{1+c}$, from (3.18) and Lemma 2.3, we obtain $n \leq \sigma_2(f) \leq m$. □

Proof of Theorem 1.12. By taking $B_j(z) \equiv 0$ $(j = 0, \dots, k-1)$ in previous theorems, we obtain that every solution $f(z) \not\equiv 0$ of (1.6) is of infinite order with $n \leq \sigma_2(f) \leq m$. It remains to show that $\sigma_2(f) = m$ or $\sigma_2(f) = n$. For that we suppose the contrary, i.e. $n < \sigma_2(f) < m$ and we prove that this implies a contradiction.

Recall the Wiman-Valiron theory [10], there is a set $E_1 \subset [1, +\infty)$ that has finite logarithmic measure, such that for $|z| = r \notin [0, 1] \cup E_1$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = (\frac{\nu(r)}{z})^j(1 + o(1)) \quad (j = 1, \dots, k-1), \quad (3.19)$$

where $\nu(r)$ is the central index of $f(z)$.

Set $\sigma_2(f)=\gamma$. From Lemma 2.5, we can take a sequence of points $\{z_p=r_pe^{i\theta_p}\}$ such that $f(z_p)=M(r_p,f)$, $\theta_p\in[0,2\pi)$, $\lim_{p\to\infty}\theta_p=\theta_0\in[0,2\pi)$, $r_p\notin[0,1]\cup E_1\cup E_2$, and for any given $\epsilon>0$, as $r_p\to\infty$, we have

$$\exp\{r_p^{\gamma-\epsilon}\}\le\nu(r_p)\le\exp\{r_p^{\gamma+\epsilon}\}. \tag{3.20}$$

From (1.6), we can write

$$-\frac{f^{(k)}}{f}=\Big(A_{k-1}(z)e^{P_{k-1}(z)}\frac{f^{(k-1)}}{f}+\dots+A_0(z)e^{P_0(z)}\Big)e^{\lambda z^m}. \tag{3.21}$$

Using (3.19) in (3.21), we obtain

$$\begin{aligned}-\nu^k(r_p)(1+o(1))&=(z_pA_{k-1}e^{P_{k-1}}\nu^{k-1}(r_p)(1+o(1))+\dots\\&\quad+z_p^{k-1}A_1e^{P_1}\nu(r_p)(1+o(1))+z_p^kA_0e^Q)e^{\lambda z_p^m}.\end{aligned} \tag{3.22}$$

Now we prove three cases separately.

Case 1. $\delta(\lambda z^m,\theta_0)=:\delta>0$. From (3.20), for p sufficiently large, we obtain

$$|-\nu^k(r_p)(1+o(1))|\le 2\exp\{kr_p^{\gamma+\epsilon}\}. \tag{3.23}$$

From Lemma 2.4 and by taking account $\gamma+\epsilon<m$, for p large enough, we have

$$\begin{aligned}\exp\{(1-\epsilon)\delta r_p^m\}&\le(z_pA_{k-1}e^{P_{k-1}}\nu^{k-1}(r_p)(1+o(1))+\dots\\&\quad+z_p^{k-1}A_1e^{P_1}\nu(r_p)(1+o(1))+z_p^kA_0e^Q)e^{\lambda z_p^m}.\end{aligned} \tag{3.24}$$

By combining (3.23) and (3.24) with (3.22), a contradiction follows.

Case 2. $\delta(\lambda z^m,\theta_0)=:\delta<0$. From (3.20), for p large enough, we have

$$\frac12\exp\{kr_p^{\gamma-\epsilon}\}\le|-\nu^k(r_p)(1+o(1))|, \tag{3.25}$$

and from Lemma 2.4, we obtain

$$\begin{aligned}&(z_pA_{k-1}e^{P_{k-1}}\nu^{k-1}(r_p)(1+o(1))+\dots\\&+z_p^{k-1}A_1e^{P_1}\nu(r_p)(1+o(1))+z_p^kA_0e^Q)e^{\lambda z_p^m}\le\exp\{(1-\epsilon)\delta r_p^m\}.\end{aligned} \tag{3.26}$$

Also a contradiction follows from the combination of (3.25) and (3.26) with (3.22) as $p\to\infty$.

Case 3. $\delta(\lambda z^m,\theta_0)=0$. Since $\lim_{p\to\infty}\theta_p=\theta_0$, then for p large enough, we obtain

$$\frac1e\le|e^{\lambda z_p^m}|\le e.$$

By Lemma 2.4, there exists $\alpha>0$ such that

$$\begin{aligned}&(z_pA_{k-1}e^{P_{k-1}}\nu^{k-1}(r_p)(1+o(1))+\dots+z_p^{k-1}A_1e^{P_1}\nu(r_p)(1+o(1))+z_p^kA_0e^Q)e^{\lambda z_p^m}\\&\le\exp\{\alpha r_p^n\}\nu^{k-1}(r_p).\end{aligned}$$

Combining this with (3.22), we obtain

$$\frac12|\nu(r_p)|\le\exp\{\alpha r_p^n\};$$

and with

$$\exp\{r_p^{\gamma-\epsilon}\}\le|\nu(r_p)|,$$

and by taking account $n<\gamma-\epsilon$, provided ϵ small enough, a contradiction follows. □

References

[1] I. Amemiya, M. Ozawa; *Non-existence of finite order solutions of* $w'' + e^{-z}w' + Q(z)w = 0$, Hokkaido Math. J., **10** (1981), 1-17.

[2] Z. X. Chen; *The growth of solutions of* $f'' + e^{-z}f' + Q(z)f = 0$*, where the order* $(Q) = 1$, Sci, China Ser. A, **45** (2002), 290-300.

[3] Z. X. Chen, K. H. Shon; *On the growth of solutions of a class of higher order linear differential equations*, Acta. Mathematica Scientia, **24** B (1) (2004), 52-60.

[4] M. Frei; *Über die Subnormalen Lösungen der Differentialgleichung* $w''+e^{-z}w'+(Konst.)w = 0$, Comment. Math. Helv. **36** (1962), 1-8.

[5] G. Gundersen; *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc., (2) **37** (1988), 88-104.

[6] G. Gundersen; *On the question of whether* $f'' + e^{-z}f' + B(z)f = 0$ *can admit a solution* $f \not\equiv 0$ *of finite order*, Proc. Roy. Soc. Edinburgh **102A** (1986), 9-17.

[7] S. Hamouda, B. Belaïdi; *On the growth of solutions of* $w^{(n)} + e^{az^m}w' + Q(z)w = 0$ *and some related extensions*, Hokkaido Math. J., Vol. **35** (2006), p. 573-586.

[8] W. K. Hayman; *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[9] J. K. Langley; *On complex oscillation and a problem of Ozawa*, Kodai Math. J. **9** (1986), 430-439.

[10] G. Valiron; *Lectures on the General Theory of Integral Functions*, New York: Chelsea, 1949.

[11] H. X. Yi and C. C. Yang; *The uniqueness theory of meromorphic functions*, Science Press, Beijing, 1995.

Nacera Berrighi

Laboratory of Pure and Applied Mathematics, University of Mostaganem, B.P. 227 Mostaganem, Algeria

E-mail address: n_berrighi@yahoo.fr

Saada Hamouda

Laboratory of Pure and Applied Mathematics, University of Mostaganem, B.P. 227 Mostaganem, Algeria

E-mail address: hamouda_saada@yahoo.fr