Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 16, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF PULLBACK ATTRACTORS FOR THE COUPLED SUSPENSION BRIDGE EQUATIONS

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Abstract. In this article, we study the existence of pullback $\mathscr{D}$-attractors for the non-autonomous coupled suspension bridge equations with hinged ends and clamped ends, respectively.

1. Introduction

In this paper, we consider the following nonlinear problems which describes a vibrating beam equation coupled with a vibrating string equation

$$\begin{gathered} u_{tt} + \alpha u_{xxxx} + \delta_1 u_t + k(u-v)^+ + f_B(u) = h_B(x,t), \quad x \in [0,L], \\ v_{tt} - \beta v_{xx} + \delta_2 v_t - k(u-v)^+ + f_S(v) = h_S(x,t), \quad x \in [0,L] \end{gathered} \tag{1.1}$$

with the simply supported boundary-value conditions

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad v(0,t) = v(L,t) = 0, \quad t \geq \tau, \tag{1.2}$$

and the initial-value conditions

$$u(\tau,x) = u_0(x), u_t(\tau,x) = u_1(x), x \in [0,L], \tag{1.3}$$

$$v(\tau,x) = v_0(x), v_t(\tau,x) = v_1(x), x \in [0,L]. \tag{1.4}$$

Where $k > 0$ denotes the spring constant of the ties, $\alpha > 0$ and $\beta > 0$ are the flexural rigidity of the structure and coefficient of tensile strength of the cable, respectively. δ_1, $\delta_2 > 0$ are constants, the force term h_B, $h_S \in L^2_{\rm loc}(\mathbb{R}; L^2(0,L))$. The nonlinear functions $f_B(u), f_S(v) \in C^2(\mathbb{R},\mathbb{R})$ satisfies the following assumptions:

- (F1) $\liminf_{|r|\to\infty} \frac{F_B(r)}{r^2} \geq 0$, $\liminf_{|r|\to\infty} \frac{F_S(r)}{r^2} \geq 0$, $\forall r \in \mathbb{R}$;
- (F2) $|f_B(r)|, |f_S(r)| \leq C_0(1+|r|^p)$, for all $p \geq 1$, $\forall r \in \mathbb{R}$;
- (F3) $\liminf_{|r|\to\infty} \frac{rf_B(r)-C_1F_B(r)}{r^2} \geq 0$, $\liminf_{|r|\to\infty} \frac{rf_S(r)-C_1F_S(r)}{r^2} \geq 0$, $\forall r \in \mathbb{R}$,

where C_0, C_1 are positive constants, $F_B(r) = \int_0^r f_B(\zeta)d\zeta$, $F_S(r) = \int_0^r f_S(\zeta)d\zeta$.

For the mathematical model of suspension bridge, there are many references to study the existence and asymptotic behavior of solutions, see [1, 4, 5, 6, 7, 8, 9, 11, 17] and references therein. For instance, Lazer and McKenna studied the nonlinear oscillation problems in a suspension bridge, and presented a (one-dimensional) mathematical model for a suspension bridge as a new problem of nonlinear analysis in [5]. Ahmed and Harbi continued to discuss this problem in

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2000 *Mathematics Subject Classification.* 35B40, 35L05, 58F12.
Key words and phrases. Coupled suspension bridge equations; pullback $\mathscr{D}$-attractors.

Submitted July 12, 2010. Published January 28, 2011.

[1], and pointed out that the system (1.1) is conservative and asymptotically stable with respect to the rest state for $k > 0$, $f_B(u) \equiv 0 \equiv f_S(v)$, and furthermore showed that the corresponding Cauchy problem has at least one weak solution. Holubová and Matas considered the more general nonlinear string-beam system in [4] and arrived at the existence and uniqueness of the weak solution by the Faedo-Galerkin methods. In 2004, Litcanu proved the existence of weak T-periodic solutions of (1.1) and obtained a regularity result when $k(u-v)^+ = \phi(u,v)$, $f_B(u) \equiv f_S(v) \equiv 0$ in [6]. Similar models have also been investigated by Malik in [11]. However, our aim is to study the longtime behavior of solutions for the suspension bridge model. In 2005, we achieved first the existence of global attractors of a weak solution for the autonomous coupled suspension bridge equations in [7]; i.e., in (1.1), $h_B(x,t)$ and $h_S(x,t)$ do not depend on the time t explicitly. In the sequel, the existence of the strong solutions and the compact global attractor have also been obtained for the autonomous coupled suspension bridge equations and the single one which the motion of the main cable is neglected, respectively, see [8, 17]. For the limit of our knowledge, the existence of the pullback attractors of (1.1) has no any results, while it is just our concern. For a good survey of the literatures dedicated to the existence of attractors for the dynamical systems we would like to mention some monograph [3, 13, 14] and so on.

About the existence of pullback attractors for the dynamical systems, it has been developed for both non-autonomous and random dynamical systems. In 2006, Caraballo et al. presented the concept of the pullback $\mathscr{D}$-attractors in [2], and obtained the abstract results verifying the existence of pullback $\mathscr{D}$-attractors, moreover, they applied their abstract results into the non-autonomous Navier-Stokes equation in an unbounded domain. Zhong [17] and Wang [16] also established some sufficient conditions for the existence of the pullback $\mathscr{D}$-attractors by using the methods introduced in [10], and achieved the existence of pullback $\mathscr{D}$-attractors for non-autonomous Sine-Gordon equations and wave equations with critical exponent, respectively. The existence of pullback $\mathscr{D}-$ attractors for the single suspension bridge equation was showed in [12]. Motivated by [2, 12, 15, 16], in this paper, we focus our attention on the existence of pullback $\mathscr{D}$-attractors for (1.1). Our main results are Theorem 3.4 and 3.5.

2. Preliminaries

With the usual notation, let $Y_0 = L^2(0,L)$, $Y_1 = H_0^1(0,L)$, $Y_2 = D(A) = H^2(0,L) \cap H_0^1(0,L)$, where $A = -\frac{\partial^2}{\partial x^2}, A^2 = \frac{\partial^4}{\partial x^4}$, and endow Y_0 with the standard scalar product and norm $(\cdot,\cdot)$, $|\cdot|$. Meanwhile, we denote $\|\cdot\|, |Au|$ be the norm of Y_1, Y_2, respectively. In addition, let λ_1 be the first eigenvalue of $Au = \lambda u, x \in [0,L]; u(0) = u(L) = 0$, the corresponding eigenfunctions $\phi_1(x)$ is positive on $[0,L]$. It's easy to know that λ_1^2 is the first eigenvalue of $A^2u = \lambda^2 u, x \in [0,L], u(0) = u(L) = u_{xx}(0) = u_{xx}(L) = 0$. Choosing $\lambda = min\{\lambda_1, \lambda_1^2\}$, by the Poincaré inequality, we have

$$\|u\|^2 \geq \lambda |u|^2, \quad \forall u \in Y_1; \quad |Au|^2 \geq \lambda \|u\|^2, \quad \forall u \in Y_2. \tag{2.1}$$

Next we iterate some definitions and abstract results concerning the pullback attractor, which is necessary to obtain our main results, please refer the reader to see [2, 15] for more details. Let (E,d) be a complete metric space, (Q,ρ) be a metric space which will be called the parameter space. We define a non-autonomous dynamical system by a cocycle mapping $\phi : \mathbb{R}_+ \times Q \times E$ which is driven by an autonomous dynamical system θ acting on a parameter space Q. Specifically, $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ is a dynamical system on Q with the properties that

(1) $\theta_0(q) = q$, for all $q \in Q$;
(2) $\theta_{t+\tau}(q) = \theta_t(\theta_\tau(q))$, for all $q \in Q$, $t, \tau \in \mathbb{R}$;
(3) the mapping (t,q)$\to \theta_t(q)$ is continuous.

Definition 2.1. A mapping ϕ is said to be a cocycle on E with respect to group θ if

(1) $\phi(0, q, x) = x$, for all $(q, x) \in Q \times E$;
(2) $\phi(t + s, q, x) = \phi(s, \theta_t(q), \phi(t, q, x))$, for all $s, t \in \mathbb{R}_+$ and all $(q, x) \in Q \times E$.

Let $\mathcal{P}(E)$ denote the family of all nonempty subsets of E, $\mathcal{B}(E)$ be the set of all bounded subsets of E, and $\mathcal{K}$ be the class of all families $\hat{D} = \{D_q\}_{q\in Q} \subset \mathcal{P}(E)$. We consider a given nonempty subclass $\mathscr{D} \subset \mathcal{K}$.

Definition 2.2. A family $\hat{B} = \{B_q\}_{q\in Q} \in \mathcal{K}$ is said to be pullback $\mathscr{D}$-absorbing if for each $q \in Q$ and $\hat{D} \in \mathscr{D}$, there exists $t_0(q, \hat{D}) \ge 0$ such that $\phi(t, \theta_{-t}(q), D_{\theta_{-t}(q)}) \subset B_q$, for all $t \ge t_0(q, \hat{D})$.

Definition 2.3. Let (θ, ϕ) be a non-autonomous dynamical system on $Q\times E$. (θ, ϕ) is said to be satisfying pullback $\mathscr{D}$-condition (C) if for any $q \in Q$, $\hat{C} \in \mathscr{D}$ and any $\epsilon > 0$, there exists a $t_0 = t_0(q, \hat{C}, \epsilon) \ge 0$ and a finite dimensional subspace E_1 of E such that

(i) $P(\bigcup_{t\ge t_0} \phi(t, \theta_{-t}(q), D_{\theta_{-t}(q)}))$ is bounded; and
(ii) $\|(I - P)\phi(t, \theta_{-t}(q), D_{\theta_{-t}(q)})\|_E \le \epsilon$, where $P : E \to E_1$ is a bounded projector.

Theorem 2.4. *Let (θ, ϕ) be a non-autonomous dynamical system on $Q \times E$. (θ, ϕ) possesses a global pullback $\mathscr{D}$-attractor $\hat{A} = \{A_q\}_{q\in Q}$ satisfying $A_q = \Lambda(\hat{D}, q) = \bigcap_{s\ge0} \overline{\bigcup_{t\ge s} \phi(t, \theta_{-t}(q), D_{\theta_{-t}(q)})}$ if*

(1) *it has a pullback $\mathscr{D}$-absorbing set $\hat{B} = \{B_q\}_{q\in Q} \in \mathscr{D}$;*
(2) *it satisfies pullback $\mathscr{D}$-Condition (C).*

3. Pullback $\mathscr{D}$-attractors in E_0

For brevity, we write $E_0 = Y_2 \times Y_0 \times Y_1 \times Y_0$, $y_0 = (u_0, u_1, v_0, v_1)$, $y = y(t) = (u(t), u_t(t), v(t), v_t(t))$. We need the following results.

Theorem 3.1 ([1, 8, 9])**.** *Suppose that $y_0 \in E_0$, $h_B, h_S \in L^2_{\rm loc}(\mathbb{R}, Y_0)$, then* (1.1)-(1.4) *has a unique solution*

$$y \in C(\mathbb{R}_\tau, E_0), \tag{3.1}$$

where $\mathbb{R}_\tau = [\tau, +\infty)$. Furthermore, $y_0 \mapsto y$ is continuous in E_0.

As in [2, 15], we denote by E_0 the space of vector function $y(x)$ with finite energy norm $\|y\|^2_{E_0} = |Au|^2 + \|v\|^2 + |u_t|^2 + |v_t|^2$. Then we can construct the non-autonomous dynamical system generated by problem $(1.1) - (1.4)$ in E_0. We consider $Q = \mathbb{R}$, $\theta_t(\tau) = t + \tau$, and define

$$\phi(t, \tau, y_0) = y(t + \tau; \tau, y_0) = \big(u(t + \tau), u_t(t + \tau), v(t + \tau), v_t(t + \tau)\big), \tag{3.2}$$

$\tau \in \mathbb{R}$, $t \ge 0, y_0 \in E_0$. Thus, thanks to Theorem 3.1, we have $\phi(t + s, \tau, y_0) = \phi(t, s + \tau, \phi(s, \tau, y_0))$, for $\tau \in \mathbb{R}, s, t \ge 0$, and the mapping $\phi(t, \tau, \cdot) : E_0 \to E_0$

defined by (3.2) is continuous. Therefore, the mapping ϕ defined by (3.2) is a continuous cocycle on E_0. Now we assume that $h_B, h_S \in L^2_{\rm loc}(\mathbb{R}; Y_0)$ with

$$\int_{-\infty}^{t} e^{\delta s}|h_B(x,s)|^2 ds < \infty, \quad \forall t \in \mathbb{R}, \tag{3.3}$$

$$\int_{-\infty}^{t} e^{\delta s}|h_S(x,s)|^2 ds < \infty, \quad \forall t \in \mathbb{R}, \tag{3.4}$$

where δ is a positive constant which will be characterized later. Let $\mathcal{R}_\delta$ be the set of all function $r : \mathbb{R} \to (0, +\infty)$ satisfying

$$\lim_{t\to-\infty} e^{\delta t} r^2(t) = 0. \tag{3.5}$$

Here $\mathscr{D}_{\delta,E_0}$ denotes the class of all families $\hat{D} = \{D(t); t \in \mathbb{R}\} \subset \mathcal{P}(E_0)$ with $D(t) \subset \bar{B}(0, r_{\hat{D}}(t))$ for some $r_{\hat{D}}(t) \in \mathcal{R}_\delta$, where $\bar{B}(0, r_{\hat{D}}(t))$ is the closed ball in E_0 centered at 0 with radius $r_{\hat{D}}(t)$. We also need the following lemmas.

Lemma 3.2 ([15]). *Suppose that the family $\{\omega_i\}_{i\in\mathbb{N}}$ and $\{\chi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of Y_2 and Y_1, respectively, which consist of the eigenvectors of A^2 and A, $h_B, h_S \in L^2_{\rm loc}(\mathbb{R}, Y_0)$ satisfy* (3.3)-(3.4). *Then*

$$\lim_{m\to\infty} \int_{-\infty}^{t} e^{\sigma s}|(I - P_m)h_B(x,s)|^2 ds = 0, \quad \forall t \in \mathbb{R}, \tag{3.6}$$

$$\lim_{m\to\infty} \int_{-\infty}^{t} e^{\sigma s}|(I - Q_m)h_S(x,s)|^2 ds = 0, \forall t \in \mathbb{R}, \tag{3.7}$$

where $P_m : Y_2 \to span\{\omega_1, \omega_2, \dots \omega_m\}$, $Q_m : Y_1 \to span\{\chi_1, \chi_2, \dots \chi_m\}$ are the orthogonal projector.

Lemma 3.3 ([8, 9]). *Assume that $f_B(u), f_S(v) \in C^2(\mathbb{R}, \mathbb{R})$ satisfies* (F2)*, moreover, $f_B(0) = f_S(0) = 0$. Then $(f_B, f_S) : Y_2 \times Y_1 \to Y_0 \times Y_0$ are continuous and compact.*

Theorem 3.4. *Suppose that* (F1)-(F3) *hold, $h_B, h_S \in L^2_{\rm loc}(\mathbb{R}, Y_0)$ with* (3.3)–(3.4)*. Then there exists a pullback $\mathscr{D}_{\delta,E_0}$-absorbing set in E_0 for the non-autonomous dynamical system (θ, ϕ) associated with* (1.1)-(1.4).

Proof. Let $t \in \mathbb{R}$, $\tau \geq 0$, and $y_0 \in E_0$ be fixed. Choose $0 < \epsilon < \epsilon_0$, where

$$\epsilon_0 = \min\{\frac{\delta_1}{4}, \frac{\delta_2}{4}, \frac{\alpha\lambda^2}{2\delta_1}, \frac{\beta\lambda^2}{2\delta_2}\}. \tag{3.8}$$

Taking the scalar product in Y_0 for the first equation of (1.1) with $\phi = u_t + \epsilon u$ and the second equation with $\varphi = v_t + \epsilon v$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\phi|^2 + \alpha|Au|^2 + \beta\|v\|^2 + |\varphi|^2) + \epsilon\alpha|Au|^2 + (\delta_1 - \epsilon)|\phi|^2 \\
&- \epsilon(\delta_1 - \epsilon)(u, \phi) + \beta\epsilon\|v\|^2 + (\delta_2 - \epsilon)|\varphi|^2 - \epsilon(\delta_2 - \epsilon)(v, \varphi) \\
&+ k((u - v)^+, \phi - \varphi) + (f_B(u), \phi) + (f_S(v), \varphi) \\
&= (h_B(t), \phi) + (h_S(t), \varphi).
\end{aligned} \tag{3.9}$$

Using Hölder and Young inequalities and (3.8), we conclude

$$\begin{aligned}
&\epsilon\alpha|Au|^2 + (\delta_1 - \epsilon)|\phi|^2 - \epsilon(\delta_1 - \epsilon)(u, \phi) + \beta\epsilon\|v\|^2 + (\delta_2 - \epsilon)|\varphi|^2 - \epsilon(\delta_2 - \epsilon)(v, \varphi) \\
&\geq \frac{\epsilon\alpha}{2}|Au|^2 + \frac{\epsilon\beta}{2}\|v\|^2 + \frac{\delta_1}{2}|\phi|^2 + \frac{\delta_2}{2}|\varphi|^2;
\end{aligned} \tag{3.10}$$

$$k((u-v)^+,\phi-\varphi)=\frac{1}{2}\frac{d}{dt}k|(u-v)^+|^2+\epsilon k|(u-v)^+|^2. \tag{3.11}$$

Due to (F3) it follows that

$$\int_0^L uf_B(u)dx - C_1\int_0^L F_B(u)dx+\frac{\epsilon}{4}|Au|^2 \geq -K_2, \quad \forall u\in Y_2,$$
$$\int_0^L vf_S(v)dx - C_1\int_0^L F_S(v)dx+\frac{\epsilon}{4}\|v\|^2 \geq -K_2, \quad \forall v\in Y_1,$$

where C_1, K_2 are positive constants. Then

$$(f_B(u),\phi)\geq \frac{d}{dt}\int_0^L F_B(u)dx+\epsilon C_1\int_0^L F_B(u)dx-\frac{\epsilon^2}{4}|Au|^2-\epsilon K_2, \tag{3.12}$$
$$(f_S(v),\varphi)\geq \frac{d}{dt}\int_0^L F_S(v)dx+\epsilon C_1\int_0^L F_S(v)dx-\frac{\epsilon^2}{4}\|v\|^2-\epsilon K_2. \tag{3.13}$$

Together with (3.10)-(3.13), from (3.9), it leads to

$$\begin{aligned}
&\frac{d}{dt}(|\phi|^2+\alpha|Au|^2+\beta\|v\|^2+|\varphi|^2+k|(u-v)^+|^2+2\int_0^L F_B(u)dx+2\int_0^L F_S(v)dx)\\
&+\epsilon(\alpha-\frac{\epsilon}{2})|Au|^2+\epsilon(\beta-\frac{\epsilon}{2})\|v\|^2+\frac{\delta_1}{2}|\phi|^2+\frac{\delta_2}{2}|\varphi|^2+2\epsilon k|(u-v)^+|^2\\
&+2\epsilon C_1\int_0^L F_B(u)dx+2\epsilon C_1\int_0^L F_S(v)dx\\
&\leq 4\epsilon K_2+\frac{2}{\delta_1}|h_B(t)|^2+\frac{2}{\delta_2}|h_S(t)|^2.
\end{aligned} \tag{3.14}$$

Provided ϵ is small enough such that $\alpha-\frac{\epsilon}{2}>\frac{\alpha}{2}$, $\beta-\frac{\epsilon}{2}>\frac{\beta}{2}$, and set $\delta=\min\{\frac{\epsilon}{2},\epsilon C_1\}$, we deduce

$$\begin{aligned}
&\frac{d}{dt}(|\phi|^2+\alpha|Au|^2+\beta\|v\|^2+|\varphi|^2+k|(u-v)^+|^2+2\int_0^L F_B(u)dx+2\int_0^L F_S(v)dx)\\
&+\delta(|\phi|^2+\alpha|Au|^2+\beta\|v\|^2+|\varphi|^2+k|(u-v)^+|^2+2\int_0^L F_B(u)dx+2\int_0^L F_S(v)dx)\\
&\leq 4\epsilon K_2+\frac{2}{\delta_1}|h_B(t)|^2+\frac{2}{\delta_2}|h_S(t)|^2.
\end{aligned}$$

By (F1) we know that there exists a positive constant K_1 such that

$$\int_0^L F_B(u)dx+\frac{\alpha}{8}|Au|^2\geq -K_1, \quad \forall u\in Y_2, \tag{3.15}$$
$$\int_0^L F_S(v)dx+\frac{\beta}{8}\|v\|^2\geq -K_1, \quad \forall v\in Y_1. \tag{3.16}$$

Therefore, by (3.15)-(3.16) it follows that

$$\begin{aligned}
E(t)&=|\phi|^2+\alpha|Au|^2+\beta\|v\|^2+|\varphi|^2+k|(u-v)^+|^2\\
&\quad+2\int_0^L F_B(u)dx+2\int_0^L F_S(v)dx+4K_1\geq 0,
\end{aligned}$$

and

$$\frac{d}{dt}E(t)+\delta E(t)\leq 4\epsilon K_2+4\delta K_1+\frac{2}{\delta_1}|h_B(t)|^2+\frac{2}{\delta_2}|h_S(t)|^2.$$

Furthermore,

$$\frac{d}{dt}(e^{\delta t}E(t)) \le e^{\delta t}(4\epsilon K_2 + 4\delta K_1 + \frac{2}{\delta_1}|h_B(t)|^2 + \frac{2}{\delta_2}|h_S(t)|^2).$$

Integrating the above inequality from $t-\tau$ to t yields

$$\begin{aligned}
E(t) &\le e^{-\delta\tau}E(t-\tau) + \frac{4(\epsilon K_2 + \delta K_1)}{\delta} + \frac{2}{\delta_1}e^{-\delta t}\int_{t-\tau}^{t} e^{\delta s}|h_B(s)|^2 ds \\
&\quad + \frac{2}{\delta_2}e^{-\delta t}\int_{t-\tau}^{t} e^{\delta s}|h_S(s)|^2 ds \\
&\le e^{-\delta\tau}E(t-\tau) + \frac{4(\epsilon K_2 + \delta K_1)}{\delta} + \frac{2}{\delta_1}e^{-\delta t}\int_{-\infty}^{t} e^{\delta s}|h_B(s)|^2 ds \\
&\quad + \frac{2}{\delta_2}e^{-\delta t}\int_{-\infty}^{t} e^{\delta s}|h_S(s)|^2 ds.
\end{aligned}$$

For any $\hat{D} \in \mathscr{D}_{\delta,E_0}, y_0 \in D(t-\tau)$, by (F2), $\int_0^L F_B(u_0)dx$ and $\int_0^L F_S(v_0)dx$ are bounded. Hence

$$\begin{aligned}
&\sup_{y_0\in D(t-\tau)} E(t-\tau) \\
&= \sup_{y_0\in D(t-\tau)} \big\{|u_1+\epsilon u_0|^2 + \alpha|Au_0|^2 + \beta\|v_0\|^2 + |v_1+\epsilon v_0|^2 + k|(u_0-v_0)^+|^2 \\
&\quad + 2\int_0^L F_B(u_0)dx + 2\int_0^L F_S(v_0)dx\big\} < \infty.
\end{aligned}$$

Using (3.15) and (3.16) again, we arrive at

$$\begin{aligned}
E(t) &= |\phi|^2 + \alpha|Au|^2 + \beta\|v\|^2 + |\varphi|^2 + k|(u-v)^+|^2 \\
&\quad + 2\int_0^L F_B(u)dx + 2\int_0^L F_S(v)dx + 4K_1 \\
&\ge |\phi|^2 + \frac{3\alpha}{4}|Au|^2 + \frac{3\beta}{4}\|v\|^2 + |\varphi|^2.
\end{aligned}$$

Therefore, if we let $\delta_0 = \min\{\frac{3\alpha}{4}, \frac{3\beta}{4}, 1\}$, $K = 4(\epsilon K_2 + \delta K_1)$, then

$$\begin{aligned}
&|\phi|^2 + |Au|^2 + |\varphi|^2 + \|v\|^2 \\
&\le \frac{1}{\delta_0}(e^{-\delta\tau}E(t-\tau) + \frac{K}{\delta} + \frac{2}{\delta_1}e^{-\delta t}\int_{-\infty}^{t} e^{\delta s}|h_B(s)|^2 ds \\
&\quad + \frac{2}{\delta_2}e^{-\delta t}\int_{-\infty}^{t} e^{\delta s}|h_S(s)|^2 ds),
\end{aligned} \tag{3.17}$$

namely, $\|\phi(\tau, t-\tau, y_0)\|_{E_0}^2$ is bounded by the above expression for all $y_0 \in D(t-\tau), t \in \mathbb{R}$ and $\tau \ge 0$. Set

$$(R_\delta(t))^2 = \frac{2}{\delta_0}\Big(\frac{K}{\delta} + \frac{2}{\delta_1}e^{-\delta t}\int_{-\infty}^{t} e^{\delta s}|h_B(s)|^2 ds + \frac{2}{\delta_2}e^{-\delta t}\int_{-\infty}^{t} e^{\delta s}|h_S(s)|^2 ds\Big), \tag{3.18}$$

and consider the family $\hat{B}_{\delta,E_0}$ of close balls in E_0 defined by

$$B_\delta(t) = \{y \in E_0 : \|y\|_{E_0}^2 \le (R_\delta(t))^2\}. \tag{3.19}$$

Thus from (3.3), (3.4) we know that $\hat{B}_{\delta,E_0}$ is a pullback $\mathscr{D}_{\delta,E_0}$ absorbing for the cocyle ϕ. $\square$

Theorem 3.5. *Suppose that $h_B, h_S \in L^2_{\rm loc}(\mathbb{R}, Y_0)$ and f_B, f_S satisfies $(F1)-(F3)$, then there exists a global pullback $\mathscr{D}_{\delta,E_0}$-attractor in E_0 for the non-autonomous dynamical system (θ, ϕ) defined by (3.2).*

Proof. Using Theorem 2.4, it is only enough to verify pullback $\mathscr{D}-$condition (C). If $\{\omega_i\}_{i=1}^{\infty}$ is orthonormal basis of Y_2, which consists of eigenvectors of A^2, it is also orthonormal basis of Y_1, Y_0. The corresponding eigenvalues are denoted by

$$0 < \nu_1 < \nu_2 \le \nu_3 \le \dots, \quad \nu_i \to \infty, i \to \infty,$$

and $A^2\omega_i = \nu_i\omega_i$ for all $i \in \mathbb{N}$. We write $V_m = \operatorname{span}\{\omega_1, \omega_2, \dots, \omega_m\}$, $P_m : Y_2 \to V_m$ is orthogonal projector. In addition, let $\{\chi_i\}_{i=1}^{\infty}$ be an orthonormal basis of Y_1 which consists of eigenvectors of A, the corresponding eigenvalue are denoted by

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots, \quad \lambda_i \to \infty,\ i \to \infty,$$

and $A\chi_i = \lambda_i\chi_i$ for all $i \in \mathbb{N}$. In fact, by the boundary value conditions (1.2), $\omega_i = \chi_i$, $\nu_i = \lambda_i^2$, $i = 1, 2, \dots$. We write $G_m = \operatorname{span}\{\chi_1, \chi_2, \dots, \chi_m\}$, $Q_m : Y_1 \to G_m$ is orthogonal projector. Then for all $u \in Y_2$, $v \in Y_1$, we make the decomposition

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2, \quad v = Q_m v + (I - Q_m)v \triangleq v_1 + v_2. \tag{3.20}$$

Taking the scalar product in Y_0 for the first equation of (1.1) with $\phi_2 = u_{2t} + \epsilon u_2$ and for the second equation with $\varphi_2 = v_{2t} + \epsilon v_2$, respectively, after a computation, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\phi_2|^2 + \alpha|Au_2|^2 + \beta\|v_2\|^2 + |\varphi_2|^2) + \epsilon\alpha|Au_2|^2 \\
&+ (\delta_1 - \epsilon)|\phi_2|^2 - \epsilon(\delta_1 - \epsilon)(u_2, \phi_2) + \beta\epsilon|\varphi_2|^2 + (\delta_2 - \epsilon)|\varphi_2|^2 - \epsilon(\delta_2 - \epsilon)(v_2, \varphi_2) \\
&+ k((u - v)^+, \phi_2 - \varphi_2) + ((I - P_m)f_B(u), \phi_2) + ((I - Q_m)f_S(v), \varphi_2) \\
&= ((I - P_m)h_B(t), \phi_2) + ((I - Q_m)h_S(t), \varphi_2).
\end{aligned} \tag{3.21}$$

Similar to the estimates of (3.10), it follows that

$$\begin{aligned}
&\epsilon\alpha|Au_2|^2 + (\delta_1 - \epsilon)|\phi_2|^2 - \epsilon(\delta_1 - \epsilon)(u_2, \phi_2) + \beta\epsilon|\varphi_2|^2 \\
&+ (\delta_2 - \epsilon)|\varphi_2|^2 - \epsilon(\delta_2 - \epsilon)(v_2, \varphi_2) \\
&\ge \frac{\epsilon\alpha}{2}|Au_2|^2 + \frac{\epsilon\beta}{2}\|v_2\|^2 + \frac{\delta_1}{2}|\phi_2|^2 + \frac{\delta_2}{2}|\varphi_2|^2.
\end{aligned} \tag{3.22}$$

Using the Hölder, Young and Poincaré inequalities, there exists a positive constant c, such that

$$\begin{aligned}
k((u - v)^+, \phi_2 - \varphi_2) &\le k|(u - v)_2| \cdot |\phi_2 - \varphi_2| \le k(|u_2| + |v_2|)(|\phi_2| + |\varphi_2|) \\
&\le \frac{ck^2}{\delta_1\nu_{m+1}}|Au_2|^2 + \frac{ck^2}{\delta_2\lambda_{m+1}}\|v_2\|^2 + \frac{\delta_1}{4}|\phi_2|^2 + \frac{\delta_2}{4}|\varphi_2|^2.
\end{aligned} \tag{3.23}$$

Together with (3.22)-(3.23), from (3.21), yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\phi_2|^2 + \alpha|Au_2|^2 + \beta\|v_2\|^2 + |\varphi_2|^2) \\
&+ (\frac{\epsilon\alpha}{2} - \frac{ck^2}{\delta_1\nu_{m+1}})|Au_2|^2 + (\frac{\epsilon\beta}{2} - \frac{ck^2}{\delta_2\lambda_{m+1}})\|v_2\|^2 + \frac{\delta_1}{4}|\phi_2|^2 + \frac{\delta_2}{4}|\varphi_2|^2 \\
&+ ((I - P_m)f_B(u), \phi_2) + ((I - Q_m)f_S(v), \varphi_2) \\
&\le ((I - P_m)h_B(t), \phi_2) + ((I - Q_m)h_S(t), \varphi_2).
\end{aligned} \tag{3.24}$$

Taking m large enough, such that $\frac{\epsilon\alpha}{2} - \frac{ck^2}{\delta_1\nu_{m+1}} \geq \frac{\epsilon\alpha}{4}$, $\frac{\epsilon\beta}{2} - \frac{ck^2}{\delta_2\lambda_{m+1}} \geq \frac{\epsilon\beta}{4}$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\phi_2|^2 + \alpha|Au_2|^2 + \beta\|v_2\|^2 + |\varphi_2|^2) + \frac{\epsilon\alpha}{4}|Au_2|^2 + \frac{\epsilon\beta}{4}\|v_2\|^2 \\
&+ \frac{\delta_1}{4}|\phi_2|^2 + \frac{\delta_2}{4}|\varphi_2|^2 + ((I-P_m)f_B(u),\phi_2) + ((I-Q_m)f_S(v),\varphi_2) \\
&\leq ((I-P_m)h_B(t),\phi_2) + ((I-Q_m)h_S(t),\varphi_2).
\end{aligned}
$$

Furthermore, there holds

$$
\begin{aligned}
&\frac{d}{dt}(|\phi_2|^2 + \alpha|Au_2|^2 + \beta\|v_2\|^2 + |\varphi_2|^2) + \frac{\epsilon\alpha}{2}|Au_2|^2 + \frac{\epsilon\beta}{2}\|v_2\|^2 + \frac{\delta_1}{4}|\phi_2|^2 + \frac{\delta_2}{4}|\varphi_2|^2 \\
&\leq \frac{8}{\delta_1}|(I-P_m)h_B(t)|^2 + \frac{8}{\delta_2}|(I-Q_m)h_S(t)|^2 + \frac{8}{\delta_1}|(I-P_m)f_B(u)|^2 \\
&+ \frac{8}{\delta_2}|(I-Q_m)f_S(v)|^2
\end{aligned}
$$

Set $\xi = \min\{\epsilon, \frac{\delta_1}{4}, \frac{\delta_2}{4}\}$, and $\chi(t) = |\phi_2|^2 + \alpha|Au_2|^2 + \beta\|v_2\|^2 + |\varphi_2|^2 > 0$. Then

$$
\begin{aligned}
\frac{d}{dt}\chi(t) + \xi\chi(t) &\leq \frac{8}{\delta_1}|(I-P_m)h_B(t)|^2 + \frac{8}{\delta_2}|(I-Q_m)h_S(t)|^2 + \frac{8}{\delta_1}|(I-P_m)f_B(u)|^2 \\
&\quad + \frac{8}{\delta_2}|(I-Q_m)f_S(v)|^2.
\end{aligned} \tag{3.25}
$$

Multiplying both sides of (3.17) with $e^{\xi t}$, we obtain

$$
\begin{aligned}
\frac{d}{dt}(e^{\xi t}\chi(t)) \leq e^{\xi t}\Big(&\frac{8}{\delta_1}|(I-P_m)h_B(t)|^2 + \frac{8}{\delta_2}|(I-Q_m)h_S(t)|^2 \\
&+ \frac{8}{\delta_1}|(I-P_m)f_B(u)|^2 + \frac{8}{\delta_2}|(I-Q_m)f_S(v)|^2\Big).
\end{aligned}
$$

Integrating over $[t-\tau, t]$, it leads to

$$
\begin{aligned}
\chi(t) &\leq e^{-\xi\tau}\chi(t-\tau) + \frac{8}{\delta_1}e^{-\xi t}\int_{t-\tau}^{t} e^{\xi s}|(I-P_m)h_B(s)|^2 ds \\
&\quad + \frac{8}{\delta_2}e^{-\xi t}\int_{t-\tau}^{t} e^{\xi s}|(I-Q_m)h_S(s)|^2 ds + \frac{8}{\delta_1}e^{-\xi t}\int_{t-\tau}^{t} e^{\xi s}|(I-P_m)f_B(u)|^2 ds \\
&\quad + \frac{8}{\delta_2}e^{-\xi t}\int_{t-\tau}^{t} e^{\xi s}|(I-Q_m)f_S(v)|^2 ds.
\end{aligned} \tag{3.26}
$$

Firstly, for any $t \in \mathbb{R}$, any $\epsilon > 0$, there exist $t_1 \in (t-\tau, t)$ and $\tau_1 > 0$ such that $u(s) = u(s, t-\tau, y_0) \in B_\delta(s), v(s) = v(s, t-\tau, y_0) \in B_\delta(s)$, for $\tau \geq \tau_1$, any $s \in [t-\tau, t_1]$, any $y_0 \in D(t-\tau)$. Also for all $\tau \geq \tau_1$,

$$
\int_{t-\tau}^{t_1} e^{-\xi(t-s)}|(I-P_m)f_B(u)|^2 ds \leq \frac{\delta_1\epsilon}{40}, \int_{t-\tau}^{t_1} e^{-\xi(t-s)}|(I-Q_m)f_S(v)|^2 ds \leq \frac{\delta_1\epsilon}{40}.
$$

Secondly, we set $\hat{R} = \max_{s\in[t_1,t]} R_\delta(s) < \infty$, then

$$
|Au(s)| = |Au(s; t-\tau, u_0)| \leq \hat{R}, \|v(s)\| = \|v(s; t-\tau, v_0)\| \leq \hat{R}
$$

for any $s \in [t_1, t]$ and any $y_0 \in D(t-\tau)$. In line with Lemma 3.3, for any $\epsilon > 0$, any $m \geq m_1, \tau \geq \tau_1$, we have

$$\int_{t_1}^{t} e^{-\xi(t-s)}|(I-P_m)f_B(u)|^2 ds \leq \frac{\delta_1\epsilon}{40}, \int_{t_1}^{t} e^{-\xi(t-s)}|(I-Q_m)f_S(v)|^2 ds \leq \frac{\delta_2\epsilon}{40}. \tag{3.27}$$

Thirdly, by Lemma 3.2, we can choose m larger enough, such that

$$\int_{t-\tau}^{t} e^{-\xi(t-s)}|(I-P_m)h_B(s)|^2 ds \leq \frac{\delta_1\epsilon}{20},$$
$$\int_{t-\tau}^{t} e^{-\xi(t-s)}|(I-Q_m)h_S(s)|^2 ds \leq \frac{\delta_2\epsilon}{20}.$$

Finally, using (3.5), there exists $\tau_2 \geq 0$ such that

$$e^{-\xi\tau}\chi(t-\tau) \leq \frac{\epsilon}{5}, \quad \forall \tau \geq \tau_2,\ y_0 \in D(t-\tau). \tag{3.28}$$

Now let $\tau_0 = \max\{\tau_1, \tau_2\}$, from (3.26)-(3.28) yields $\chi(t) \leq \epsilon$. Therefore, it is easy to see that

$$\|\phi_2(\tau, t-\tau, y_0)\|_{E_0}^2 \leq \epsilon, \quad \forall \tau \geq \tau_0, y_0 \in D(t-\tau).$$

The proof is complete. □

We remark that our main results is also true for (1.1) with fixed boundary-value conditions ends:

$$u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0, \quad v(0,t) = v(L,t) = 0, \quad t \geq \tau.$$

In this case, we put $Y_2 = H_0^2(0, L)$. Theorem 3.4 and 3.5, as well as their proofs, are remain valid without any changes.

Acknowledgements. The authors are grateful to the anonymous referees for their valuable comments. This work was partly supported by grant 11061030 from the NSFC, grant 0801-02 from the Education Department Foundation of Gansu Province, and the Fundamental Research Funds for the Gansu Universities.

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