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UME'S U-DISTANCE AND ITS RELATION WITH BOTH (PS)-CONDITION AND COERCIVITY

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Abstract. In this article, we study the connection between the u-distance and a new Palais-Smale condition of compactness. We compare this Palais-Smale condition with the coercivity.

1. Introduction and preliminaries

In 1997, Zhong [17, 18] generalized the Ekeland variational principle and proved the existence of minimal points for Gâteaux-differentiable functions under weak (PS) conditions. The following theorem is well-known and we name it Zhong's variational principle (ZVP).

Theorem 1.1 ([17, 18])**.** *Let (X, d) be a complete metric space, $x_0 \in X$ fixed and $f : X \to (-\infty, \infty]$ a proper lower semicontinuous function which is bounded from below. Let $h : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function such that*

$$\int_0^\infty \frac{1}{1 + h(r)}\, dr = +\infty.$$

Then, for every $\varepsilon > 0$, every $y \in X$ such that

$$f(y) < \int_{x\in X} f(x) + \varepsilon,$$

and $\lambda > 0$, there exists some point $z \in X$ such that

(i) $f(z) \le f(y)$,
(ii) $d(x_0, z) \le r_0 + r^*$,
(iii) $f(x) \ge f(z) - \frac{\varepsilon}{\lambda(1+h(d(x_0,z)))} \cdot d(z, x)$, *for all $x \in X$, where $r_0 = d(x_0, y)$, and r^* is such that*

$$\int_{r_0}^{r_0+r^*} \frac{1}{1 + h(t)}\, dt \ge \lambda.$$

In 2010, Ume [15] introduced a new concept of distance called u-distance, which generalizes some distances anterior studied (see e.g., ω-distance [9, 16], Tataru's distance [13], τ-distance [11]) and expanded the celebrated Ekeland's variational principle.

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In Section 2, we present a generalization of Zhong's variational principle using Ume's u-distance. In Section 3, we define a new Palais-Smale condition related to above variational principle and we study the existence of the minimal point for Gâteaux-differentiable functions. In the last section, we deal with the relation between new Palais-Smale condition and the coercivity, following a techniques which is based on u-distance. Our results extend and improve other known results due to Zhong [17, 18], Ekeland [6, 7] and Costa & Silva [5].

For the beginning, we present some results needed in our approach. First, we recall Ume's [15] concept of generalized distance in metric spaces.

Definition 1.2. Let (X, d) be a metric space. A function $p : X \times X \to \mathbb{R}_+$ is called u-distance on X if there exists a map $\Theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that the following conditions hold:

(U1) $p(x, z) \le p(x, y) + p(y, z)$, for all $x, y, z \in X$;

(U2) $\Theta(x, y, 0, 0) = 0$ and $\Theta(x, y, s, t) \ge \min\{s, t\}$ for all $x, y \in X$, $s, t \in \mathbb{R}_+$, and for every $x \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$|\Theta(x, y, s, t) - \Theta(x, y, s_0, t_0)| < \varepsilon$$

if $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in \mathbb{R}_+$ whenever $y \in X$;

(U3) $\lim_n x_n = x$ and $\limsup_n \{\Theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \ge n\} = 0$ imply

$$p(y, x) \le \liminf_{n\to\infty} p(y, x_n) \quad \text{for } y \in X;$$

(U4) The four equalities

$$\limsup_n \{p(x_n, w_m) : m \ge n\} = 0, \quad \limsup_n \{p(y_n, z_m) : m \ge n\} = 0,$$
$$\lim_n \Theta(x_n, w_n, s_n, t_n) = 0, \quad \lim_n \Theta(y_n, z_n, s_n, t_n) = 0$$

imply $\lim_n \Theta(w_n, z_n, s_n, t_n) = 0$; or the four equalitires

$$\limsup_n \{p(w_m, x_n) : m \ge n\} = 0, \quad \limsup_n \{p(z_m, y_n) : m \ge n\} = 0,$$
$$\lim_n \Theta(x_n, w_n, s_n, t_n) = 0, \quad \lim_n \Theta(y_n, z_n, s_n, t_n) = 0$$

imply $\lim_n \Theta(w_n, z_n, s_n, t_n) = 0$;

(U5) The two equalities

$$\lim_n \Theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0,$$
$$\lim_n \Theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0$$

imply $\lim_n d(x_n, y_n) = 0$; or the two equalities

$$\lim_n \Theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) = 0,$$
$$\lim_n \Theta(a_n, b_n, p(z_n, a_n), p(y_n, b_n)) = 0$$

imply $\lim_n d(x_n, y_n) = 0$.

Example 1.3 ([15])**.** Let X be a space with norm $\|\cdot\|$. Then the function $p : X \times X \to \mathbb{R}_+$ defined by $p(x, y) = \|x\|$ is a u-distance on X, but it is not a τ-distance on X, in Suzuki's sense [11].

Example 1.4 ([15])**.** Let p be a u-distance on a metric space (X,d) and let c be a real positive number. Then a function $q : X \times X \to \mathbb{R}_+$ defined by $q(x,y) = cp(x,y)$ for every $x, y \in X$ is also a u-distance on X.

By means of the generalized u-distance, Ume obtained in [15] the following version of Ekeland's variational principle. This result will play a crucial role in the proof of our variational principle.

Theorem 1.5 ([15])**.** *Let (X,d) be a complete metric space, let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function which is bounded from below, and let $p : X \times X \to \mathbb{R}_+$ be a u-distance on X. Then the following two statements hold:*

(1) *For each $x \in X$ with $f(x) < \infty$, there exists $v \in X$ such that $f(v) \le f(x)$ and $f(w) > f(v) - p(v,w)$, for all $w \in X\setminus v\}$.*
(2) *For each $\varepsilon > 0$, $\lambda > 0$ and $x \in X$ with $p(x,x) = 0$ and $f(x) < \inf_{a\in X} f(a) + \varepsilon$, there exists $v \in X$ such that*
$$f(v) \le f(x), \quad p(x,v) \le \lambda,$$
$$f(w) > f(v) - \frac{\varepsilon}{\lambda}\cdot p(v,w), \quad \text{for all } w \in X\setminus\{v\}.$$

2. A generalization of Zhong's variational principle

We start this section by extending a result by Suzuki [12], using the u-distance.

Proposition 2.1. *Let (X,d) be a complete metric space and let $p : X \times X \to \mathbb{R}_+$ be a u-distance on X. Let $q : X \times X \to \mathbb{R}_+$ be a function such that*

(a) *$q(x,z) \le q(x,y) + q(y,z)$ for all $x, y, z \in X$;*
(b) *q is lower semicontinuous in its second argument;*
(c) *$q(x,y) \ge p(x,y)$ for all $x, y \in X$.*

Then q is also a u-distance.

Proof. Assumption (a) is equivalently with $(\mathrm{U1})_q$. Let $\Theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying (U2)–(U5). Clearly, $(\mathrm{U3})_q$ follows from (b). Now, we assume that
$$\begin{gathered}\limsup_n \{q(x_n, w_m) : m \ge n\} = 0,\\ \limsup_n \{q(y_n, z_m) : m \ge n\} = 0,\\ \lim_n \Theta(x_n, w_n, s_n, t_n) = 0,\\ \lim_n \Theta(y_n, z_n, s_n, t_n) = 0.\end{gathered} \tag{2.1}$$
By (2.1) and (c), we have
$$\limsup_n \{p(x_n, w_m) : m \ge n\} = 0,$$
$$\limsup_n \{p(y_n, z_m) : m \ge n\} = 0.$$
Therefore, by (U4), we find $\lim_n \Theta(w_n, z_n, s_n, t_n) = 0$, and derive $(\mathrm{U4})_q$.

Next, we assume that
$$\lim_n \Theta(w_n, z_n, q(w_n, x_n), q(z_n, x_n)) = 0, \tag{2.2}$$
$$\lim_n \Theta(w_n, z_n, q(w_n, y_n), q(z_n, y_n)) = 0. \tag{2.3}$$

Applying again (c) in (2.2) and (2.3), we obtain

$$\lim_n \Theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0,$$
$$\lim_n \Theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0.$$

By (U5), we have $\lim_n d(x_n, y_n) = 0$, and $(\mathrm{U5})_q$, is also verified. □

Next, we establish a more general variational principle [1, 14], which is an extension of both Ekeland's and Zhong's variational principles.

Theorem 2.2. *Let (X, d) be a complete metric space, $a \in X$ be a fixed point and let $p : X \times X \to \mathbb{R}_+$ be a u-distance on X lower semicontinuous in its second argument. Let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function which is bounded from below and let $b : [0, \infty) \to (0, \infty)$ be a non-increasing continuous function such that*

$$B(t) = \int_0^t b(r)dr,$$

where B is a C^1 function from $\mathbb{R}_+$ to itself and $B(\infty) = +\infty$. Let $y \in X$ be such that $p(y, y) = 0$ and

$$f(y) > \inf_{x\in X} f(x). \tag{2.4}$$

Then, for $\epsilon_0 > 0$, there exists $z \in X$ such that

(i) *$f(z) \le f(y)$,*
(ii) *$p(a, z) \le \beta(y) + \beta^*$,*
(iii) *$f(x) > f(z) - \frac{\epsilon_0}{\lambda} b(\beta(z))p(z, x)$, for all $x \in X$ where $\beta(.) = p(a, .)$, and β^* is such that*

$$\int_{\beta(y)}^{\beta(y)+\beta^*} b(t)\, dt \ge \alpha(y), \tag{2.5}$$

with $\alpha(y) = f(y) - \inf x \in X f(x) \ge \lambda > 0$.

Proof. First, we define the function $q : X \times X \to \mathbb{R}_+$ by

$$q(x, y) := \int_{p(a,x)}^{p(a,x)+p(x,y)} b(t)\, dt.$$

Since b is non-increasing, for $(x, z) \in X \times X$, we deduce

$$\begin{aligned} q(x, z) &= \int_{p(a,x)}^{p(a,x)+p(x,z)} b(t)\, dt \\ &\le \int_{p(a,x)}^{p(a,x)+p(x,y)+p(y,z)} b(t)\, dt \\ &= \int_{p(a,x)}^{p(a,x)+p(x,y)} b(t)\, dt + \int_{p(a,x)+p(x,y)}^{p(a,x)+p(x,y)+p(y,z)} b(t)\, dt \\ &\le \int_{p(a,x)}^{p(a,x)+p(x,y)} b(t)\, dt + \int_{p(a,y)}^{p(a,y)+p(y,z)} b(t)\, dt \\ &= q(x, y) + q(y, z). \end{aligned}$$

In addition, q is obviously lower semicontinuous in its second variable. On the other hand, we have

$$\begin{aligned} q(x,y) &= \int_{p(a,x)}^{p(a,x)+p(x,y)} b(t)\,dt \\ &= B(p(a,x)+p(x,y)) - B(p(a,x)) \\ &\geq b(p(a,x)+p(x,y))p(x,y). \end{aligned} \tag{2.6}$$

Taking into account the definition of function b, we obtain boundedness from below,

$$b(p(a,x)+p(x,y)) > b(\infty) \geq M \geq 0 \tag{2.7}$$

Combining (2.6) and (2.7), we deduce

$$q(x,y) \geq Mp(x,y).$$

Since $Mp(x,y)$ is a u-distance and the assumptions of Proposition 2.1 are verified, $q(x,y)$ is also u-distance.

Now, from (2.4) and (2.5), we obtain

$$\begin{aligned} 0 < \lambda \leq f(y) - \inf_{x\in X} f(x) &= \alpha(y) \\ \leq \int_{\beta(y)}^{\beta(y)+\beta^*} b(t)\,dt &= \int_0^{\beta^*} b(u+\beta(y))\,du \\ \leq \int_0^{\beta^*} b(u)du &= B(\beta^*). \end{aligned} \tag{2.8}$$

So by the above inequality,

$$f(y) \leq \inf_{x\in X} f(x) + B(\beta^*),$$

and the Theorem 1.5 is applicable to $q(x,y)$ for $\varepsilon = B(\beta^*) > 0$ and $\lambda = \alpha(y) > 0$. Therefore, there exists $z \in X$ such that

$$f(z) \leq f(y), \tag{2.9}$$

$$q(y,z) \leq \alpha(y) \tag{2.10}$$

$$f(x) > f(z) - \frac{B(\beta^*)}{\alpha(y)} \cdot q(z,x), \quad \forall x \neq z,\ x \in X. \tag{2.11}$$

By (U1), we know that

$$p(a,z) \leq p(a,y) + p(y,z) = \beta(y) + p(y,z). \tag{2.12}$$

On the other hand, from (2.5) and (2.10) it follows that

$$B(\beta(y)+p(y,z)) - B(\beta(y)) \leq \alpha(y) \leq B(\beta(y)+\beta^*) - B(\beta(y)).$$

Thereby, we find that

$$p(y,z) \leq \beta^*, \tag{2.13}$$

because B is a nondecreasing function. Thus, (ii) follows from (2.12) and (2.13). Moreover, since

$$q(z,x) = \int_{p(a,z)}^{p(a,z)+p(z,x)} b(t)\,dt \leq b(p(a,z))p(z,x) = b((\beta(z)))p(z,x); \tag{2.14}$$

multiplying by (-1) and, using (2.8) and (2.11), for $0 < B(\beta^*) \leq \epsilon_0$, we obtain

$$f(x) > \quad f(z) - \frac{B(\beta^*)}{\alpha(y)} \cdot q(z,x) \geq f(z) - \frac{\epsilon_0}{\lambda} q(z,x) \quad \geq f(z) - \frac{\epsilon}{\lambda} b((\beta(z)))p(z,x),$$

for all $x \in X$, and (iii) is verified. This completes the proof. $\square$

Remark 2.3. Let a, f, b, p, $\alpha(y)$, $\beta(y)$, β^*, and X be as in Theorem 2.2.

(i) When $a = y$, $b(t) \equiv 1$, $\beta^* = \lambda$, $\epsilon_0 > \alpha(y) \geq \lambda > 0$, and $p(x,y) = d(x,y)$, Theorem 2.2 reduces to Ekeland's variational principle (EVP) [6, 7].

(ii) Take $a = x_0$,
$$b(t) = \frac{1}{1+h(t)},$$
where $h : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function such that
$$\int_0^\infty \frac{1}{1+h(r)} dr = +\infty,$$
$\epsilon_0 > \alpha(y) \geq \lambda > 0$, $\beta(y) = d(x_0, y) = r_0$, $\beta^* = r^*$ and $p(x,y) = d(x,y)$. Therefore, Theorem 2.2 implies Theorem 1.1.

3. The b-(PS) condition and the existence of a minimal point

Throughout this section X denotes a Banach space. We recall that a function $f : X \to (-\infty,\infty]$ is called Gâteaux differentiable at $x \in X$ with $f(x) < \infty$ if there exists a continuous linear functional $f'(x)$ such that
$$\lim_{t\to 0} \frac{f(x+ty) - f(x)}{t} = \langle f'(x), y\rangle$$
holds for every $y \in X$.

In the following, we assume that $f : X \to (-\infty,\infty]$ is Gâteaux differentiable.

Theorem 3.1. *Let $a \in X$ be fixed and $p : X \times X \to \mathbb{R}_+$ a u-distance on X lower semicontinuous in its second argument. Let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function which is bounded from below and let $b : [0,\infty) \to (0,\infty)$ be a nonincreasing continuous function such that*
$$B(t) = \int_0^t b(r) dr,$$
where B is a C^1 function from $\mathbb{R}_+$ to itself such that $B(\infty) = +\infty$. Let $y \in X$ be such that $p(y,y) = 0$ and
$$f(y) > \inf_{x\in X} f(x).$$
Then, for every $\epsilon > 0$, there exists $z \in X$ such that

(i') *$f(z) \leq f(y)$,*

(ii') *$\beta(z) \leq \beta(y) + \beta^*$,*

(iii') *$\|f'(z)\|/b(\beta(z)) \leq \epsilon$ for all $x \in X$, where $\beta(.) = p(a,.)$, and β^* is a real number such that*
$$\int_{\beta(y)}^{\beta(y)+\beta^*} b(t)\, dt \geq \alpha(y),$$
with $\alpha(y) = f(y) - \inf_{x\in X} f(x) > 0$.

Proof. We have the hypotheses of Theorem 2.2. So, applying this theorem, we obtain (i') and (ii') from (i) and (ii). Moreover, (iii) guaranties that there exists $z \in X$ such that
$$f(x) \geq f(z) - \frac{\epsilon}{\lambda} b(\beta(z)) p(z,x), \quad \text{for all } x \in X, \tag{3.1}$$

where $0 < \lambda \le \alpha(y)$. Choose $x = z + ty$ with $\|y\| = 1$ in (3.1) and obtain

$$\frac{f(z+ty) - f(z)}{t} \ge -\frac{\epsilon}{\lambda}\frac{b(\beta(z))p(z, z+ty)}{t}, \tag{3.2}$$

for every $t > 0$. Let λ be such that

$$\lim_{t\to 0} \frac{p(z, z+ty)}{t} \le \lambda. \tag{3.3}$$

Then, letting $t \to 0$ in (3.2) and using (3.3), we conclude that

$$\langle f'(z), y\rangle \ge -\epsilon \cdot b(\beta(z)), \tag{3.4}$$

for all $y \in X$ with $\|y\| = 1$. Since (3.4) is true for $\pm y$, we deduce that

$$|\langle f'(z), y\rangle| \le \epsilon \cdot b(\beta(z)). \tag{3.5}$$

Now, from (3.5), we obtain

$$\|f'(z)\| = \sup_{y\in X, \|y\|=1} \frac{|\langle f'(z), y\rangle|}{\|y\|} \le \epsilon \cdot b(\beta(z)),$$

and the claim (iii') holds. □

Corollary 3.2. *Suppose that the hypotheses of Theorem 3.1 are verified. Then there exists a minimizing sequence* $\{z_n\}_n$ *of* f *such that*

$$f(z_n) < \inf_{x\in X} f(x) + \epsilon,$$
$$\|f'(z_n)\|/b(\beta(z_n)) \to 0.$$

The proof follows form taking $\epsilon = \frac{1}{n}$, $n = 1, 2, \dots$ in Theorem 3.1.

Let $\mathcal{B}$ be the set of all non-increasing and strictly positive continuous functions $b : [0, \infty) \to (0, \infty)$ such that

$$\int_0^\infty b(t)\, dt = \infty.$$

Let $p : X \times X \to \mathbb{R}_+$ be a u-distance on X lower semicontinuous in its second variable with $p(x, x) = 0$ $\forall x \in X$, $a \in X$ a fixed point and $\beta : X \to \mathbb{R}_+$ defined by $\beta(x) = p(a, x)$.

Definition 3.3. Let $f : X \to (-\infty, +\infty]$ be a C^1 function, $c \in \mathbb{R}$ and $b \in \mathcal{B}$.

- f is said to satisfy the b-(PS) condition if any sequence $\{x_n\}_n$ in X such that $\{f(x_n)\}$ is bounded and $\|f'(x_n)\|/b(\beta(x_n)) \to 0$ has a convergent subsequence.
- f is said to satisfy the b-(PS)$_c$ condition if any sequence $\{x_n\}_n$ in X such that $f(x_n) \to c$ and $\|f'(x_n)\|/b(\beta(x_n)) \to 0$ has a convergent subsequence.

Remark 3.4. Suppose that $\beta(x) = d(a, x)$.

- Then the b-(PS) condition is the Schechter-(PS) condition [10].
- If b is constant, then the b-(PS) condition is the usual (PS) condition.
- If $b(t) = 1/(1 + t)$, then the b-(PS) condition is the Cerami-(PS) condition [4].
- If $b(t) = 1/(1+h(t))$, where $h : [0, \infty) \to [0, \infty)$ is a non-decreasing function, then the b-(PS) condition is the Zhong-(WPS) condition [17, 18].

Theorem 3.5. *If* f *is bounded below and satisfying the* b-(PS) *condition, then* f *has a minimal point.*

Proof. By Corollary 3.2, there is a minimizing sequence $\{z_n\}_n$ in X such that $f(z_n) < \inf_{x\in X} f(x) + \epsilon$ and $\|f'(z_n)\|/b(\beta(z_n)) \to 0$. The b-(PS) condition implies that $\{z_n\}_n$ has a subsequence $\{z_{n_k}\}_k$ convergent to some point z^*. Since f is lower semicontinuous, we obtain

$$\inf_X f \le f(z^*) \le \liminf_{k\to\infty} f(z_{n_k}) \le \inf_X f.$$

Therefore, $f(z^*) = \inf_X f$. ☐

4. The b-(PS) condition versus coercivity

Using the method of gradient flows, Li [8] first observed that the (PS) condition implies the coercivity for C^1 functionals bounded from below. Using Ekeland's variational principle, Caklovic, Li and Willem [3] proved the same result for a Gâteaux differentiable functional which is lower semicontinuous. The same conclusion was also proved by Costa and Silva [5] and Brezis and Nirenberg [2] for C^1functionals by also employing Ekeland's principle. Using ZVP, Zhong [17] studied the connection between (WPS) and coercivity. A similar result was established by Suzuki [11], using τ-distance.

In this section, we discuss the relation between the b-(PS) condition and coercivity. We recall that a function $f : X \to (-\infty, \infty]$ is said to be coercive if

$$\lim_{r\to\infty} \inf_{\|x\|\ge r} f(x) = \infty.$$

For our aim, we first prove the following lemma.

Lemma 4.1. *Let $p : X \times X \to \mathbb{R}_+$ be a u-distance on X and $f : X \to \mathbb{R}$ is a Gâteaux differentiable function. Suppose that there are $\xi \ge 0$, $\delta > 0$ and either of the following conditions is satisifed:*

- *$f(y) \ge f(x) - \xi p(x, y)$ for all $y \in X$ with $0 < p(x, y) < \delta$; or*
- *$f(y) \le f(x) + \xi p(x, y)$ for all $y \in X$ with $0 < p(x, y) < \delta$.*

Then $\|f'(x)\| \le \xi$.

Proof. Assume that

$$f(y) \ge f(x) - \xi p(x, y) \tag{4.1}$$

for all $y \in X$ with $0 < p(x, y) < \delta$. Set $y = x + \delta z$ in (4.1), and infer that

$$f(x + \delta z) - f(x) \ge -\xi p(x, x + \delta z) > -\xi\delta \tag{4.2}$$

Then,

$$\frac{f(x + \delta z) - f(x)}{\delta} > -\xi. \tag{4.3}$$

Taking the limit as $\delta \to 0$, we obtain

$$\langle f'(x), y\rangle \ge -\xi. \tag{4.4}$$

As (4.4) holds for both of $\pm y$, we derive

$$|\langle f'(x), y\rangle| \le \xi. \tag{4.5}$$

Then, for all $y \in X$ with $\|y\| = 1$, the inequality (4.5) implies that

$$\|f'(x)\| = \sup_{y\in X,\|y\|=1} \frac{|\langle f'(x), y\rangle|}{\|y\|} = \sup_{y\in X,\|y\|=1} |\langle f'(x), y\rangle| \le \xi,$$

and the desired claim holds. ☐

Next, we consider a more suitable version of Theorem 1.5, for our purpose.

Theorem 4.2. *Let (X,d) be a complete metric spaces, let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function which is bounded from below, and let $p : X\times X \to \mathbb{R}_+$ be a u-distance on X lower semicontinuous in its second argument. Then for $\varepsilon > 0$ and $x \in X$ with $f(x) < \infty$ and $p(x,x) = 0$, there exists $v \in X$ such that*

(i) $f(v) \le f(x) - \varepsilon p(x,v)$;
(ii) $f(w) > f(v) - \varepsilon p(v,w)$, *for all* $w \in X\backslash\{v\}$.

For the sake of completeness, we supply a proof of the equivalence between Theorems 1.5 and 4.2.

Proof. $\Leftarrow$ Let the assumptions of Theorem 1.5 be satisfied. Obviously, the conclusion of (1) follows by Theorem 4.2. For (2), applying again Theorem 4.2 with $\varepsilon = \frac{e}{\lambda}$, we deduce that

$$p(x,v) \le \frac{\lambda}{e}(f(x) - f(v)) \le \frac{\lambda}{e} e \le \lambda.$$

Hence the conclusion of Theorem 1.5 is valid.

$\Rightarrow$ Now, suppose that Theorem 1.5 holds. Let $x \in X$ with $f(x) < \infty$ and $\varepsilon > 0$ be given. Fix any $e > f(x) - \inf_{a\in X} f(a)$ and set $\lambda = \frac{e}{\varepsilon}$. Consider

$$M(x) = \{v \in X \mid f(v) \le f(x) - \varepsilon p(x,v)\}.$$

By the lower semicontinuity of f and $p(x,.)$, the set $M(x)$ is closed. Furthermore, $M(x)$ is nonempty as $x \in M(x)$. Applying Theorem 1.5 (2) for the chosen e, λ and for $M(x)$ instead of X one finds $v \in X$ such that

$$f(v) \le f(x), \quad p(x,v) \le \lambda,$$
$$f(w) > f(v) - \frac{e}{\lambda}\cdot p(v,w), \quad \text{for all } w \in M(x)\backslash\{v\}.$$

Since $v \in M(x)$, then (i) holds.

To show (ii) it is sufficient to check that

$$f(w) > f(v) - \frac{e}{\lambda}\cdot p(v,w), \quad \text{for all } w \notin M(x).$$

By the definition of $M(x)$, the property $w \notin M(x)$ means that

$$f(w) > f(x) - \varepsilon p(x,w).$$

From this and (i) we easily deduce (ii) and then obtain Theorem 4.2. □

We are in position to state the main result of this section. The proof follows a technique developed by Suzuki in [11].

Theorem 4.3. *Let X be a Banach space, $a \in X$ fixed, and let $p : X \times X \to \mathbb{R}_+$ be a symmetric u-distance on X, lower semicontinuous in its second argument and such that $p(x,x) = 0$ for all $x \in X$. Let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous function which is bounded from below and let $b : [0,\infty) \to (0,\infty)$ be a non-increasing continuous function such that*

$$B(t) = \int_0^t b(r)\,dr,$$

where B is a function from $\mathbb{R}_+$ to itself such that $B(\infty) = +\infty$. Let $a \in X$ be fixed and $\beta : X \to \mathbb{R}_+$ defined by $\beta(x) = p(a, x)$. Assume that f is Gâteaux differentiable at every point $x \in X$ with $f(x) \in \mathbb{R}$. If

$$\alpha = \liminf_{\beta(y)\to\infty} f(y) \in \mathbb{R},$$

then there exists a sequence $\{z_n\}_n$ in X such that

(a) $\lim_{n\to\infty} \beta(z_n) = \infty$;
(b) $\lim_{n\to\infty} f(z_n) = \alpha$;
(c) $\lim_{n\to\infty} \|f'(z_n)\|/b(\beta(z_n)) = 0$.

Proof. We shall show only the following: for every $\varepsilon > 0$, there exists $v \in X$ satisfying $\beta(v) \geq \frac{1}{\varepsilon}$, $|f(v) - \alpha| \leq \varepsilon$ and $\|f'(v)\|/b(\beta(v)) \leq \varepsilon$. Fix $\varepsilon > 0$ and define a function $\chi : [0, \infty) \to [0, \infty)$ by

$$\chi(t) = \frac{1}{2} b(t + 1) \tag{4.6}$$

for $t \in [0, \infty)$. Then χ is non-increasing, and

$$\int_0^\infty \chi(t)\,dt = \frac{1}{2}\int_0^\infty b(t + 1)\,dt = \frac{1}{2}\int_1^\infty b(t)\,dt = \infty.$$

We also determine a function $h : X \to (-\infty, +\infty]$ by

$$h(x) = \max\{f(x), \alpha - 2\varepsilon\} \tag{4.7}$$

for $x \in X$. Then it is obvious that h is proper lower semicontinuous and bounded from below. We choose $r, r' \in \mathbb{R}$ with $\frac{1}{\varepsilon} < r < r'$, $1 < r$,

$$\inf_{\beta(y)\geq r} f(y) > \alpha - \varepsilon, \tag{4.8}$$

$$\int_r^{r'} \chi(t)\,dt = 3. \tag{4.9}$$

We also choose $u \in X$ with

$$\beta(u) > r', \quad f(u) < \alpha + \varepsilon. \tag{4.10}$$

We note that $h(u) = f(u)$ because $\beta(u) > r' > r$. We know from the earlier that the function $q : X \times X \to \mathbb{R}_+$, defined by

$$q(u, v) = \int_{\beta(u)}^{\beta(u)+p(u,v)} \chi(t)\,dt \tag{4.11}$$

is a u-distance. So, by Proposition 2.1, the function $s : X \times X \to \mathbb{R}_+$, defined by

$$s(u, v) = q(u, v) + q(v, u) \tag{4.12}$$

is also a u-distance. Thereby, by Theorem 4.2, there exists $v \in X$ such that

$$h(v) \leq h(u) - \varepsilon s(u, v), \tag{4.13}$$

$$h(w) > h(v) - \varepsilon s(v, w), \quad \forall w \neq v. \tag{4.14}$$

Arguing by contradiction, we assume that $\beta(v) < r$. Moreover, we have

$$\beta(v) < r < r' < \beta(u). \tag{4.15}$$

Also, from (4.11), (4.12) and (4.13), we successively obtain

$$\begin{aligned} \alpha-2\varepsilon \le h(v) &\le h(u)-\varepsilon\int_{\beta(u)}^{\beta(u)+p(u,v)}\chi(t)\,dt-\varepsilon\int_{\beta(v)}^{\beta(v)+p(u,v)}\chi(t)\,dt \\ &\le h(u)-\varepsilon\int_{\beta(v)}^{\beta(v)+p(u,v)}\chi(t)\,dt \\ &\le h(u)-\varepsilon(H(\beta(v)+p(u,v))-H(\beta(v))), \end{aligned} \tag{4.16}$$

where H is a primitive of χ. Using that H is nondecreasing in (4.16), we obtain

$$\alpha-2\varepsilon \le h(u)-\varepsilon(H(\beta(u))-H(\beta(v))) = h(u)-\varepsilon\int_{\beta(v)}^{\beta(u)}\chi(t)\,dt. \tag{4.17}$$

Then, by (4.15), (4.17) and (4.10), we obtain

$$\alpha-2\varepsilon \le h(u)-\varepsilon\int_{r}^{r'}\chi(t)\,dt = f(u)-3\varepsilon < \alpha-2\varepsilon,$$

which is a contradiction. Therefore,

$$\beta(v)\ge r>\frac{1}{\varepsilon},$$

and (a) holds. Thus, we have $h(v)=f(v)$ and

$$\alpha-\varepsilon<\inf_{\beta(y)\ge r} f(y)\le f(v)\le f(u)<\alpha+\varepsilon.$$

This implies

$$|f(v)-\alpha|\le\varepsilon,$$

that is (b). For (c), from (4.11) , (4.12) and (4.14) and the non-increasing property of χ, we infer

$$\begin{aligned} h(w) &> h(v)-\varepsilon\int_{\beta(v)}^{\beta(v)+p(v,w)}\chi(t)\,dt-\varepsilon\int_{\beta(w)}^{\beta(w)+p(v,w)}\chi(t)\,dt \\ &\ge h(v)-\varepsilon(\chi(\beta(v))+\chi(\beta(w)))\cdot p(v,w), \end{aligned} \tag{4.18}$$

for $w\in X$, $w\ne v$. Since f is lower semicontinuous and $f(v)>\alpha-2\varepsilon$, there exists $\delta\in(0,1)$ such that $f(w)>\alpha-2\varepsilon$ for $w\in X$ with $p(v,w)<\delta$. Hence, for $w\in X$ with $0<p(v,w)<\delta$, since $h(w)=f(w)$ and

$$\beta(w)=p(a,w)\ge p(a,v)-p(w,v)>\beta(v)-\delta>\beta(v)-1>0,$$

we derive

$$\begin{aligned} f(w) &> f(v)-\varepsilon(\chi(\beta(v))+\chi(\beta(v)-1))\cdot p(v,w) \\ &\ge f(v)-2\varepsilon\chi(\beta(v)-1)\cdot p(v,w) \\ &= f(v)-\varepsilon b(\beta(v))\cdot p(v,w). \end{aligned} \tag{4.19}$$

By means of Lemma 4.1, we reach

$$\|f'(v)\|\le\varepsilon b(\beta(v)),$$

and (c) is verified too. The proof is complete. □

Corollary 4.4. *Let X be a Banach space. Let $f: X\to(-\infty,\infty]$ be a proper lower semicontinuous function which is bounded from below. Assume that f is Gâteaux differentiable at every point $x\in X$ with $f(x)\in\mathbb{R}$. If f satisfies the b-$(PS)_c$ condition for all $c\in\mathbb{R}$, then f is coercive; i.e., $f(x)\to\infty$ as $\beta(x)\to\infty$.*

Proof. Suppose the contrary; then $\alpha = \liminf_{\beta(x)\to\infty} f(x) \in \mathbb{R}$. By Theorem 4.3, there exists a sequence $\{z_n\}_n$ in X such that $\beta(z_n) \to \infty$, $f(z_n) \to \alpha$ and $\|f'(z_n)\|/b(\beta(z_n)) \to 0$. Then, the b-(PS)$_\alpha$ condition implies that $\{z_n\}_n$ has a convergent subsequence, which clearly leads to a contradiction. □

Remark 4.5. Corollary 4.4 generalizes the result proved by [8] using a gradient flow, by Costa-Silva [5], Caklovic-Li-Willem [3] and Brezis-Nirenberg [2] using EVP, and by Zhong [17] using ZVP.

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