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GOURSAT PROBLEM FOR THE YANG-MILLS-VLASOV SYSTEM IN TEMPORAL GAUGE

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Abstract. This article studies the characteristic Cauchy problem for the Yang-Mills-Vlasov (YMV) system in temporal gauge, where the initial data are specified on two intersecting smooth characteristic hypersurfaces of Minkowski spacetime $(\mathbb{R}^4, \eta)$. Under a $\mathcal{C}^\infty$ hypothesis on the data, we solve the initial constraint problem and the evolution problem. Local in time existence and uniqueness results are established thanks to a suitable combination of the method of characteristics, Leray's Theory of hyperbolic systems and techniques developed by Choquet-Bruhat for ordinary spatial Cauchy problems related to (YMV) systems.

1. Introduction

The purpose of this article is to solve, locally in time in the Minkowski spacetime $(\mathbb{R}^4, \eta)$, the Cauchy problem for the Yang-Mills-Vlasov (YMV) system, where the initial data are prescribed on two intersecting smooth characteristic hypersurfaces. Such problems are often referred to as non-linear Goursat problems [2, 3, 4, 8, 9, 10, 12, 13, 14, 21, 22, 23, 26, 27, 29, 30].

The Maxwell-Vlasov system and its generalization to the non Abelian charge provided by the (YMV) system play a fundamental role in numerous physical situations; possibly coupled with Einstein Equations, they govern the dynamic of various species of plasma in the absence of collisions. A plasma is a collection of charged particles of various species moving at high speed under forces which they have generated and maintained. The statistical distribution of these particles is described by a density function of particles subjected to the Vlasov equation. Here the particles have a non Abelian charge called a Yang-Mills charge and are submitted to forces generated by a Yang-Mills field which is solution of Yang-Mills Equations with a current generated by the density of particles. According to physicists, the matter which is lying in the Universe is almost made up with plasmas such as the interior of a star, reactor in fusion, ionosphere, solar winds, nebulous galaxies, the plasmas of quarks and gluons of the primordial Universe.

Maxwell-Vlasov or Yang-Mills-Vlasov systems, possibly coupled with Einstein Equations, are mostly studied by authors in the setting of ordinary spatial Cauchy

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problems [1, 7, 15, 16, 17, 18, 19, 20, 24]. It is however important and useful to consider the study of these same equations in the setting of a Goursat problem; i.e., where the initial hypersurface is a characteristic cone [5], or the union of two intersecting smooth characteristic hypersurfaces. Indeed initial data on a characteristic cone correspond to a "physical" Cauchy problem more natural than the ordinary spatial Cauchy problem, because they provide an ideal mathematical representation of the measure at the present moment of the physical field studied. Goursat problems arise also naturally in delicate physical situations where radiation phenomena appear. In this latter case the solutions considered must be global or semi-global since they must be defined in a neighborhood of null infinity.

The Goursat problem for the (YMV) system splits into two sub-problems: the initial constraint problem and the evolution problem. For a suitable choice of free data, we solve globally the initial constraint problem thanks to a hierarchy of algebraic-integral-differential relations deduced from the (YMV) system. Thereafter the solutions so obtained are used as initial data for the evolution problem. The fundamental partial differential equations (PDE) for the evolution problem consist of the Vlasov equation (verified by the density of particles f) and a hyperbolic symmetric system of first order extracted from the Yang-Mills equations and the related Bianchi identities.

Thanks to domain of dependence arguments, we transform this problem into an ordinary Cauchy problem with zero data on a spatial hypersurface, which we solve using a suitable combination of the classical method of characteristics, Leray's Theory of hyperbolic systems [25] and techniques developed in [7] for the ordinary spatial Cauchy problems associated with (YMV) systems.

For sake of clarity and simplicity, the Goursat problem considered here for the (YMV) system in temporal gauge is studied under $\mathcal{C}^\infty$ assumptions on the data. The study of solutions of finite differentiability class would normally require the use of a functional setting of non isotropic weighted Sobolev spaces defined by cumbersome norms, which would considerably complicate the analysis of this problem.

The work is subdivided in four sections. In section 2, we set the geometrical framework and describe the PDE under consideration. Section 3 is devoted to the resolution of the initial constraint problem. Section 4 is devoted to the determination of the restrictions, to both initial characteristic hypersurfaces, of the derivatives of all order of the possible $\mathcal{C}^\infty$ solution of the evolution problem; this is an important step towards the transformation of the problem under consideration into an ordinary Cauchy problem with zero initial data. The concern of section 5 is the resolution of the evolution problem.

2. Geometric setting, equations and mathematical formulation

2.1. Geometric setting and the unknown functions.

Throughout this article, we use the Einstein summation convention of repeated indices, e. g.,

$$a_\alpha b^\alpha = \sum_\alpha a_\alpha b^\alpha; \quad \alpha = 0, 1, 2, 3.$$

Unless otherwise is specified, Greek indexes range from 0 to 3 and Latin ones from 1 to 3.

The fundamental geometric setting of this work is the Minkowski spacetime $(\mathbb{R}^4, \eta)$, where the Minkowski metric η is of signature $(+, -, -, -)$. Let $(x^\alpha) = (x^0, x^i)$, the global canonical coordinates system on $\mathbb{R}^4$, where $x^0 = t$ is the time coordinate and the x^i are the spatial coordinates. Let $\sqcup$ be a compact domain of $\mathbb{R}^4$ with boundary $\partial\sqcup$ which is contained in the half-space $x^0 \geq 0$. Let H_1 and H_2 be two hypersurfaces defined as follows

$$H_1 = \{(x^0, x^1, x^2, x^3) \in \sqcup,\ x^0 + x^1 = 0,\ x^0 \geq 0\},$$
$$H_2 = \{(x^0, x^1, x^2, x^3) \in \sqcup,\ x^0 - x^1 = 0,\ x^0 \geq 0\}.$$

Set $H = H_1 \cup H_2$ and

$$I = H_1 \cap H_2 = \{(x^0, x^1, x^2, x^3) \in \sqcup, x^0 = x^1 = 0\}.$$

Denote by B the unique compact domain of $\mathbb{R}^2$ such that

$$I = \{0, 0, x^2, x^3) \in \mathbb{R}^4,\ (x^2, x^3) \in B\}.$$

It is assumed that $H \subset \partial\sqcup$ and that $\partial\sqcup\backslash H$ is a hypersurface of $\sqcup$, piecewise smooth, spatial or null at each of its points, with unit normal exterior to the domain $\sqcup$ which is future oriented. For every $t \geq 0$, set

$$\sqcup_t = \{(x^0, x^1, x^2, x^3) \in \sqcup : x^0 \leq t\};$$
$$\omega_t = \sqcup \cap \{x^0 = t\}; \quad I_t^r = H_r \cap \{x^0 = t\},\ r = 1, 2.$$

Denote by A a Yang-Mills potential represented by a 1-form on $\sqcup$ which takes its values in an $N-$dimensional real Lie algebra $\mathbf{K}$ of a Lie group G, endowed with an Ad- invariant scalar product denoted by a dot (.).

In the global canonical coordinates (x^α) of $\mathbb{R}^4$ and an orthonormal basis (ε_a) of $\mathbf{K}$, A reads:

$$A = A_\alpha dx^\alpha,\ \text{with } A_\alpha = A_\alpha^a \cdot \varepsilon_a,\ a = 1, 2, \dots, N.$$

We say that A verifies the temporal gauge condition if $A_0 = 0$ in $\sqcup$. Denote by F the Yang-Mills field associated to A, i. e., the curvature of A. It is represented by a $\mathbf{K}$-valued antisymmetric 2-form of type Ad, defined on $\sqcup$ by

$$F = dA + \frac{1}{2}[A, A]. \tag{2.1}$$

The $2-$ form F in $\sqcup$ is of type Ad, which means that, if $F_{(i)}$ and $F_{(j)}$ are respectively the representatives in gauges s_i and s_j of the $2-$ form F, then the relations between these two representatives is $F_{(i)} = Ad(u_{ij}^{-1})F_{(j)}$ where u_{ij}^{-1} is roughly the transition function between the two gauges s_i and s_j.

In a global canonical coordinates (x^α) and in the basis (ε_a), (2.1) reads

$$F_{\alpha\beta}^a = \partial_\alpha A_\beta^a - \partial_\beta A_\alpha^a + [A_\alpha, A_\beta]^a, \tag{2.2}$$

with

$$F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \quad [A_\alpha, A_\beta]^a \equiv C_{bc}^a A_\alpha^b A_\beta^c.$$

where the C_{bc}^a are the structure constants of the Lie group G and $[,]$ denotes the Lie brackets of the Lie algebra $\mathbf{K}$.

The 2-form F verifies Bianchi identities

$$\widehat{\nabla}_\alpha F_{\beta\mu} + \widehat{\nabla}_\beta F_{\mu\alpha} + \widehat{\nabla}_\mu F_{\alpha\beta} \equiv 0, \tag{2.3}$$

and the relation

$$\widehat{\nabla}_\alpha \widehat{\nabla}_\beta F^{\alpha\beta} \equiv 0,$$

where $\widehat{\nabla}_\alpha$ is the gauge covariant derivative defined by

$$\widehat{\nabla}_\alpha = \partial_\alpha + [A_\alpha,] \quad \text{with } \partial_\alpha = \frac{\partial}{\partial x^\alpha}.$$

The trajectories of a particle with momentum p and charge q in a Yang-Mills field F defined in $(\sqcup, \eta)$ verify the differential system

$$\begin{aligned}\frac{dx^\alpha}{ds} &= p^\alpha \\ \frac{dp^\alpha}{ds} &= p^\beta q.F^\alpha_\beta \\ \frac{dq^a}{ds} &= -p^\alpha [A_\alpha, q]^a.\end{aligned} \tag{2.4}$$

This system expresses the fact that tangent vector Y to the trajectory of a particle in $\mathbf{P} = T\sqcup \times K$ is $Y = (p, P, Q)$, with

$$p = (p^\alpha), \quad P = (P^\alpha) \equiv (p^\beta q.F^\alpha_\beta), \quad Q = (Q^a) \equiv (-p^\alpha C^a_{bc} q^c A^b_\alpha).$$

If the particles have the same rest mass m and a charge q of given size e, then their phase space, that is the domain described by their trajectories is a subset $\mathbf{P}_{m,e}$ of $T\sqcup \times K$ with equations

$$p^0 = \Big(m^2 + \sum_{i=1}^{3} (p^i)^2\Big)^{1/2}, \quad q.q = e. \tag{2.5}$$

Coordinates system on $\mathbf{P}_{m,e}$ is given by

$$x^0 \equiv t, x^i, p^i, q^L, \quad i = 1, 2, 3;\ L = 1, 2, \dots, N-1.$$

$\mathbf{P}_{m,e}$ can then be identified with $\sqcup \times \mathbb{R}^3 \times O \equiv \widehat{\sqcup}$.

Set

$$P_x = \{x\} \times \mathbb{R}^3 \times O \simeq \mathbb{R}^3 \times O.$$

Let f be a distribution (or density) function for charged particles, that is a positive scalar function defined on $\mathbf{P}_{m,e}$. f satisfies the Vlasov equation if in the coordinates (x^0, x^i, p^i, q^L) considered on $\mathbf{P}_{m,e}$, it holds that

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + P^i \frac{\partial f}{\partial p^i} + Q^L \frac{\partial f}{\partial q^L} = 0, \quad L = 1, 2, \dots, N-1. \tag{2.6}$$

The current generated by the distribution function f of particles is represented by a $\mathbf{K}$- valued vector field J, which is of type Ad by gauge transformation, defined at a point $x \in \sqcup$ by

$$J^\beta(x) = \int_{\mathbf{P}_x} p^\beta q f \omega_p \omega_q, \tag{2.7}$$

where $\omega_p = \frac{1}{p_0} dp^1 dp^2 dp^3$ is the Leray form induced by the volume element $dp^0 dp^1 dp^2 dp^3$ on $T_x\sqcup$ and ω_q is the Leray form induced on O by an Ad-invariant volume element on $\mathbf{K}$.

Remark 2.1. As the Lie algebra $\mathbf{K}$ has an Ad-invariant scalar product, if f satisfies the Vlasov equation in $\sqcup$, then we have

$$\widehat{\nabla}_\alpha J^\alpha = 0 \quad \text{in } \sqcup\,.$$

2.2. **Definition of Yang-Mills-Vlasov system.** The Yang-Mills equations read

$$\widehat{\nabla}_\alpha F^{\alpha\beta} = J^\beta \quad \text{in } \sqcup,$$

where

$$\widehat{\nabla}_\alpha F^{\alpha\beta} \equiv \partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}].$$

By definition, the "complete Yang-Mills-Vlasov system" is the following system defined in $\widehat{\sqcup}$ and with unknown function (A, F, f),

$$\begin{gathered} \widehat{\nabla}_\alpha F^{\alpha\beta} = J^\beta \\ \widehat{\nabla}_\alpha F_{\beta\mu} + \widehat{\nabla}_\beta F_{\mu\alpha} + \widehat{\nabla}_\mu F_{\alpha\beta} = 0 \\ p^\alpha \frac{\partial f}{\partial x^\alpha} + P^i \frac{\partial f}{\partial p^i} + Q^L \frac{\partial f}{\partial q^L} = 0 \end{gathered} \tag{2.8}$$

By definition, the reduced system, in temporal gauge, extracted from (2.8) is the system of unknown function (A, F, f), defined in $\widehat{\sqcup}$ by

$$\begin{gathered} A_0 = 0 \\ \widehat{\nabla}_\alpha F^{\alpha i} = J^i \\ \widehat{\nabla}_0 F_{ij} + \widehat{\nabla}_i F_{j0} + \widehat{\nabla}_j F_{0i} = 0 \\ p^\alpha \frac{\partial f}{\partial x^\alpha} + P^i \frac{\partial f}{\partial p^i} + Q^L \frac{\partial f}{\partial q^L} = 0 \\ \partial_0 A_i = F_{0i}, \end{gathered} \tag{2.9}$$

where $i, j = 1, 2, 3$; $\alpha = 0, 1, 2, 3$; $L = 1, 2, \dots, N-1$.

The evolution problem for the (YMV) system in temporal gauge with initial data prescribed on the two intersecting smooth characteristic hypersurfaces H_1 and H_2 consists in solving the reduced system (2.9) under the initial conditions:

$$A_i\big|_H = a_i, \quad F^{0i}\big|_H = b^i, \quad F_{ij}\big|_H = \Phi_{ij}, \quad f\big|_{\widehat{H}} = \varphi \tag{2.10}$$

where $\widehat{H} = H \times \mathbb{R}^3 \times O$.

The initial constraint problem consists in studying how to generally prescribe the initial data of the conditions (2.10) such that the unique solution of the evolution problem (2.9), (2.10) is also solution of the complete system (2.8) of the Yang-Mills-Vlasov Equations (and satisfies the temporal gauge condition).

3. The initial constraint problem

The following useful notation will be needed. For every function (or tensor field) v defined in the domain $\sqcup$, we denote by $[v]_r$ the restriction to H_r of v, $r = 1, 2$, and $[v]$ the restriction to H of v; i.e,

$$\begin{gathered} [v]_1(x^1, x^2, x^3) = v(-x^1, x^1, x^2, x^3) \quad \text{on } H_1, \\ [v]_2(x^1, x^2, x^3) = v(x^1, x^1, x^2, x^3) \quad \text{on } H_2, \\ [v](x^1, x^2, x^3) = v(|x^1|, x^1, x^2, x^3) \quad \text{on } H. \end{gathered}$$

Theorem 3.1. *Consider $V = (A_0 \equiv 0, A_i, F^{0i}, F_{ij}, f)$ a $\mathcal{C}^\infty$ solution, defined in a neighborhood $\widetilde{\sqcup}$ of $\widehat{H}$ in $\widehat{\sqcup}$, of the reduced system* (2.9) *defined above.*

(1) Set

$$\begin{gathered} A_i\big|_H = a_i = \begin{cases} \overline{a}_i & \text{on } H_1 \\ \widetilde{a}_i & \text{on } H_2; \end{cases} \quad F^{0i}\big|_H = b^i = \begin{cases} \overline{b}^i & \text{on } H_1 \\ \widetilde{b}^i & \text{on } H_2; \end{cases} \\ F_{ij}\big|_H = \Phi_{ij} = \begin{cases} \overline{\Phi}_{ij} & \text{on } H_1 \\ \widetilde{\Phi}_{ij} & \text{on } H_2; \end{cases} \quad f\big|_{\widehat{H}} = \varphi = \begin{cases} \overline{\varphi} & \text{on} \widehat{H}_1 \\ \widetilde{\varphi} & \text{on } \widehat{H}_2, \end{cases} \end{gathered} \tag{3.1}$$

where $i, j = 1, 2, 3$. Then the functions $\overline{a}_i, \overline{b}^i$ and $\overline{\Phi}_{ij}$ (resp. $\widetilde{a}_i, \widetilde{b}^i$ and $\widetilde{\Phi}_{ij}$) are $\mathcal{C}^\infty$ on H_1 (resp. H_2) and $\overline{\varphi}$ (resp. $\widetilde{\varphi}$) is $\mathcal{C}^\infty$ on $\widehat{H}_1$ (resp. $\widehat{H}_2$). Furthermore, these functions satisfy the following compatibility conditions:

$$\begin{gathered} \overline{a}_i = \widetilde{a}_i, \overline{b}^i = \widetilde{b}^i \quad \text{on } H_1 \cap H_2,\ i = 1,2,3 \\ \overline{\varphi} = \widetilde{\varphi} \quad \text{on } \widehat{H}_1 \cap \widehat{H}_2 \\ \partial_1 \overline{a}_i - \partial_1 \widetilde{a}_i = 2\overline{b}^i = 2\widetilde{b}^i \quad \text{on } H_1 \cap H_2,\ i = 1,2,3 \end{gathered} \tag{3.2}$$

The functions a_i, b^i and φ are continuous on $\widehat{H} = \widehat{H}_1 \cup \widehat{H}_2$, and the functions Φ_{ij} verify $\Phi_{ij} = -\Phi_{ji}$. The functions Φ_{ij}, a_i, b^i are linked by the algebraic differential relations

$$\begin{gathered} \Phi_{1j} = \begin{cases} \overline{\Phi}_{1j} = \partial_1 \overline{a}_j - \partial_j \overline{a}_1 + [\overline{a}_1, \overline{a}_j] - \overline{b}^j & \text{on } H_1,\ j = 2,3; \\ \widetilde{\Phi}_{1j} = \partial_1 \widetilde{a}_j - \partial_j \widetilde{a}_1 + [\widetilde{a}_1, \widetilde{a}_j] + \widetilde{b}^j & \text{on } H_2. \end{cases} \\ \Phi_{23} = \begin{cases} \overline{\Phi}_{23} = \partial_2 \overline{a}_3 - \partial_3 \overline{a}_2 + [\overline{a}_2, \overline{a}_3] & \text{on } H_1 \\ \widetilde{\Phi}_{23} = \partial_2 \widetilde{a}_3 - \partial_3 \widetilde{a}_2 + [\widetilde{a}_2, \widetilde{a}_3] & \text{on } H_2. \end{cases} \end{gathered} \tag{3.3}$$

(2) Furthermore $V = (A_0 \equiv 0, A_i, F^{0i}, F_{ij}, f)$, with $i, j = 1, 2, 3$, verifies the equation $\widehat{\nabla}_\alpha F^{\alpha 0} = J^0$ on $\widehat{H}$, if and only if the function $\overline{b}^1$ (resp. $\widetilde{b}^1$) is solution of the following Cauchy problem (3.4) *(resp.* (3.5)*),*

$$\begin{gathered} \partial_1 \overline{b}^1 + [\overline{a}_1, \overline{b}^1] = -\Big(\overline{J}^0 + \overline{J}^1 + \partial_2 \overline{\Psi}_{12} + \partial_3 \overline{\Psi}_{13} + [\overline{a}_2, \overline{\Psi}_{12}] + [\overline{a}_3, \overline{\Psi}_{13}]\Big) \\ \text{on } \widehat{H}_1; \\ \overline{b}^1(0, x^2, x^3) = \frac{1}{2}(\partial_1 \overline{a}_1 - \partial_1 \widetilde{a}_1)(0, x^2, x^3), \quad \forall (x^2, x^3) \in B \end{gathered} \tag{3.4}$$

and

$$\begin{gathered} \partial_1 \widetilde{b}^1 + [\widetilde{a}_1, \widetilde{b}^1] = \Big(\widetilde{J}^0 - \widetilde{J}^1 + \partial_2 \widetilde{\Psi}_{12} + \partial_3 \widetilde{\Psi}_{13} + [\widetilde{a}_2, \widetilde{\Psi}_{12}] + [\widetilde{a}_3, \widetilde{\Psi}_{13}]\Big) \\ \text{on } \widehat{H}_2; \\ \widetilde{b}^1(0, x^2, x^3) = \frac{1}{2}(\partial_1 \overline{a}_1 - \partial_1 \widetilde{a}_1)(0, x^2, x^3), \quad \forall (x^2, x^3) \in B \end{gathered} \tag{3.5}$$

with

$$\begin{gathered} \overline{\Psi}_{12} = \partial_1 \overline{a}_2 - \partial_2 \overline{a}_1 + [\overline{a}_1, \overline{a}_2], \quad \overline{\Psi}_{13} = \partial_1 \overline{a}_3 - \partial_3 \overline{a}_1 + [\overline{a}_1, \overline{a}_3], \\ \overline{J}^\alpha = \int_{\mathbb{R}^3 \times O} p^\alpha q \overline{\varphi} \omega_p \omega_q, \quad \alpha = 0, 1; \\ \widetilde{\Psi}_{12} = \partial_1 \widetilde{a}_2 - \partial_2 \widetilde{a}_1 + [\widetilde{a}_1, \widetilde{a}_2], \quad \widetilde{\Psi}_{13} = \partial_1 \widetilde{a}_3 - \partial_3 \widetilde{a}_1 + [\widetilde{a}_1, \widetilde{a}_3], \\ \widetilde{J}^\alpha = \int_{\mathbb{R}^3 \times O} p^\alpha q \widetilde{\varphi} \omega_p \omega_q, \quad \alpha = 0, 1. \end{gathered}$$

Remark 3.2. We observe that the PDE appearing in (3.4) and (3.5) are in fact linear ordinary differential equations of the scalar variable x^1, smoothly depending on the parameters x^2 and x^3.

Proof of Theorem 3.1. (1) As restrictions to the smooth surface H_1 (resp H_2) of the $\mathcal{C}^\infty$ functions A_i, F^{0i}, F_{ij}, the functions $\overline{a}_i, \overline{b}^j$ $\overline{\Phi}_{ij}$ (resp. $\widetilde{a}_i, \widetilde{b}^j$, $\widetilde{\Phi}_{ij}$) are $\mathcal{C}^\infty$ on H_1 (resp. H_2). Likewise, as restrictions to the smooth surface $\widehat{H}_1$ (resp. $\widehat{H}_2$) of the function f which is $\mathcal{C}^\infty$ on $\hat{\sqcup}$, $\overline{\varphi}$ (resp. $\widetilde{\varphi}$) is $\mathcal{C}^\infty$ on $\widehat{H}_1$ (resp. $\widehat{H}_2$).

The first and the second compatibility conditions of (3.2) are obviously satisfied. Moreover, by considering the restrictions to H_r $(r = 1, 2)$ of A_i $(i = 1, 2, 3)$, we have:

$$\begin{gathered}
\partial_1[A_i]_1 = -[\partial_0 A_i]_1 + [\partial_1 A_i]_1 \quad \text{on } H_1, \\
\partial_1[A_i]_2 = [\partial_0 A_i]_2 + [\partial_1 A_i]_2 \quad \text{on } H_2, \\
\partial_j[A_i]_r = [\partial_j A_i]_r \quad \text{on } H_r,\ r = 1, 2,\ j = 2, 3,\ i = 1, 2, 3.
\end{gathered} \tag{3.6}$$

To show the third compatibility condition of (3.2), we observe that on $H_1 \cap H_2$ the following relations are valid: $[\partial_1 A_i]_1 = [\partial_1 A_i]_2$ and $[\partial_0 A_i]_1 = [\partial_0 A_i]_2$. We then deduce in view of first and second relations of (3.6) that

$$2[\partial_0 A_i]_r = \partial_1[A_i]_2 - \partial_1[A_i]_1 \quad \text{on } H_1 \cap H_2, \quad r = 1, 2.$$

Then, using the fact that $[\partial_0 A_i]_1 = -F^{0i}\big|_{H_1} = -\overline{b}^i$ and $[\partial_0 A_i]_2 = -F^{0i}\big|_{H_2} = -\widetilde{b}^i$, we deduce the third relation of (3.2),

$$\partial_1 \overline{a}_i - \partial_1 \widetilde{a}_i = 2\overline{b}^i = 2\widetilde{b}^i \quad \text{on } H_1 \cap H_2,\ i = 1, 2, 3.$$

The algebraic differential relations stated in (3.3) follow directly from definitions (2.2), (3.1) and relations (3.6).

(2) To show that $\overline{b}^1$ is solution of (3.4) if $\widehat{\nabla}_\alpha F^{\alpha 0} = J^0$ on $\hat{H}$, we consider the restrictions to $\widehat{H}_1$ of equations $\widehat{\nabla}_\alpha F^{\alpha 1} = J^1$ and $\widehat{\nabla}_\alpha F^{\alpha 0} = J^0$. By adding these restrictions and in view of definitions (3.1) and the relations (3.3), we obtain

$$\partial_1 \overline{b}^1 + [\overline{a}_1, \overline{b}^1] = -\Big(\overline{J}^0 + \overline{J}^1 + \partial_2 \overline{\Psi}_{12} + \partial_3 \overline{\Psi}_{13} + [\overline{a}_2, \overline{\Psi}_{12}] + [\overline{a}_3, \overline{\Psi}_{13}]\Big) \quad \text{on } \widehat{H}_1.$$

Moreover, we know that the third relation of (3.2), for $i = 1$, gives

$$\overline{b}^1(0, x^2, x^3) = \frac{1}{2}(\partial_1 \overline{a}_1 - \partial_1 \widetilde{a}_1)(0, x^2, x^3), \quad \forall (x^2, x^3) \in B.$$

We then deduce that $\overline{b}^1$ solves (3.4).

By the same process, subtracting the restriction to $\widehat{H}_2$ of the equation $\widehat{\nabla}_\alpha F^{\alpha 1} = J^1$ from that of the equation $\widehat{\nabla}_\alpha F^{\alpha 0} = J^0$, we obtain in view of the third relation of (3.2) that $\widetilde{b}^1$ solves problem (3.5). Conversely, if $\overline{b}^1$ (resp $\widetilde{b}^1$)is solution of the problem (3.4) (resp (3.5)), then it is obvious that we have $\widehat{\nabla}_\alpha F^{\alpha 0} = J^0$ on $\widehat{H}$. □

3.1. **Precise statement of the initial constraint problem: the choice of the free data.** Let $T \in \mathbb{R}_+^*$ such that $T \leq \sup\{x^0 = t, (x^0, x^i) \in \sqcup\}$, T given. We assume the temporal gauge condition $A_0 = 0$ in $\sqcup_T$.

(a) **Free data of the initial constraint problem**: Consider the arbitrary functions:

$$\begin{gathered}\overline{a}_i(x^1,x^2,x^3)\,\mathcal{C}^\infty \text{ function, } (-x^1,x^1,x^2,x^3)\in H_1,\ i=1,2,3;\\ \widetilde{a}_i(x^1,x^2,x^3)\,\mathcal{C}^\infty \text{ function, } (x^1,x^1,x^2,x^3)\in H_2;\\ \overline{b}^j(x^1,x^2,x^3)\,\mathcal{C}^\infty \text{ function, } (-x^1,x^1,x^2,x^3)\in H_1, j=2,3;\\ \widetilde{b}^j(x^1,x^2,x^3)\,\mathcal{C}^\infty \text{ function, } (x^1,x^1,x^2,x^3)\in H_2, j=2,3;\\ \overline{\varphi}(x^1,x^2,x^3,p^i,q^L)\mathcal{C}^\infty \text{ function, } (-x^1,x^1,x^2,x^3,p^i,q^L)\in \widehat{H}_1, L=1,\dots N-1;\\ \widetilde{\varphi}(x^1,x^2,x^3,p^i,q^L)\mathcal{C}^\infty \text{ function, } (x^1,x^2,x^3,p^i,q^L)\in \widehat{H}_2,\ L=1,\dots,N-1;\\ \text{with } \overline{\varphi} \text{ and } \widetilde{\varphi} \text{ having compact support.}\end{gathered}\tag{3.7}$$

These functions satisfy the compatibility conditions:

$$\begin{gathered}\overline{a}_i(0,x^2,x^3)=\widetilde{a}_i(0,x^2,x^3),\quad \text{where } (x^2,x^3)\in B,\ i=1,2,3;\\ \overline{b}^j(0,x^2,x^3)=\widetilde{b}^j\left(0,x^2,x^3\right),\ j=2,3;\\ (\partial_1\overline{a}_j-\partial_1\widetilde{a}_j)(0,x^2,x^3)=2\overline{b}^j(0,x^2,x^3)=2\widetilde{b}^j(0,x^2,x^3);\\ \overline{\varphi}\left(0,x^2,x^3,p^i,q^L\right)=\widetilde{\varphi}\left(0,x^2,x^3,p^i,q^L\right),\ L=1,\dots,N-1.\end{gathered}\tag{3.8}$$

(b) For the reduced system (2.9), as initial conditions, we consider

$$\begin{gathered}A_i\big|_H=a_i=\begin{cases}\overline{a}_i & \text{on } H_1\\ \widetilde{a}_i & \text{on } H_2;\end{cases}\quad F^{0i}\big|_H=b^i=\begin{cases}\overline{b}^i & \text{on} H_1\\ \widetilde{b}^i & \text{on } H_2;\end{cases}\\ F_{ij}\big|_H=\Phi_{ij}=\begin{cases}\overline{\Phi}_{ij} & \text{on } H_1\\ \widetilde{\Phi}_{ij} & \text{on } H_2;\end{cases}\quad f\big|_{\widehat{H}}=\varphi=\begin{cases}\overline{\varphi} & \text{on } \widehat{H}_1\\ \widetilde{\varphi} & \text{on } \widehat{H}_2;\end{cases}\quad i,j=1,2,3\end{gathered}\tag{3.9}$$

where: Φ_{ij} are given by relation (3.3) of theorem 3.1 and b^1 is such that $\overline{b}^1$ (resp. $\widetilde{b}^1$) is the unique solution of (3.4) (resp. (3.5)) of theorem 3.1, with

$$\overline{b}^1(0,x^2,x^3)=\widetilde{b}^1(0,x^2,x^3)=\frac{1}{2}(\partial_1\overline{a}_1-\partial_1\widetilde{a}_1)(0,x^2,x^3).\tag{3.10}$$

Our goal is now to show that every $\mathcal{C}^\infty$ solution of the reduced system (2.9) subjected to initial conditions (3.9), (3.10) and compatibility conditions (3.8) is in fact $\mathcal{C}^\infty$ solution of the complete system (2.8) of Yang-Mills-Vlasov Equations.

Theorem 3.3. *Every $\mathcal{C}^\infty$ solution $V=(A_0\equiv 0,A_i,F^{0i},F_{ij},f)$, defined in a neighborhood $\widetilde{\sqcup}$ of $\widehat{H}$ in $\widehat{\sqcup}$, for the reduced system* (2.9) *with initial conditions and compatibility conditions* (3.9), (3.10) *and* (3.8) *associated to free data* (3.7) *and such that the support of φ is compact, is solution of the complete system* (2.8) *of Yang-Mills-Vlasov Equations in the domain $\widehat{\sqcup}$.*

Proof. Let $V=(A_0\equiv 0,A_i,F^{0i},F_{ij},f)$ be a $\mathcal{C}^\infty$ solution, defined in a neighborhood $\widetilde{\sqcup}$ of $\widehat{H}$, for the reduced system (2.9) with the initial conditions described as above. We give the proof in three steps:

Step 1. We show that F is the curvature of A. Set $\Omega=dA$, that is the curvature of A. In the global canonical coordinates system on $\sqcup$ it holds that

$$\Omega_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha+[A_\alpha,A_\beta],\quad \alpha,\beta=0,1,2,3.$$

In view of the temporal gauge condition $A_0 = 0$ in $\widetilde{\sqcup}$ and the reduced system (2.9) which is verified in $\widetilde{\sqcup}$, this implies that

$$\Omega_{0i} = \partial_0 A_i = F_{0i} \quad \text{in } \widetilde{\sqcup},\ i = 1,2,3.$$

It remains to prove that $\Omega_{ij} = F_{ij}$ in $\widetilde{\sqcup}$, $i, j = 1,2,3$. Since V solves the reduced system (2.9), it holds that

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} + [A_i, F_{j0}] + [A_j, F_{0i}] = 0 \ \text{ in } \widetilde{\sqcup}, \quad i,j = 1,2,3.$$

In view of (2.9), it holds that $F_{0i} = \partial_0 A_i$. Inserting this latter relation in the previous one and integrating with respect to the variable x^0 on $[|x^1|, t]$, we obtain

$$\begin{aligned} &\Big(F_{ij} - \partial_i A_j + \partial_j A_i - [A_i, A_j]\Big)(t, x^1, x^2, x^3) \\ &- \Big(F_{ij} - \partial_i A_j + \partial_j A_i - [A_i, A_j]\Big)(|x^1|, x^1, x^2, x^3) = 0. \end{aligned}$$

Now the initial conditions (3.9) and (3.10) imply that

$$\big(- F_{ij} + \partial_i A_j - \partial_j A_i + \ [A_i, A_j]\big)(|x^1|, x^1, x^2, x^3) = 0.$$

Hence, we obtain

$$\begin{aligned} F_{ij}\left(x^0, x^1, x^2, x^3\right) &= \Big(\partial_i A_j - \partial_j A_i + [A_i, A_j]\Big)(x^0, x^1, x^2, x^3) \\ &= \Omega_{ij}(x^0, x^1, x^2, x^3). \end{aligned} \tag{3.11}$$

Step 2. We must verify that

$$\widehat{\nabla}_k F_{ij} + \widehat{\nabla}_i F_{jk} + \widehat{\nabla}_j F_{ki} = 0 \quad k, i, j = 1,2,3.$$

This is obvious since the first step provides

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta], \quad \forall \alpha, \beta = 0,1,2,3.$$

This implies Bianchi identities

$$\widehat{\nabla}_\alpha F_{\beta\gamma} + \widehat{\nabla}_\beta F_{\gamma\alpha} + \widehat{\nabla}_\gamma F_{\alpha\beta} = 0, \quad \alpha, \beta, \gamma = 0,1,2,3.$$

The result then follows.

Step 3. We show that V satisfies the equation $\widehat{\nabla}_\alpha F^{\alpha 0} = J^0$ at each point $(x^0, x^i) \in \widetilde{\sqcup}$ with $x^1 \neq 0$. Indeed, the equation $\widehat{\nabla}_\alpha F^{\alpha i} = J^i$, which is verified according to (2.9), can be written as follows

$$\partial_0 F^{0i} = J^i - \partial_j F^{ji} - [A_j, F^{ji}], \quad i, j = 1,2,3.$$

By integrating this latter expression with respect to x^0 on $[|x^1|, t]$ and differentiating with respect to x^i, we obtain

$$\begin{aligned} &\partial_i F^{0i}(t, ., x^3) \\ &= \partial_i\{F^{0i}(|x^1|, ., x^3)\} + \partial_i\Big\{\int_{|x^1|}^t \{J^i - \partial_j F^{ji} - [A_j, F^{ji}]\}(\tau, ., x^3)\Big\} d\tau. \end{aligned} \tag{3.12}$$

Direct calculations give

$$\partial_i\big\{F^{0i}\left(|x^1|, x^1, x^2, x^3\right)\big\} = \partial_0 F^{0i}\left(|x^1|, x^1, x^2, x^3\right).\varepsilon_i + \partial_i F^{0i}\left(|x^1|, x^1, x^2, x^3\right), \tag{3.13}$$

with

$$\varepsilon_i = \begin{cases} \varepsilon & \text{if } i = 1 \\ 0 & \text{if } i = 2,3; \end{cases} \qquad \varepsilon = \begin{cases} 1 & \text{if } x^1 > 0 \\ -1 & \text{if } x^1 < 0; \end{cases}$$

and

$$
\begin{aligned}
&\partial_i\Big\{\int_{|x^1|}^{t}\{J^i-\partial_jF^{ji}-[A_j,F^{ji}]\}\big(\tau,x^1,x^2,x^3\big)\,d\tau\Big\}\\
&=\int_{|x^1|}^{t}\{\partial_iJ^i-\partial_i\partial_jF^{ji}-\partial_i[A_j,F^{ji}]\}\big(\tau,x^1,x^2,x^3\big)\,d\tau\\
&\quad-\varepsilon\{J^1-\partial_jF^{j1}-[A_j,F^{j1}]\}\big(|x^1|,x^1,x^2,x^3\big)\,.
\end{aligned}
\tag{3.14}
$$

Now, as V is solution of the system (2.9) in $\widetilde{\sqcup}$, f satisfies Vlasov equation in $\widetilde{\sqcup}$ and this implies $\widehat{\nabla}_\beta J^\beta=0$ in $\widetilde{\sqcup}$; i. e.,

$$
\partial_iJ^i=-\partial_0J^0-[A_i,J^i],\quad i=1,2,3.
$$

By inserting this latter relation into (3.14) and in view of the identity ($\partial_i\partial_jF^{ji}=0$) we gain

$$
\partial_i\Big\{\int_{|x^1|}^{t}\{J^i-\partial_lF^{li}-[A_l,F^{li}]\}\big(\tau,x^1,x^2,x^3\big)\,d\tau\Big\}=Z,
\tag{3.15}
$$

where

$$
\begin{aligned}
Z=&\int_{|x^1|}^{t}\{-\partial_0J^0-[A_i,J^i]-\partial_i[A_l,F^{li}]\}\big(\tau,x^1,x^2,x^3\big)\,d\tau\\
&-\{J^1-\partial_jF^{j1}-[A_j,F^{j1}]\}\big(|x^1|,x^1,x^2,x^3\big)\,.\varepsilon.
\end{aligned}
$$

Equation (3.12) then becomes

$$
\begin{aligned}
&\partial_iF^{0i}\big(t,x^1,x^2,x^3\big)\\
&=\partial_0F^{01}\big(|x^1|,x^1,x^2,x^3\big)\,\varepsilon+\partial_iF^{0i}\big(|x^1|,x^1,x^2,x^3\big)-J^0\big(t,x^1,x^2,x^3\big)\\
&\quad+J^0\big(|x^1|,x^1,x^2,x^3\big)-\big\{J^1-\partial_jF^{j1}-[A_j,F^{j1}]\big\}\big(|x^1|,x^1,x^2,x^3\big)\,\varepsilon\\
&\quad-\int_{|x^1|}^{t}\big\{[A_i,J^i]+\partial_i[A_l,F^{li}]\big\}\big(\tau,x^1,x^2,x^3\big)\,d\tau.
\end{aligned}
\tag{3.16}
$$

The relation $\widehat{\nabla}_\alpha F^{\alpha 1}=J^1$ in $\widetilde{\sqcup}$, which follows from (2.9) can be written as

$$
\partial_0F^{01}=J^1-\partial_jF^{j1}-[A_j,F^{j1}].
\tag{3.17}
$$

Inserting (3.17) into (3.16), we obtain

$$
\begin{aligned}
\partial_iF^{0i}\big(x^0,x^1,x^2,x^3\big)=&\,\partial_iF^{0i}\big(|x^1|,x^1,x^2,x^3\big)\\
&-\int_{|x^1|}^{t}\Big\{[A_i,J^i]+\partial_i[A_l,F^{li}]\Big\}\big(\tau,x^1,x^2,x^3\big)\,d\tau\\
&-J^0\big(t,x^1,x^2,x^3\big)+J^0\big(|x^1|,x^1,x^2,x^3\big)\,.
\end{aligned}
\tag{3.18}
$$

Likewise, the relation $\widehat{\nabla}_\alpha F^{\alpha i}=J^i$ reads

$$
\partial_0F^{0i}+\partial_lF^{li}+[A_l,F^{li}]=J^i,\quad i,l=1,2,3.
$$

Setting

$$
L=\int_{|x^1|}^{t}[A_i,J^i]d\tau,
$$

it follows that

$$\begin{aligned} L &= \int_{|x^1|}^{t} \big\{[A_i, \partial_0 F^{0i}] + [A_i, \partial_l F^{li}] + [A_i, [A_l, F^{li}]]\big\} d\tau \\ &= \int_{|x^1|}^{t} \Big\{\partial_0 [A_i, F^{0i}] - [\partial_0 A_i, F^{0i}] + [A_i, \partial_l F^{li}] + [A_i, [A_l, F^{li}]]\Big\} d\tau \\ &= [A_i, F^{0i}]\left(t, x^1, x^2, x^3\right) - [A_i, F^{0i}]\left(|x^1|, x^1, x^2, x^3\right) \\ &\quad + \int_{|x^1|}^{t} \Big\{[A_i, \partial_l F^{li}] + [A_i, [A_l, F^{li}]]\Big\} d\tau, \end{aligned} \tag{3.19}$$

thanks to

$$[\partial_0 A_i, F^{0i}] = [F_{0i}, F^{0i}] = \sum_{i=1}^{3} [-F^{0i}, F^{0i}] = 0.$$

The insertion of (3.19) into (3.18) gives

$$\begin{aligned} &\partial_i F^{0i}\left(t, x^1, x^2, x^3\right) \\ &= -[A_i, F^{0i}](t, ., x^3) + [A_i, F^{0i}]\left(|x^1|, ., x^3\right) \\ &\quad + \partial_i F^{0i}\left(|x^1|, ., x^3\right) - J^0(t, ., x^3) + J^0\left(|x^1|, ., x^3\right) - R, \end{aligned} \tag{3.20}$$

where

$$R = \int_{|x^1|}^{t} \Big\{[A_i, \partial_l F^{li}] + \partial_i [A_l, \partial_l F^{li}] + [A_i, [A_l, F^{li}]]\Big\} d\tau.$$

By permuting indexes l and i in $[A_i, \partial_l F^{li}]$ and using the fact that F is antisymmetric, we obtain

$$[A_i, \partial_l F^{li}] + \partial_i [A_l, F^{li}] = [\partial_i A_l, F^{li}].$$

Thus

$$R = \int_{|x^1|}^{t} \Big\{[\partial_i A_l, F^{li}] + [A_i, [A_l, F^{li}]]\Big\} d\tau. \tag{3.21}$$

Moreover it holds that

$$F^{li} = F_{li} = \partial_l A_i - \partial_i A_l + [A_l, A_i].$$

By inserting this latter relation in (3.21), using several times Jacobi identity and the fact that Lie bracket $[,]$ is antisymmetric, we obtain $R = 0$. But in view of theorem 3.1, V satisfies the equation

$$\widehat{\nabla}_\alpha F^{\alpha 0}\big|_{\widehat{H}} = J^0\big|_{\widehat{H}}.$$

This implies

$$\left(\partial_i F^{0i} + [A_i, F^{0i}] + J^0\right)\left(|x^1|, x^1, x^2, x^3\right) = 0. \tag{3.22}$$

Thus (3.20) becomes

$$\widehat{\nabla}_\alpha F^{\alpha 0}(t, x^1, x^2, x^3) = J^0(t, x^1, x^2, x^3) \quad \text{with } x^1 \neq 0,$$

which implies, by an obvious continuity argument,

$$\widehat{\nabla}_\alpha F^{\alpha 0}\left(t, x^1, x^2, x^3\right) = J^0\left(t, x^1, x^2, x^3\right) \quad \text{with } x^1 = 0.$$

Consequently

$$\widehat{\nabla}_\alpha F^{\alpha 0} = J^0 \quad \text{in } \widetilde{\sqcup}.$$

This completes the proof. $\square$

4. Determination of restrictions

In this section we determine restrictions to the initial characteristic hypersurfaces $\widehat{H}_1$ and $\widehat{H}_2$ of derivatives of all order of a possible $\mathcal{C}^\infty$ solution of the evolution problem (2.9) subjected to initial conditions (3.8), (3.9), (3.10). associated with free data. We will use Rendall's method [29] which consists in transforming the Goursat problem under consideration into an ordinary Cauchy problem with zero data specified on the spatial hypersurface $x^0 = 0$. An important step is the determination of the restrictions to $\widehat{H}_1$ and $\widehat{H}_2$ of the derivatives of all order of the possible $\mathcal{C}^\infty$ solutions of the evolution problem (2.9) subject to (3.8), (3.9), (3.10), which is the goal of the present section. To reach it, it will be useful to reinforce hypothesis of free data $\overline{\varphi}$ and $\widetilde{\varphi}$

Hypothesis I. If $m > 0$, we assume:

(i) $\overline{\varphi}$ (resp $\widetilde{\varphi}$) is smooth on $\widehat{H}_1$ (resp $\widehat{H}_2$) and $\operatorname{supp}(\overline{\varphi})$ (resp $\operatorname{supp}(\widetilde{\varphi})$) is compact.

(ii) $\operatorname{supp}(\overline{\varphi}) \cap \{\widehat{H}_1 \cap \widehat{H}_2\} = \emptyset$ and $\operatorname{supp}(\widetilde{\varphi}) \cap \{\widehat{H}_1 \cap \widehat{H}_2\} = \emptyset$.

If $m = 0$, we add (i) and (ii) the following hypothesis

(iii) $\operatorname{supp}(\overline{\varphi}) \subset \{X = (-x^1, x^1, x^2, x^3, p^i, q^L) : (p^2)^2 + (p^3)^2 > 0\}$ and $\operatorname{supp}(\widetilde{\varphi}) \subset \{X = (x^1, x^1, x^2, x^3, p^i, q^L) : (p^2)^2 + (p^3)^2 > 0\}$.

Remark 4.1. From hypothesis (i), (ii), (iii), we have the following:

If $m \neq 0$, for every $X = (-x^1, x^1, x^2, x^3, p^i, q^L) \in \operatorname{supp}(\overline{\varphi})$, we have $p^0 + p^1 \neq 0$; and for every $X = (x^1, x^1, x^2, x^3, p^i, q^L) \in \operatorname{supp}(\widetilde{\varphi})$ we have $p^0 - p^1 \neq 0$.

If Hypothesis (i) holds, that is the compactness of the supports of $\overline{\varphi}$ and $\widetilde{\varphi}$, then

$$\inf_{X \in \operatorname{supp}(\overline{\varphi})} |p^0 + p^1| = \min_{X \in \operatorname{supp}(\overline{\varphi})} |p^0 + p^1| = C_1 > 0,$$
$$\inf_{X \in \operatorname{supp}(\widetilde{\varphi})} |p^0 - p^1| = \min_{X \in \operatorname{supp}(\widetilde{\varphi})} |p^0 - p^1| = C_2 > 0.$$

Analogously, if $m = 0$ hypothesis (i) and (iii) again imply

$$\inf_{X \in \operatorname{supp}(\overline{\varphi})} |p^0 + p^1| = \min_{X \in \operatorname{supp}(\overline{\varphi})} |p^0 + p^1| = C_1 > 0$$
$$\inf_{X \in \operatorname{supp}(\widetilde{\varphi})} |p^0 - p^1| = \min_{X \in \operatorname{supp}(\widetilde{\varphi})} |p^0 - p^1| = C_2 > 0.$$

These remarks are crucial to establish the unique determination of the restrictions to $\widehat{H}_r$ $(r = 1, 2)$ of the derivatives of all order of a possible $\mathcal{C}^\infty$ solution f of the Vlasov equation.

Hypothesis (ii) is a sufficient condition for showing that

$$[\frac{\partial^l f}{(\partial x^0)^l}]_1 = [\frac{\partial^l f}{(\partial x^0)^l}]_2 \quad \text{on } \widehat{H}_1 \cap \widehat{H}_2,\ \forall l \in \mathbb{N},$$

which are necessary conditions for a function $\mathcal{C}^\infty$ in a neighborhood of $\widehat{H}_1 \cap \widehat{H}_2$

4.1. Determination of the restrictions.

In this section we determine the restriciotns to $\widehat{H}$ of the first derivatives of any possible $\mathcal{C}^\infty$ solution of the evolution problem. Let $V = \big(A_0 \equiv 0, A_i, F^{0i}, F_{ij}, f\big)$ be a $\mathcal{C}^\infty$ solution, defined in the neighborhood $\widetilde{\sqcup}$ of $\widehat{H}$ in $\widehat{\sqcup}$, of the evolution problem. We will use the equations of system (2.9) to determine the restrictions to $\widehat{H}_r (r = 1, 2)$ of the first order derivatives of V.

(i) To determine $[\partial_0 A_i]_1$, we use the equation $\partial_0 A_l = F_{0l}$ of (2.9), to obtain

$$[\partial_0 A_i]_1 = -\overline{b}^i, \quad i = 1, 2, 3. \tag{4.1}$$

(ii) To determine $[\frac{\partial f}{\partial x^0}]_1$, we use the fourth equation of (2.9) which implies in view of Remark 4.1,

$$[\frac{\partial f}{\partial x^0}]_1 = \frac{-1}{p^0 + p^1}\{p^{i_0}\frac{\partial \overline{\varphi}}{\partial x^{i_0}} + \overline{P}^{j_0}\frac{\partial \overline{\varphi}}{\partial p^{j_0}} + \overline{Q}^{L_0}\frac{\partial \overline{\varphi}}{\partial q^{L_0}}\}, \tag{4.2}$$

(iii) To determine $[\partial_0 F^{01}]_1$, we consider the restriction to $\widehat{H}_1$ of the equation $\widehat{\nabla}_\alpha F^{\alpha 1} = J^1$ which implies

$$[\partial_0 F^{01}]_1 = \overline{J}^1 + \sum_{j=1}^{3}\partial_j \overline{\Psi}_{1j} - \partial_j \overline{b}^j + \sum_{j=1}^{3}[\overline{a}_j, \overline{\Psi}_{1j} - \overline{b}^j], \ \ j \neq 1, \tag{4.3}$$

with

$$\overline{\Psi}_{1j} = \partial_1 \overline{a}_j - \partial_j \overline{a}_1 + [\overline{a}_1, \overline{a}_j], \quad \bar{J}^1 = \int_{\mathbb{R}^3 \times O} p^1 q \overline{\varphi}\omega_p \omega_q.$$

(iv) To determine $[\partial_0 F_{ij}]_1$ with $i, j \neq 1$, we consider the restriction to H_1 of Bianchi identities (third equation of (2.9)) which leads to

$$[\partial_0 F_{ij}]_1 = -\partial_i \overline{b}^j - [\overline{a}_i, \overline{b}^j] + \partial_j \overline{b}^i + [\overline{a}_j, \overline{b}^i], \quad i, j = 2, 3; \ i \neq j. \tag{4.4}$$

(v) It remains to determine $[\partial_0 F^{1i}]_1$ and $[\partial_0 F^{0i}]_1$ for $i \neq 1$. Consider the following equations extracted from system (2.9),

$$\widehat{\nabla}_\alpha F^{\alpha i} = J^i, \quad \widehat{\nabla}_0 F_{i1} + \widehat{\nabla}_i F_{10} + \widehat{\nabla}_1 F_{0i} = 0, \quad \text{for } i \neq 1.$$

Differentiating the sum of the latter equalities with respect to x^0 and taking the restriction to H_1 of the relation obtained, we gain

$$\partial_1\Big\{[\partial_0 F^{1i}]_1 - [\partial_0 F^{0i}]_1\Big\} + [\overline{a}_1, [\partial_0 F^{1i}]_1 - [\partial_0 F^{0i}]_1] = \overline{D}, \tag{4.5}$$

with $\overline{D}$ a known function on $\widehat{H}_1$. Moreover, it holds that

$$\begin{gathered} F^{1i}(-x^1, x^1, x^2, x^3) = (\overline{\Psi}_{1i} - \overline{b}^i)(x^1, x^2, x^3) \quad \text{on } H_1 \\ F^{1i}(x^1, x^1, x^2, x^3) = \big(\widetilde{\Psi}_{1i} + \widetilde{b}^i\big)(x^1, x^2, x^3) \quad \text{on } H_2, \ i \neq 1, \end{gathered}$$

By differentiating with respect to x^1 the above relations, and using the fact that the F^{1i} and their derivatives are continuous on $I = H_1 \cap H_2$, for $i \neq 1$, we have

$$[\partial_0 F^{1i}]\left(0, 0, x^2, x^3\right) = \frac{1}{2}\Big\{\big(\partial_1 \widetilde{\Psi}_{1i} + \partial_1 \widetilde{b}^i\big) - \big(\partial_1 \overline{\Psi}_{1i} - \partial_1 \overline{b}^i\big)\Big\}\left(0, 0, x^2, x^3\right), \tag{4.6}$$

Analogously, we obtain

$$[\partial_0 F^{0i}]\left(0, 0, x^2, x^3\right) = \frac{1}{2}\Big\{\partial_1 \widetilde{b}^i - \partial_1 \overline{b}^i\Big\}\left(0, 0, x^2, x^3\right), i \neq 1, \tag{4.7}$$

From (4.6) and (4.7), for $i \neq 1$, we have

$$\{[\partial_0 F^{1i}]_1 - [\partial_0 F^{0i}]_1\}(0, x^2, x^3) = \frac{1}{2}\{\partial_1 \widetilde{\Psi}_{1i} - \partial_1 \overline{\Psi}_{1i} + 2\partial_1 \overline{b}^i\}(0, x^2, x^3), \tag{4.8}$$

We then deduce $[\partial_0 F^{1i}]_1 - [\partial_0 F^{0i}]_1$ on H_1 as the unique solution of the Cauchy problem (4.5), (4.8). We can then set

$$[\partial_0 F^{1i}]_1 - [\partial_0 F^{0i}]_1 = \overline{C} \tag{4.9}$$

where $\overline{C}$ is now a known smooth function on H_1. To determine $\big([\partial_0 F^{1i}]_1, [\partial_0 F^{0i}]_1\big)$ on H_1, considering now the restriction to H_1 of the equation $\widehat{\nabla}_\alpha F^{\alpha i} = J^i$, $i \neq 1$, we obtain

$$[\partial_0 F^{0i}]_1 + [\partial_0 F^{1i}]_1 = [J^i]_1 - \partial_j [F^{ji}]_1 - [[A_j, F^{ji}]]_1, \tag{4.10}$$

for $i = 2, 3$, $j = 1, 2, 3$. The relations (4.9) and (4.10) then determine $[\partial_0 F^{1i}]_1$ and $[\partial_0 F^{0i}]_1$. By the same process, we can uniquely determine $[\partial_0 A_i]_2$ (for $i = 1, 2, 3$), $[\partial_0 f]_2$, $[\partial_0 F^{01}]_2, [\partial_0 F_{ij}]_2$, for $i, j = 2, 3$, $[\partial_0 F^{1i}]_2$, $[\partial_0 F^{0i}]_2$ for $i = 2, 3$ and these functions are $\mathcal{C}^\infty$ on $\widehat{H}_2$.

We have then proved the following proposition.

Proposition 4.2. *Let $V = (A_0 \equiv 0, A_i, F_{ij}, F^{0i}, f)$ a $\mathcal{C}^\infty$ solution, defined in a neighborhood $\widetilde{\sqcup}$ of $\widehat{H}$ in $\widehat{\sqcup}$ of the evolution problem* (2.9) *subject to* (3.8), (3.9), (3.10) *such that initial datum φ satisfies hypothesis (I). Then the restrictions to $\widehat{H}$ of all first order derivatives of V, that is $[\partial_0 A_i]$, $[\partial_0 F_{ij}]$, $[\partial_0 F^{0i}]$ and $[\partial_0 f]$, are uniquely determined on $\widehat{H}$. These functions are continuous on $\widehat{H}$and are $\mathcal{C}^\infty$ on $\widehat{H}_r, (r = 1, 2)$. Moreover,* $\operatorname{supp}[\partial_0 f]$ *is compact and contained in the support of φ.*

4.2. **Determination of derivatives.** In this section we determine derivatives all order of any possible $\mathcal{C}^\infty$ solution of the evolution problem (2.9) subject to (3.8), (3.9), (3.10).

Let $V = \big(A_0 \equiv 0, A_i, F^{0i}, F_{ij}, f\big)$ be a $\mathcal{C}^\infty$ solution, defined on a neighborhood $\widetilde{\sqcup}$ of $\widehat{H}$ in $\widehat{\sqcup}$, of the evolution problem. We want to show, for every $k \in \mathbb{N}$, that $[\partial_0^k V]$ is uniquely determined, is continuous on $\widehat{H}$ and that $[\partial_0^k V]_r$ is $\mathcal{C}^\infty$ on $\widehat{H}_r$, $r = 1, 2$. This can obviously be done, by induction on k, by considering suitable combinations of k order derivatives with respect to x^0 or x^1 of equations of the reduced system (2.9) and by using continuity of V and its derivatives of all order in the neighborhood of $I = H_1 \cap H_2$. We then obtain the following proposition which generalizes proposition 4.2.

Proposition 4.3. *(a) Let $V = \big(A_0 \equiv 0, A_i, F^{0i}, F_{ij}, f\big)$ be a $\mathcal{C}^\infty$ solution, defined in a neighborhood $\widetilde{\sqcup}$ of $\widehat{H}$ in $\widehat{\sqcup}$, of the evolution problem* (2.9) *subject to* (3.8), (3.9), (3.10) *defined in a neighborhood such that initial data φ satisfies hypothesis (I). Then the restrictions to $\widehat{H}$ of derivatives of order l of V, that is $[\partial_0^l A_i]$, $[\partial_0^l F_{ij}]$, $[\partial_0^l F^{0i}]$ and $[\partial_0^l f]$, $l \in \mathbb{N}$, are uniquely determined on $\widehat{H}$. These functions are continuous on $\widehat{H}$ and are $\mathcal{C}^\infty$ on $\widehat{H}_r, r = 1, 2$. Moreover, for every $l \in \mathbb{N}$,* $\operatorname{supp}[\partial_0^l f]$ *is compact and contained in* $\operatorname{supp}\varphi$.

(b) Moreover, if W is a $\mathcal{C}^\infty$ function defined in a neighborhood of $\widehat{H}$ in $\widehat{\sqcup}$ such that for every $l \in \mathbb{N}$, $[\partial_0^l W] = [\partial_0^l V]$ on $\widehat{H}$, then W satisfies on $\widehat{H}$ the reduced system (2.9) *and its derivatives of all orders.*

In the next section, we will use the following convenient notation.

$$[\partial_0^k V] = \big(\Lambda_i^{(k)}, \Xi_i^{(k)}, \Omega_{ij}^{(k)}, f^{(k)}\big), k \in \mathbb{N},$$

with

$$\Lambda_i^{(k)} = \begin{cases} \overline{\Lambda}_i^{(k)} & \text{on } H_1 \\ \widetilde{\Lambda}_i^{(k)} & \text{on } H_2; \end{cases} \quad E_i^{(k)} = \begin{cases} \overline{E}_i^{(k)} & \text{on } H_1 \\ \widetilde{E}_i^{(k)} & \text{on } H_2; \end{cases}$$

$$\Omega_{ij}^{(k)} = \begin{cases} \overline{\Omega}_{ij}^{(k)} & \text{on } H_1 \\ \widetilde{\Omega}_{ij}^{(k)} & \text{on } H_2; \end{cases} \quad f^{(k)} = \begin{cases} \overline{f}^{(k)} & \text{on } \widehat{H}_1 \\ \widetilde{f}^{(k)} & \text{on } \widehat{H}_2; \end{cases}$$

$$\overline{\Lambda}_i^{(k)} = [\partial_0^k A_i]_1, \quad \overline{E}_i^{(k)} = [\partial_0^k F^{0i}]_1, \quad \overline{\Omega}_{ij}^{(k)} = [\partial_0^k Fij]_1, \quad \overline{f}^{(k)} = [\partial_0^k f]_1,$$
$$\widetilde{\Lambda}_i^{(k)} = [\partial_0^k A_i]_2, \quad \widetilde{E}_i^{(k)} = [\partial_0^k F^{0i}]_2, \quad \widetilde{\Omega}_{ij}^{(k)} = [\partial_0^k F_{ij}]_2, \quad \widetilde{f}^{(k)} = [\partial_0^k f]_2.$$

We also use the notation

$$[\partial_0^k V] = \Phi^{(k)} = \begin{cases} \overline{\Phi}^{(k)}(x^1, x^2, x^3) & \text{on } \widehat{H}_1 \\ \widetilde{\Phi}^{(k)}(x^1, x^2, x^3) & \text{on } \widehat{H}_2, \end{cases} \tag{4.11}$$

where

$$\overline{\Phi}^{(k)} \equiv \big(\overline{\Lambda}_i^{(k)}, \overline{E}_i^{(k)}, \overline{\Omega}_{ij}^{(k)}, \overline{f}^{(k)}\big), \quad \widetilde{\Phi}^{(k)} \equiv \big(\widetilde{\Lambda}_i^{(k)}, \widetilde{E}_i^{(k)}, \widetilde{\Omega}_{ij}^{(k)}, \widetilde{f}^{(k)}\big), \quad \forall k \in \mathbb{N}.$$

5. Resolution of problem (2.9), (3.8), (3.9), (3.10)

The goal of this section is to solve the evolution problem (2.9), (3.8), (3.9), (3.10), where the initial data φ satisfies the support condition of hypothesis (I). The method used consists in reducing this problem into an ordinary Cauchy problem with zero data assigned on the spatial hypersurface $x^0 = 0$, which we solve thanks to a suitable combination of the classical characteristics method, the Leray's theory [25] of hyperbolic systems and techniques of solution developed in [7] for the ordinary Cauchy problem associated to Yang-Mills-Vlasov equations. According to section 4, the evolution problem at hand, of unknown $V = \big(A_0 \equiv 0, A_i, F^{0i}, F_{ij}, f\big)$, is equivalent to the following Goursat problem defined in $\widehat{\sqcup}_T = \sqcup_T \times \mathbb{R}^3 \times O$,

$$\begin{gathered} \widehat{\nabla}_\alpha F^{\alpha i} = J^i \\ \widehat{\nabla}_0 F_{ij} + \widehat{\nabla}_i F_{j0} + \widehat{\nabla}_j F_{0i} = 0 \\ p^\alpha \frac{\partial f}{\partial x^\alpha} + P^i \frac{\partial f}{\partial p^i} + Q^L \frac{\partial f}{\partial q^L} = 0, \\ \partial_0 A_i = F_{0i}, \\ [\partial_0^k V] = \Phi^{(k)} = \begin{cases} \overline{\Phi}^{(k)} & \text{on } \widehat{H}_1 \\ \widetilde{\Phi}^{(k)} & \text{on } \widehat{H}_2, \end{cases} \end{gathered} \tag{5.1}$$

where $L = 1, \dots, N-1$; $\alpha = 0, 1, 2, 3$; $i, j = 1, 2, 3$; $k \in \mathbb{N}$.

Proceeding as in [11] and [29], we transform problem (5.1) into a Goursat problem defined in $\widehat{\sqcup}_T$ with zero initial data on $\widehat{H}$, by introducing a new unknown function $V_1 = \big(C_i, D^{0i}, D_{ij}, v\big)$ such that $V = W + V_1$, where the auxiliary function $W = \big(B_i, G^{0i}, G_{ij}, h\big)$ must be a $\mathcal{C}^\infty$ function on $\widehat{\sqcup}$ such that

$$[\partial_0^l W] = \Phi^{(l)}, \quad \forall l \in \mathbb{N}; \tag{5.2}$$

i. e., for all $l \in \mathbb{N}$,

$$[\partial_0^l B_i] = \Lambda_i^{(l)}, \quad [\partial_0^l G^{0i}] = E_i^{(l)}, \quad [\partial_0^l G_{ij}] = \Omega_{ij}^{(l)}, \quad [\partial_0^l h] = f^{(l)}\,. \tag{5.3}$$

As in [11] and [21] the construction of the function W is made thanks to some variants of classical Borel lemma [28]. The function W so constructed is defined not only in the domain $\widehat{\sqcup}_T$, but also in the whole domain $\widehat{\Omega}_T^\varepsilon = \Omega_T^\varepsilon \times \mathbb{R}^3 \times O$, where Ω_T^ε is the maximal subdomain of

$$\mathcal{L} = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4, 0 \le x^0 \le T, -T \le x^1 \le T, (x^2, x^3) \in B\}$$

containing $\sqcup_T$, of which the future boundary $\partial\Omega_T^{(f)}$ contains $\partial \sqcup_T \setminus H$ and of which the past boundary is equal to $\sqcup_T \cap \{x^0 = 0\}$. The component h of W is $\mathcal{C}^\infty$ with a compact support, since the $f^{(l)}$ have their supports contained in $\operatorname{supp}\varphi$ which is compact. The transformed problem of unknown V_1 can also be written in the domain $\widehat{\sqcup}_T$ as follows:

$$\begin{aligned} \partial_0 D^{0i} + \partial_j D^{ji} + [B_j + C_j, D^{ji}] + [C_j, G^{ji}] - \int_{\mathbb{R}^3\times O} qp^i v w_p w_q = X^i \\ \partial_0 D^{ij} - \partial_i D^{j0} - \partial_j D^{0i} - [B_i + C_i, D^{j0}] - [B_j + C_j, D^{0i}] \\ + [C_i, G_{j0}] + [C_j, G_{0i}] = Z_{0ij}, \\ p^\alpha \frac{\partial v}{\partial x^\alpha} + p^\mu q\left(G^i_\mu + D^i_\mu\right)\frac{\partial v}{\partial p^i} - p^\alpha\left([B_\alpha, q]^L + [C_\alpha, q]^L\right)\frac{\partial v}{\partial q^L} \\ + p^\mu q D^i_\mu \frac{\partial h}{\partial p^i} - p^\alpha [C_\alpha, q]^L \frac{\partial h}{\partial q^L} = \omega, \\ \partial_0 C_i - D_{0i} = E_{0i} \\ [\partial_0^l V_1]_{\widehat{H}_T} = 0, \quad \forall l \in \mathbb{N}, \end{aligned} \tag{5.4}$$

where

$$\begin{gathered} X^i \equiv \int_{\mathbb{R}^3\times O} qp^i h w_p w_q - \partial_0 G^{0i} - \partial_j G^{ji} - [B_j, G^{ji}], \\ Z_{0ij} \equiv -\partial_0 G_{0i} - \partial_i G_{j0} - \partial_j G_{0i} - [B_i, G_{j0}] - [B_j, G_{0i}], \\ \omega \equiv -p^\alpha \frac{\partial h}{\partial x^\alpha} - p^\mu q G^i_\mu \frac{\partial h}{\partial p^i} + p^\alpha [B_\alpha, q]^L \frac{\partial h}{\partial q^L}, \\ E_{0i} \equiv G_{0i} - \partial_0 B_i. \end{gathered}$$

Remark 5.1. According to (5.2), and in view of proposition 4.3 the terms of the right hand side of the system (5.4); i.e., X^i, Z_{0ij}, ω and E_{0i} and their derivatives of all order vanish on $\widehat{H}$.

Consider now the functions $\overline{X}^i$, $\overline{Z}_{0ij}$, $\overline{\omega}$ and $\overline{E}_{0i}$ which are the following continuations on $\widehat{\Omega}_T$ of the functions X^i, Z_{0ij}, ω and E_{0i}:

$$\begin{gathered} \overline{X}^i = \begin{cases} X^i & \text{on } \widehat{\sqcup}_T \\ 0 & \text{on } \widehat{\Omega}_T \backslash \widehat{\sqcup}_T; \end{cases} \quad \overline{Z}_{0ij} = \begin{cases} Z_{0ij} & \text{on } \widehat{\sqcup}_T \\ 0 & \text{on } \widehat{\Omega}_T \backslash \widehat{\sqcup}_T; \end{cases} \\ \overline{\omega} = \begin{cases} \omega & \text{on } \widehat{\sqcup}_T \\ 0 & \text{on } \widehat{\Omega}_T \backslash \widehat{\sqcup}_T; \end{cases} \quad \overline{E}_{0i} = \begin{cases} E_{0i} & \text{on } \widehat{\sqcup}_T \\ 0 & \text{on } \widehat{\Omega}_T \backslash \widehat{\sqcup}_T. \end{cases} \end{gathered}$$

Remark 5.2. In view of Remark 5.1, the functions $\overline{X}^i$, $\overline{Z}_{0ij}$, $\overline{\omega}$ and $\overline{E}_{0i}$ are $\mathcal{C}^\infty$ on $\widehat{\Omega}_T$.

To study problem (5.4), we will first consider the following ordinary Cauchy problem of unknown $\overline{V}_1 = \left(\overline{C}_i, \overline{D}^{0i}, \overline{D}^{ij}, \overline{v}\right)$, defined on $\widehat{\Omega}_T$, with zero initial data given on the spatial hyperplane $x^0 = 0$:

$$
\begin{aligned}
&\partial_0 \overline{D}^{0i} + \partial_j \overline{D}^{ji} + [B_j + \overline{C}_j, \overline{D}^{ji}] + [\overline{C}_j, G^{ji}] - \int_{\mathbb{R}^3\times O} qp^i \overline{v} w_p w_q = \overline{X}^i,\\
&\partial_0 \overline{D}^{ij} - \partial_i \overline{D}^{j0} - \partial_j \overline{D}^{0i} - [B_i + \overline{C}_i, \overline{D}^{j0}]\\
&- [B_j + \overline{C}_j, \overline{D}^{0i}] + [\overline{C}_i, G_{j0}] + [\overline{C}_j, G_{0i}] = \overline{Z}_{0ij},\\
&p^\alpha \frac{\partial \overline{v}}{\partial x^\alpha} + p^\mu q\big(G^i_\mu + \overline{D}^i_\mu\big)\frac{\partial \overline{v}}{\partial p^i} - p^\alpha\left([B_\alpha, q]^L + [\overline{C}_\alpha, q]^L\right)\frac{\partial \overline{v}}{\partial q^L}\\
&+ p^\mu q \overline{D}^i_\mu \frac{\partial h}{\partial p^i} - p^\alpha [\overline{C}_\alpha, q]^L \frac{\partial h}{\partial q^L} = \overline{\omega},\\
&\partial_0 \overline{C}_i - \overline{D}_{0i} = \overline{E}_{0i},\\
&\overline{V}_1 = 0 \text{ on } x^0 = 0.
\end{aligned}
\tag{5.5}
$$

We will show, in the last step, by using mostly the techniques of solution of [7], that problem (5.5) admits in a domain $\widehat{\Omega}_{T_1} = \widehat{\Omega}_T \cap \{x^0 \le T_1\}$, $T_1 \in]0, T_0]$, small enough, a unique $\mathcal{C}^\infty$ solution $\overline{V}_1$ such that the support of $\overline{V}_1$ is contained in $\widehat{\sqcup}_{T_1}$ with the $\operatorname{supp} \overline{v}$ compact.

We will then deduce that $V_1 = \overline{V}_1\big|_{\widehat{\sqcup}_{T_1}}$ is the unique solution of problem (5.4), consequently, the $\mathcal{C}^\infty$ function $V = W + V_1$ will be the unique solution of the evolution problem (2.9) subject to ((3.8), (3.9), (3.10) in the domain $\widehat{\sqcup}_{T_1}$.

5.1. **Resolution of problem** (5.5)**.** We write problem (5.5) defined in $\widehat{\Omega}_T$ in the following appropriate form:

$$
\begin{aligned}
&\partial_0 \overline{D}^{0i} + \partial_j \overline{D}^{ji} = F_1\big(X, \overline{C}_j, \overline{D}^{0i}, \overline{D}^{ji}, \overline{v}\big),\\
&\partial_0 \overline{D}^{ij} - \partial_i \overline{D}^{j0} - \partial_j \overline{D}^{0i} = F_2\big(X, \overline{C}_j, \overline{D}^{0i}, \overline{D}^{ji}, \overline{v}\big),\\
&p^\alpha \frac{\partial \overline{v}}{\partial x^\alpha} + p^\mu q\big(G^i_\mu + \overline{D}^i_\mu\big)\frac{\partial \overline{v}}{\partial p^i} - p^\alpha\left([B_\alpha, q]^L + [\overline{C}_\alpha, q]^L\right)\frac{\partial \overline{v}}{\partial q^L}\\
&= F_3\big(X, \overline{C}_j, \overline{D}^{0i}, \overline{D}^{ji}, \overline{v}\big),\\
&\partial_0 \overline{C}_i = F_4\big(X, \overline{C}_j, \overline{D}^{0i}, \overline{D}^{ji}, \overline{v}\big) \quad \text{on } \widehat{\Omega}_T,\\
&\overline{V}_1 = 0 \quad \text{on } x^0 = 0, \quad \text{where } X = (t, x^1, x^2, x^3, p^i, q^L) \in \widehat{\Omega}_T\,.
\end{aligned}
\tag{5.6}
$$

Definition 5.3. We will call linearized problem associated to problem (5.6), to the given $\mathcal{C}^\infty$ functions $\widehat{C}_j$, $\widehat{D}^{0i}$, $\widehat{D}^{ji}$, $\widehat{v}$ defined on $\widehat{\Omega}_T$, the following linear problem defined in $\widehat{\Omega}_T$, of unknown $\overline{V}_1 = \big(\overline{C}_i, \overline{D}^{0i}, \overline{D}^{ij}, \overline{v}\big)$:

$$
\partial_0 \overline{D}^{0i} + \partial_j \overline{D}^{ji} = F_1\big(X, \widehat{C}_j, \widehat{D}^{0i}, \widehat{D}^{ji}, \widehat{v}\big), \tag{5.7}
$$

$$
\partial_0 \overline{D}^{ij} - \partial_i \overline{D}^{j0} - \partial_j \overline{D}^{0i} = F_2\big(X, \widehat{C}_j, \widehat{D}^{0i}, \widehat{D}^{ji}, \widehat{v}\big), \tag{5.8}
$$

$$
\partial_0 \overline{C}_i = F_4\big(X, \widehat{C}_j, \widehat{D}^{0i}, \widehat{D}^{ji}, \widehat{v}\big) \quad \text{on } \widehat{\Omega}_T, \tag{5.9}
$$

$$
\begin{aligned}
&p^\alpha \frac{\partial \overline{v}}{\partial x^\alpha} + p^\mu q\big(G^i_\mu + \overline{D}^i_\mu\big)\frac{\partial \overline{v}}{\partial p^i} - p^\alpha\left([B_\alpha, q]^L + [\overline{C}_\alpha, q]^L\right)\frac{\partial \overline{v}}{\partial q^L}\\
&= F_3\big(X, \widehat{C}_j, \widehat{D}^{0i}, \widehat{D}^{ji}, \widehat{v}\big),
\end{aligned}
\tag{5.10}
$$

$$
\overline{V}_1 = 0 \quad \text{on } x^0 = 0, \quad \text{where } X = (t, x^1, x^2, x^3, p^i, q^L) \in \widehat{\Omega}_T\,. \tag{5.11}
$$

Proposition 5.4. *If $\widehat{v} \in \mathcal{C}_{\prime}^{\infty}(\widehat{\Omega}_T)$ and $\widehat{C}, \widehat{D} \in \mathcal{C}^{\infty}(\Omega_T)$, then the linear system* (5.7), (5.8), (5.9) *under initial condition $\overline{C} = 0$, $\overline{D} = 0$, on $x^0 = 0$, admits in the domain Ω_T a unique $\mathcal{C}^{\infty}$ solution with support contained in $\sqcup_T$.*

Proof. The subsystem (5.7), (5.8), (5.9) is a linear symmetric hyperbolic system of first order with unknown $\overline{C}$ and $\overline{D}$ of which the right hand side is $\mathcal{C}^{\infty}$ with support contained in $\widehat{\sqcup}_T$. Thanks to Leray's theory [25] of hyperbolic systems, this system possesses in the domain Ω_T a unique $\mathcal{C}^{\infty}$ solution with support contained in the future emission of $\widehat{\sqcup}_T$ which is equal to $\widehat{\sqcup}_T$. Hence $\operatorname{supp} \overline{C} \subset \widehat{\sqcup}_T$, and $\operatorname{supp} \overline{D} \subset \widehat{\sqcup}_T$. $\square$

Proposition 5.5. *If $\widehat{C}$ and $\widehat{D}$ are $\mathcal{C}^{\infty}$ on Ω_T, then the linear equation* (5.10) *under the initial condition $\overline{v} = 0$ on $x^0 = 0$ admits in $\widehat{\Omega}_T$ a unique $\mathcal{C}^{\infty}$ solution $\overline{v}$ with compact support contained in $\widehat{\sqcup}_T$.*

We will mostly use the classical method of characteristics to solve the problem considered in proposition 5.5. The differential characteristic system associated to the first order PDE (5.10) is in fact

$$d\overline{x}^0 = \frac{\overline{p}^0 d\overline{x}^i}{\overline{p}^i} = \frac{\overline{p}^0 d\overline{p}^i}{\overline{p}^0 \overline{q}\big(G^i_\mu + \overline{D}^i_\mu\big)} = \frac{d\overline{q}^L}{-\overline{p}^i [B_i + \overline{C}_i, \overline{q}]^L} = d\tau, \tag{5.12}$$

with $\overline{p}^0 = \big(m^2 + \sum (\overline{p}^i)^2\big)^{1/2}$. The solutions of (5.12) will be called characteristic curves. The proof of proposition 5.5 is based on the following lemma.

Lemma 5.6. *(1) If $m > 0$ and $\widehat{D}$ is differentiable in Ω_T, and $\widehat{D}$ and its derivatives are bounded on Ω_T, then the solution $\tau \mapsto X(x_0^0, x_0^i, p_0^i, q_0^L, \tau)$ of the differential characteristic system* (5.12) *such that $X\big(x_0^0, x_0^i, p_0^i, q_0^L, 0\big) = X\big(x_0^0, x_0^i, p_0^i, q_0^L\big)$ is defined in the interval $]-x_0^0, T - x_0^0[$.*

If moreover $\widehat{C}$ and $\widehat{D}$ are $\mathcal{C}^{\infty}$ on Ω_T, the function: $X\big(x_0^0, x_0^i, p_0^i, q_0^L, 0\big) \mapsto X\big(x_0^0, x_0^i, p_0^i, q_0^L, \tau\big)$ is $\mathcal{C}^{\infty}$.

(2) If $m = 0$, $\operatorname{supp} \varphi \subset \{p^0 > 0\}$ and $p_0^0 > 0$, then the solution of the differential system (5.12) *is defined in the interval $]-x_0^0, \min(T - x_0^0, \varepsilon)[$, where ε is a strictly positive real number depending only on the bounds $\widehat{D}$ on Ω_T and on p_0^0.*

For a proof of the above lemma, see [7, Theorem of section 2].

Proof of Proposition 5.5. From the methods of characteristics and Lemma 5.6, the linear problem under consideration has, in the domain $\widehat{\Omega}_T$, a unique $\mathcal{C}^{\infty}$ solution given by

$$\begin{aligned} \overline{v}\left(x_0^0, x_0^i, p_0^i, q_0^L\right) &= \overline{v}\left(\overline{x}^0(0), \overline{x}^i(0), \overline{p}^i(0), \overline{q}^L(0)\right) \\ &= \int_{-x_0^0}^{0} [\frac{1}{\overline{p}^0(s)} G_3\left(\overline{x}^0(s), \overline{x}^i(s), \overline{p}^i(s), \overline{q}^L(s)\right)] ds, \end{aligned} \tag{5.13}$$

with

$$G_3(X) = \begin{cases} -p^\alpha(X)\frac{\partial h(X)}{\partial x^\alpha} - (p^0 q G^{i0}(X) - p^j q G_{ij}(X))\frac{\partial h(X)}{\partial p^i} \\ +p^\alpha [B_\alpha, q]^L(X)\frac{\partial h(X)}{\partial q^L} - (p^0 q \widehat{D}^{i0}(X) \\ -p^j q \widehat{D}_{ij}(X))\frac{\partial h(X)}{\partial p^i} + p^\alpha [\widehat{C}_\alpha, q]^L(X)\frac{\partial h(X)}{\partial q^L}, & \text{in } \widehat{\sqcup}_T, \\ 0 & \text{in } \widehat{\Omega}_T \backslash \widehat{\sqcup}_T. \end{cases} \tag{5.14}$$

The support of G_3 is compact, contained in $\widehat{\sqcup}_T$, since the support of h is compact.

We will now show that the support of $\overline{v}$ is contained in $\widehat{\sqcup}_T$. Let $(x_0^0, x_0^i, p_0^i, q_0^L) \in \widehat{\Omega}_T \setminus \widehat{\sqcup}_T$. It suffices, in view of the fact that $\operatorname{supp} G_3$ is contained in $\widehat{\sqcup}_T$ (see 5.14), to show that the part of the characteristic curve $\tau \mapsto X\left(x_0^0, x_0^i, p_0^i, q_0^L, \tau\right)$ originating from $\left(x_0^0, x_0^i, p_0^i, q_0^L\right)$ and corresponding to parameters $s \in]-x_0^0, 0]$ is entirely contained in $\widehat{\Omega}_T \setminus \widehat{\sqcup}_T$. For so doing, it suffices to show, by setting

$$X\left(x_0^0, x_0^i, p_0^i, q_0^L, \tau\right) \equiv \left(x^0(\tau), x^1(\tau), p^i(\tau), q^L(\tau)\right),$$

that $|\overline{x}^1(\tau)| > \overline{x}^0(\tau)$ for all $\tau \in]-x_0^0, 0]$; this is an obvious consequence of the following relations:

$$x_0^0 < |x_0^1|, \quad \overline{x}^1(\tau)\tau) = x_0^1 + \int_\tau^0 (\frac{\overline{p}^1}{\overline{p}^0})(s)ds, \quad |\frac{\overline{p}^1}{\overline{p}^0}| \leq 1, \forall \tau \in]-x_0^0, 0].$$

We will now show that $\operatorname{supp} \overline{v}$ is compact.

According to the relations (5.13) and (5.14), $\operatorname{supp} \overline{v}$ is contained in an ε -closed neighborhood of $\operatorname{supp} h$, where ε is a positive real number depending only on T and the bounds of the continuous and bounded functions $\overline{q}\big(G^{i0} + \overline{D}^{i0}\big) - \frac{\overline{P}^j}{\overline{p}^0}\overline{q}\left(G_{ij} + \overline{D}_{ij}\right)$ on Ω_T. We then deduce that $\operatorname{supp} \overline{v}$ is compact, as support of h is compact. $\square$

5.2. **Functional spaces used for the resolution of** (5.4)**.** Let s be an integer and k a given real number such that $s > 4$ and $k > 3/2$.

Definition 5.7 ([7])**.** Let $\bar{D} = (\bar{D}_{\lambda\mu})$ denote a 2-form defined on $\sqcup$. $E^s(\Omega_T)$ is the closure of $\mathcal{C}^\infty(\Omega_T)$ with respect to the norm

$$\|\bar{D}\|_{E^s(\Omega_T)} = \sup_{0 \leq \tau \leq T} \|\bar{D}\|_s^\tau,$$

with

$$\|\overline{D}\|_s^\tau = \Big\{ \int_{\omega_\tau} \sum_{|r| \leq s} \left|\partial^r \overline{D}\right|^2 \mu_\tau \Big\}^{1/2}, \quad |\partial^r \overline{D}|^2 = \sum_{\lambda \leq \mu} (\partial^r \overline{D}_{\lambda\mu})^2,$$

where

$$\mu_\tau = dx^1 dx^2 dx^3, \quad \partial^r = \sum_{|\alpha| \leq r} \frac{\partial^{|\alpha|}}{(\partial x^0)^{\alpha_0} (\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} (\partial x^3)^{\alpha_3}},$$

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3), \quad |\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3\,.$$

$E^{s,k}(\widehat{\Omega_T})$ is the closure of $\mathcal{C}_0^\infty(\widehat{\Omega}_T)$ with respect to the norm

$$\|\overline{v}\|_{E^{s,k}(\Omega_T)} = \sup_{0 \leq \tau \leq T} \|\overline{v}\|_{s,k}^\tau,$$

with

$$\|\overline{v}\|_{s,k}^\tau = \Big\{ \sum_{|l| \leq s} \int_{\widehat{\omega}_\tau} (p^0)^{2k + 2(\widehat{l} + \widetilde{l}) + 1} (D^l \overline{v})^2 \beta_\tau \Big\}^{1/2},$$

where

$$\widehat{\omega}_\tau = w_\tau \times \mathbb{R}^3 \times \mathcal{O}, \quad \beta_\tau = dx^1 dx^2 dx^3 dp^1 dp^2 dp^3 dq^1 dq^2 \dots dq^{N-1},$$

$$D^l = \sum_{|\alpha| \leq l} \frac{\partial^{|\alpha|}}{(\partial x^0)^{\alpha_0} (\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} (\partial x^3)^{\alpha_3} (\partial p^1)^{\alpha_4} \dots (\partial p^3)^{\alpha_6} (\partial q^1)^{\alpha_7} \dots (\partial q^{N-1})^{\alpha_{N+6}}}.$$

Remark 5.8. The functional spaces defined above are the same as those defined in [7].

5.3. **Energy inequalities for the linearized problems.** Problem (5.5) is an ordinary Cauchy problem defined in $\widehat{\Omega}_T$ with zero data specified on the spatial hyperplane $x^0 = 0$. We can establish for the smooth solution of this problem the same energy inequalities as those given in [7]. This energy inequalities will be expressed in the functional spaces

$$H^{s,k}(\widehat{\Omega}_T) \equiv (E^s(\Omega_T))^9 \times E^{s,k}(\widehat{\Omega}_T).$$

Proposition 5.9. *If $\overline{C}$ and $\overline{D} \in \mathcal{C}^\infty(\Omega_T)$ satisfy the linearized Yang-Mills problem (5.7), (5.8), (5.9) under the initial condition $\overline{C}_i = 0$, $\overline{D}^{0i} = 0$, $\overline{D}_{ij} = 0$ on $x^0 = 0$, then for every $t \in]0, T]$, the following inequality is satisfied*

$$\|\overline{C}\|_{E^s(\Omega_t)} + \|\overline{D}\|_{E^s(\Omega_t)} \leq Ct\big[\|F(X, \widehat{C}, \widehat{D}, \widehat{v})\|_{\mathcal{H}^{s,k}(\widehat{\Omega}_t)}\big], \tag{5.15}$$

where C is a positive constant depending only on T.

For a proof of the above proposition see [7].

Proposition 5.10. *If $\widehat{v} \in \mathcal{C}_0^\infty(\widehat{\Omega}_T)$ satisfies the linearized Vlasov equation (5.10) with initial condition $\overline{v} = 0$ on $x^0 = 0$, then, for all $t \in]0, T]$, the following inequality is valid*

$$\|\overline{v}\|_{E^{s,k}(\widehat{\Omega}_t)} \leq Ct\big[\|F(X, \widehat{C}, \widehat{D})\|_{E^s(\Omega_t)}\big], \tag{5.16}$$

where C is a positive constant depending only on T.

For a proof of the above proposition see [7]. To complete the resolution of problem (5.4), let us consider

$$\begin{gathered} g_0 : (\mathcal{C}^\infty(\Omega_T))^9 \times \mathcal{C}_0^\infty(\widehat{\Omega}_T) \longrightarrow (\mathcal{C}^\infty(\Omega_T))^9 \times \mathcal{C}_0^\infty(\widehat{\Omega}_T) \\ \widehat{V} = \big(\widehat{C}_j, \widehat{D}^{0i}, \widehat{D}_{ij}, \widehat{v}\big) \longmapsto \overline{V}_1 = \big(\overline{C}_j, \overline{D}^{0i}, \overline{D}_{ij}, \overline{v}\big), \end{gathered}$$

where $\overline{V}_1$ is the unique solution of the linearized problem (5.7), (5.8), (5.9), (5.10), (5.11).

By using the denseness of $(\mathcal{C}^\infty(\Omega_T))^9 \times \mathcal{C}_0^\infty(\widehat{\Omega}_T)$ in $H^{s,k}(\widehat{\Omega}_T)$, and propositions 5.4 and 5.5, we can obviously show that g_0 can be extended to a function:

$$\begin{gathered} g : H^{s,k}(\widehat{\Omega}_T) \to H^{s,k}(\widehat{\Omega}_T) \\ \widehat{V} \mapsto \overline{V_1}, \end{gathered}$$

where $\overline{V_1}$ is now the unique solution of the linearized problem (5.7), (5.8), (5.9), (5.10), (5.11), with $\widehat{V}$ belonging to $H^{s,k}(\widehat{\Omega}_T)$. Then we show using again propositions 5.4 and 5.5 that there exist some constants $R > 0$, large enough, $T_0 \in]0, T]$, small enough, such that g is a contraction from the closed ball $B(0, R)$ of the Banach space $H^{s,k}(\widehat{\Omega}_T)$ into itself; g then has a unique fixed point $\overline{V}_1 = \big(\overline{C}_i, \overline{D}^{0i}, \overline{D}_{ij}, \overline{v}\big)$, $\operatorname{supp} \overline{V}_1 \subset \widehat{\sqcup}_{T_1}$ with $\operatorname{supp} \overline{v}$ compact. We can also show by a classical argument [6] that $\overline{V}_1 \in (\mathcal{C}^\infty(\Omega_{T_0}))^9 \times \mathcal{C}_0^\infty(\widehat{\Omega}_{T_0})$. $\overline{V}_1$ is then the unique solution of the problem (5.5)

We have then proved the following theorem.

Theorem 5.11. *There exists $T_0 \in]0,T]$, small enough, such that the evolution problem* (2.9), *(* (3.8), (3.9), (3.10), *with initial data φ satisfying hypothesis* (I), *admits in the domain $\widehat{\sqcup}_{T_0}$ a unique $\mathcal{C}^\infty$ solution.*

We sum up the whole work in the following Theorem.

Theorem 5.12. *For any free data $\overline{a}_i, \overline{b}^j$, $(i = 1,2,3;\ j = 2,3)$ $\mathcal{C}^\infty$ on H_1, $\overline{\varphi}$ $\mathcal{C}^\infty$ on $\widehat{H}_1$, and $\widetilde{a}_i, \widetilde{b}^j$ $\mathcal{C}^\infty$ on H_2, $\widetilde{\varphi}$ $\mathcal{C}^\infty$ on $\widehat{H}_2$ and satisfying the following compatibility conditions*

$$\overline{a}_i(0,x^2,x^3) = \widetilde{a}_i(0,x^2,x^3), \quad \text{where } (x^2,x^3) \in B,\ i = 1,2,3;$$
$$\overline{b}^j(0,x^2,x^3) = \widetilde{b}^j(0,x^2,x^3), \quad j = 2,3;$$
$$\big(\partial_1\overline{a}_j - \partial_1\widetilde{a}_j\big)(0,x^2,x^3) = 2\overline{b}^j(0,x^2,x^3) = 2\widetilde{b}^j(0,x^2,x^3);$$
$$\overline{\varphi}(0,x^2,x^3,p^i,q^L) = \widetilde{\varphi}\left(0,x^2,x^3,p^i,q^L\right), \quad (p^i,q^L) \in \mathbb{R}^3 \times O,\ L = 1,\dots,N-1.$$

there exists $T_0 \in]0,T]$, small enough, such that the complete system (2.8) *of Yang-Mills-Vlasov equations admits, in the domain $\widehat{\sqcup}_{T_0}$, a unique $\mathcal{C}^\infty$ solution $V = (A_0 = 0, A_i, F^{0i}, F_{ij}, f)$ satisfying the following conditions:*

$$A_i\big|_H = a_i = \begin{cases} \overline{a}_i & on H_1 \\ \widetilde{a}_i & on H_2, \end{cases} \quad i = 1,2,3;$$
$$F^{0j}\big|_H = b^j = \begin{cases} \overline{b}^j & on H_1 \\ \widetilde{b}^j & on H_2, \end{cases} \quad j = 2,3;$$
$$f\big|_{\widehat{H}} = \varphi = \begin{cases} \overline{\varphi} & on\ \widehat{H}_1 \\ \widetilde{\varphi} & on\ \widehat{H}_2. \end{cases}$$

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