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POSITIVE NONDECREASING SOLUTIONS FOR A MULTI-TERM FRACTIONAL-ORDER FUNCTIONAL DIFFERENTIAL EQUATION WITH INTEGRAL CONDITIONS

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Abstract. In this article, we prove the existence of positive nondecreasing solutions for a multi-term fractional-order functional differential equations. We consider Cauchy boundary problems with: nonlocal conditions, two-point boundary conditions, integral conditions, and deviated arguments.

1. Introduction

Problems with non-local conditions have been extensively studied by several authors in the previous two decades; see for example [1]-[6], [9]-[17] and references therein. In this work we study the existence of nondecreasing solutions for the fractional differential equation

$$x'(t) = f(t, D^{\alpha_1} x(m_1(t)), D^{\alpha_2} x(m_2 t)), \dots, D^{\alpha_n} x(m_n(t))), \quad \alpha_i \in (0,1), \tag{1.1}$$

a.e. $t \in (0,1)$, with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = \beta \sum_{j=1}^{p} b_j x(\eta_j), \tag{1.2}$$

where $a_k, b_j > 0$, $\tau_k \in (a,c)$, $\eta_j \in (d,b)$, $0 < a < c \le d < b < 1$, $\sum_{k=1}^m a_k \neq \beta \sum_{j=1}^p b_j$ and β is parameter.

As applications, we prove the existence of at least one nondecreasing solution for the Cauchy problem of (1.1) with the nonlocal integral condition

$$\int_a^c x(s) d\phi(s) = \beta \int_d^b x(s) d\psi(s), \quad 0 < a < c \le d < b < 1, \tag{1.3}$$

where ϕ and ψ are nondecreasing functions. Also we prove the existence of at least one positive nondecreasing solution for the Cauchy problems of (1.1) with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0, \tau_k \in (a,c) \subset (0,1), \tag{1.4}$$

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and with the integral condition

$$\int_a^c x(s)d\phi(s) = 0, (a,c) \subset (0,1). \tag{1.5}$$

As another applicatin, we problems with deviated arguments $m_i(t) \leq t$, $i = 1, 2 \dots n)$.

2. Preliminaries

Let $L^1 = L^1(I)$ denote the class of Lebesgue integrable functions on the interval $I = [0,1]$ and $\Gamma(\cdot)$ denote the usual gamma function.

Definition 2.1. The fractional-order integral of the function $f \in L^1[a,b]$, of order $\beta > 0$, is defined by (see [19])

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds.$$

The Caputo fractional-order derivative of $f(t)$ of order $\alpha \in (0,1]$ is defined as (see [18, 19])

$$D_a^\alpha f(t) = I_a^{1-\alpha} \frac{d}{dt} f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} f(s)\, ds.$$

Theorem 2.2 (Schauder fixed point theorem [7])**.** *Let E be a Banach space and Q be a convex subset of E, and $T : Q \to Q$ is compact, continuous map, Then T has at least one fixed point in Q.*

Theorem 2.3 (Kolmogorov compactness criterion [8])**.** *Let $\Omega \subseteq L^p[0,1]$, $1 \leq p < \infty$. If*

(i) *Ω is bounded in $L^p[0,1]$, and*
(ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$,*

then Ω is relatively compact in $L^p[0,1]$, where

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)\, ds.$$

3. Main results

We consider firstly the fractional-order functional integral equation

$$y(t) = f(t, I^{1-\alpha_1} m_1'(t) y(m_1(t)), \dots, I^{1-\alpha_n} m_n'(t) y(m_n(t))). \tag{3.1}$$

A function y is called a solution of the fractional-order functional integral equation (3.1) if $y \in L^1[0,1]$ and satisfies (3.1).

In this article, we use the following assumption:

(i) $f : [0,1] \times R_+^n \to R_+$ is a function with the following properties:
  (a) for each $t \in [0,1], f(t,.)$ is continuous,
  (b) for each $x \in R_+^n, f(.,x)$ is measurable;
(ii) there exists an integral function $a \in L^1[0,1]$ and constants $q_i > 0$, $i = 1, 2$, such that

$$|f(t,x)| \leq a(t) + \sum_{i=1}^n q_i |x_i|, \text{for each } t \in [0,1], x \in R_n;$$

(iii) $m_i : [0,1] \to [0,1]$ are absolutely continuous functions;

(iv)
$$\sum_{i=1}^{n}\frac{q_i}{\Gamma(2-\alpha_i)}<1.$$

Theorem 3.1. *Assume* (i)-(iv). *Then* (3.1) *has at least one positive solution* $y\in L^1[0,1]$.

Proof. Define the operator T associated with (3.1) by
$$Ty(t)=f(t,I^{1-\alpha_1}m_1'(t)y(m_1(t)),\dots,I^{1-\alpha_n}m_n'(t)y(m_n(t))).$$
Let $B_r^+=\{y\in R^+:\|y\|_{L^1}\le r\}\subset L^1$,
$$r=\frac{\|a\|}{1-\sum_{i=1}^{n}\frac{q_i}{\Gamma(2-\alpha_i)}}.$$
Let y be an arbitrary element in B_r^+. Then from the assumptions (i) and (ii), we obtain
$$\begin{aligned}
\|Ty\|_{L_1}&=\int_0^1|Ty(t)|dt\\
&=\int_0^1|f(t,I^{1-\alpha_1}m_1'(t)y(m_1(t)),\dots,I^{1-\alpha_n}m_n'(t)y(m_n(t)))|dt\\
&\le\int_0^1|a(t)|dt+\sum_{i=1}^{n}q_i\int_0^1\int_0^t\frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)}|y(m_i(s))|dm_i(s)dt\\
&\le\|a\|_{L^1}+\sum_{i=1}^{n}q_i\int_0^1\Big(\int_s^1\frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)}dt\Big)|y(m_i(s))|dm_i(s)\\
&\le\|a\|_{L^1}+\sum_{i=1}^{n}q_i\int_{m_i(0)}^{m_i(1)}\frac{1}{\Gamma(2-\alpha_i)}|y(m_i(s))|dm_i(s)\\
&\le\|a\|_{L^1}+\sum_{i=1}^{n}q_i\int_0^1\frac{1}{\Gamma(2-\alpha_i)}|y(u)|du\\
&\le\|a\|_{L^1}+\sum_{i=1}^{n}\frac{q_i}{\Gamma(2-\alpha_i)}\|y\|_{L_1}\le r,
\end{aligned}$$
which implies that the operator T maps B_r^+ into itself.

Assumption (i) implies that T is continuous. Now, we will show that T is compact. Let Ω be a bounded subset of B_r^+. Then $T(\Omega)$ is bounded in $L^1[0,1]$; i.e., condition (i) of Theorem 2.3 is satisfied. It remains to show that $(Ty)_h\to Ty$ in $L^1[0,1]$ as $h\to 0$, uniformly with respect to $Ty\in T\Omega$. Now
$$\begin{aligned}
&\|(Ty)_h-Ty\|_{L^1}\\
&=\int_0^1|(Ty)_h(t)-(Ty)(t)|dt\\
&=\int_0^1\Big|\frac{1}{h}\int_t^{t+h}(Ty)(s)\,ds-(Ty)(t)\Big|dt\\
&=\int_0^1\Big(\frac{1}{h}\int_t^{t+h}|(Ty)(s)-(Ty)(t)|\,ds\Big)dt
\end{aligned}$$

$$\leq \int_0^1 \frac{1}{h} \int_t^{t+h} \Big| f(s, I^{1-\alpha_1} m_1'(s) y(m_1(s)), \dots, I^{1-\alpha_n} m_n'(s) y(m_n(s))) \\ - f(t, I^{1-\alpha_1} m_1'(t) y(m_1(t)), \dots, I^{1-\alpha_n} m_n'(t) y(m_n(t))) \Big| \, ds \, dt.$$

Now, by assumption (ii), $y \in \Omega$ implies $f \in L^1[0,1]$; then

$$\frac{1}{h} \int_t^{t+h} \big| f(s, I^{1-\alpha_1} m_1'(s) y(m_1(s)), \dots) - f(t, I^{1-\alpha_1} m_1'(t) y(m_1(t)), \dots) \big| \, ds \, dt \to 0.$$

Therefore, by Theorem 2.3, $T(\Omega)$ is relatively compact; that is, T is compact, then the operator T has a fixed point in B_r^+, which proves the existence of positive solution $y \in B_r^+ \subset L^1[0,1]$ of equation (3.1). □

Let $AC[0,1]$ be the class of absolutely continuous functions defined on $[0,1]$. For the existence of solution for the nonlocal problem (1.1)-(1.2), we have the following result.

Theorem 3.2. *Under the assumptions of Theorem 3.1, problem* (1.1)-(1.2) *has at least one solution* $x \in AC[0,1]$.

Proof. Let $y(t) = x'(t)$, then

$$x(t) = x(0) + I y(t), \tag{3.2}$$

$$x'(m_i(t)) = m_i'(t) y(m_i(t)), \tag{3.3}$$

and y is the solution of the fractional-order functional integral equation (3.1). Let $t = \tau_k$ in equation (3.2). We obtain

$$x(\tau_k) = \int_0^{\tau_k} y(s)\, ds + x(0),$$

$$\sum_{k=1}^m a_k x(\tau_k) = \sum_{k=1}^m a_k \int_0^{\tau_k} y(s)\, ds + x(0) \sum_{k=1}^m a_k.$$

Let $t = \eta_j$ in equation (3.2). We obtain

$$x(\eta_j) = \int_0^{\eta_j} y(s)\, ds + x(0),$$

$$\sum_{j=1}^p b_j x(\eta_j) = \sum_{j=1}^p bj \int_0^{\eta_j} y(s)\, ds + x(0) \sum_{j=1}^p b_j.$$

From (1.2), we obtain

$$\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)\, ds + x(0) \sum_{k=1}^m a_k = \beta \sum_{j=1}^p b_j \int_0^{\eta_j} y(s)\, ds + x(0) \beta \sum_{j=1}^p b_j.$$

Then

$$x(0) = A\Big(\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)\, ds - \beta \sum_{j=1}^p b_j \int_0^{\eta_j} y(s)\, ds \Big),$$

$$A = (\beta \sum_{j=1}^p b_j - \sum_{k=1}^m a_k)^{-1},$$

and

$$x(t) = A\Big(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\,ds\Big) + \int_0^t y(s)\,ds, \tag{3.4}$$

which, by Theorem 3.1, has at least one solution $x \in AC(0,1)$.

Now, from equation (3.4), we have

$$x(0) = \lim_{t\to 0^+} x(t) = A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds - A\beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\,ds$$

and

$$x(1) = \lim_{t\to 1^-} x(t) = A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds - A\beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\,ds + \int_0^1 y(s)\,ds,$$

from which we deduce that (3.4) has at least one solution $x \in AC[0,1]$.

Next we differentiate (3.4), to obtain

$$\begin{gathered}\frac{dx}{dt} = y(t),\\ D^{\alpha_i} x(m_i(t)) = I^{1-\alpha_i}\frac{d}{dt}x(m_i(t)) = I^{1-\alpha_i} m_i'(t) y(m_i(t)),\\ x'(t) = f(t, D^{\alpha_1}x(t), D^{\alpha_2}x(t), \dots, D^{\alpha_n}x(t)).\end{gathered}$$

By direct calculation, we can prove that (3.4) satisfies the nonlocal condition (1.2). This completes the proof. □

From the above theorem we have the following corollaries.

Corollary 3.3. *Under the assumptions of Theorem 3.1, the solution of* (1.1)-(1.2) *is nondecreasing.*

Proof. Let $t_1, t_2 \in (0,1)$ and $t_1 < t_2$, then from (3.4) we have

$$\begin{aligned} x(t_1) &= A\Big(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\,ds\Big) + \int_0^{t_1} y(s)\,ds\\ &\le A\Big(\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds - \beta \sum_{j=1}^{p} b_j \int_0^{\eta_j} y(s)\,ds\Big) + \int_0^{t_2} y(s)\,ds\\ &= x(t_2)\end{aligned}$$

and the solution of the nonlocal problem (1.1)-(1.2) is nondecreasing. □

Corollary 3.4. *Under the assumptions of Theorem 3.1, problem* (1.1) *with the nonlocal condition*

$$\sum_{k=1}^{m} a_k x(\tau_k) = 0, \quad \tau_k \in (a,c) \subset (0,1). \tag{3.5}$$

has at least one nondecreasing solution $x \in AC[0,1]$, represented by

$$x(t) = \int_0^t y(s)\,ds - A^* \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds, \quad A^* = (\sum_{k=1}^{m} a_k)^{-1}. \tag{3.6}$$

This solution is positive in the interval $[c,1]$.

Proof. Letting $\beta = 0$ in (1.2) and (3.4), then from Theorem 3.2 we deduce that the nonlocal problem (1.1) and (3.5) has at least one nondecreasing solution given by (3.6). Let $t \in [c, 1]$, then $\tau_k < t$ and

$$A^* \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds \le A^* \sum_{k=1}^{m} a_k \int_0^{t} y(s)\,ds = \int_0^t y(s)\,ds,$$

which proves that the solution (3.6) is positive in $[c, 1]$. □

Corollary 3.5. *Under the assumptions of Theorem 3.1, the two point problem*

$$x'(t) = f(t, D^{\alpha_1} x(m_1(t)), D^{\alpha_2} x(m_2(t)), \dots, D^{\alpha_n} x(m_n(t))), \alpha_i \in (0, 1),$$
$$a.e. t \in (0, 1),$$
$$x(\tau) = \beta x(\eta), \quad \tau, \eta \in (a, c) \subset (0, 1).$$

has at least one nondecreasing solution $x \in AC[0, 1]$ *represented by*

$$x(t) = A(\int_0^{\tau} y(s)\,ds - \beta \int_0^{\eta} y(s)\,ds) + \int_0^t y(s)\,ds, A = (\beta - 1)^{-1}. \tag{3.7}$$

This solution is positive in the interval $[c, 1]$.

4. Integral condition

Let$x \in AC[0, 1]$ be the solution of the nonlocal problem (1.1)-(1.2). Let$a_k = \phi(\tau_k) - \phi(\tau_{k-1})$, $t_k \in (\tau_{k-1}, \tau_k)$, $a = \tau_0 < \tau_1 < \tau_2, \dots < \tau_m = c$ and $b_j = \psi(\eta_j) - \psi(\eta_{j-1})$, $t_j \in (\eta_{j-1}, \eta_j)$, $d = \eta_0 < \eta_1 < \eta_2, \dots < \eta_p = b$ then the nonlocal condition (1.2) will be

$$\sum_{k=1}^{m} (\phi(\tau_k) - \phi(\tau_{k-1})) x(t_k) = \beta \sum_{j=1}^{p} (\psi(\eta_j) - \psi(\eta_{j-1})) x(t_j).$$

From the continuity of the solution x of (1.1)-(1.2) we can obtain

$$\lim_{m \to \infty} \sum_{k=1}^{m} (\phi(\tau_k) - \phi(\tau_{k-1})) x(t_k) = \beta \lim_{p \to \infty} \sum_{j=1}^{p} (\psi(\eta_j) - \psi(\eta_{j-1})) x(t_j).$$

and the nonlocal condition (1.2) transformed to the integral condition

$$\int_a^c x(s) d\phi(s) = \beta \int_d^b x(s) d\psi(s). \tag{4.1}$$

Also from the continuity of the function $Iy(t)$, where y is the solution of (3.1), we deduce that the solution (3.4) will be

$$x(t) = (\beta(b - d) - (c - a))^{-1} \Big(\int_a^c \int_0^s y(\theta)\,d\phi(\theta)\,ds - \beta \int_d^b \int_0^s y(\theta) d\psi(\theta)\,ds \Big) + \int_0^t y(s)\,ds.$$

Theorem 4.1. *Under the assumptions of Theorem 3.2, there exists at least one nondecreasing solution* $x \in AC[0, 1]$ *of the nonlocal problem with integral condition,*

$$x'(t) = f(t, D^{\alpha_1} x(m_1(t)), D^{\alpha_2} x(m_2(t)), \dots, D^{\alpha_n} x(m_n(t))), \alpha_i \in (0, 1),$$
$$a.e. t \in (0, 1),$$

$$\int_a^c x(s)\,ds = \beta \int_d^b y(s)\,ds, \quad \beta(b-d) \neq (c-a).$$

Letting $\beta = 0$ in (4.1), the we can easily prove the following corollary.

Corollary 4.2. *Under the assumptions of Theorem 3.2, the nonlocal problem*

$$x'(t) = f(t, D^{\alpha_1}x(m_1(t)), D^{\alpha_2}x(m_2(t)), \dots, D^{\alpha_n}x(m_n(t))),$$
$$\alpha_i \in (0,1), \ a.e.t \in (0,1),$$
$$\int_a^c x(s)\,ds = 0, \quad (a,c) \subset (0,1)$$

has at least one nondecreasing solution $x \in AC[0,1]$ *represented by*

$$x(t) = \int_0^t y(s)\,ds - (c-a)^{-1} \int_a^c \int_0^s y(\theta)d\theta\,ds.$$

This solution is positive in the interval $[c,1]$.

5. Equations with deviated arguments

As a first example, let $m_i(t) = \beta_i t$, $\beta_i \in (0,1)$, then $m_i : [0,1] \to [0,1]$ is absolutely continuous deviated functions and all our results here can be applied to the multi-term fractional-order functional differential equation with deviated arguments

$$x'(t) = f(t, D^{\alpha_1}x(\beta_1 t), D^{\alpha_2}x(\beta_2 t), \dots, D^{\alpha_n}x(\beta_n t)), \alpha_i \in (0,1), \quad \text{a.e. } t \in (0,1). \tag{5.1}$$

As a second example, let $m_i(t) = t^{\gamma_i}$, $\gamma_i \geq 1$, then $m_i : [0,1] \to [0,1]$ is absolutely continuous deviated functions and all our results here can be applied to the multi-term fractional-order functional differential equation with deviated arguments

$$x'(t) = f(t, D^{\alpha_1}x(t^{\gamma_1}), D^{\alpha_2}x(t^{\gamma_2}), \dots, D^{\alpha_n}x(t^{\gamma_n})), \alpha_i \in (0,1), \quad \text{a.e. } t \in (0,1). \tag{5.2}$$

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