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OSCILLATION OF SOLUTIONS TO ODD-ORDER NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this note, we establish some new comparison theorems and Philos-type criteria for oscillation of solutions to the odd-order nonlinear neutral functional differential equation

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)x^{\alpha}(\sigma(t))=0,$$

where $0\le p(t)\le p_0<\infty$ and $\alpha\ge 1$.

1. Introduction

This paper is concerned with the oscillation and asymptotic behavior of solutions to the odd-order nonlinear neutral differential equation

$$\big[x(t)+p(t)x(\tau(t))\big]^{(n)}+q(t)x^{\alpha}(\sigma(t))=0, \tag{1.1}$$

where $n\ge 3$ is an odd integer, $\alpha\ge 1$ is the ratio of odd positive integers, $p(t),q(t)\in C([t_0,\infty))$ and

- (H1) $q(t)>0$, $0\le p(t)\le p_0<\infty$;
- (H2) $\tau(t)=a+bt$, with $b>0$, $\sigma(t)\in C([t_0,\infty))$, $\tau(t)\le t$, $\tau\circ\sigma=\sigma\circ\tau$, $\lim_{t\to\infty}\sigma(t)=\infty$.

We set $z(t)=x(t)+p(t)x(\tau(t))$. By a solution of (1.1), we mean a function $x(t)\in C([T_x,\infty))$, $T_x\ge t_0$, which has the property $z(t)\in C^n([T_x,\infty))$ and satisfies (1.1) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (1.1) which satisfy $\sup\{|x(t)|: t\ge T\}>0$ for all $T\ge T_x$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

Since the differential equations have important applications in the natural sciences, technology and population dynamics, there is a permanent interest in obtaining sufficient conditions for the oscillation or nonoscillation of the solutions of various types of even-order/odd-order differential equations; see references in this article, and their references.

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For the oscillation of odd-order neutral differential equations, see e.g., [3, 4, 5, 6, 7, 10, 11, 12, 13, 20, 21, 24, 25, 27, 29]. They studied the oscillatory behavior of odd-order neutral differential equations

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)x(\sigma(t))=0,$$
$$[x(t)+p(t)x(t-\tau)]^{(n)}+q(t)h(x(t-\sigma))=0,$$

and established some oscillatory and asymptotic criteria for the case when $-1 \le p(t) \le 1$.

To the best of our knowledge, the study of oscillatory behavior of odd-order neutral differential equations has not been sufficient. In this paper, we try to obtain some new oscillation results for (1.1). To prove our results, we use the following definition and remarks.

Definition 1.1. Consider the sets $\mathbb{D}_0=\{(t,s): t>s\ge t_0\}$ and $\mathbb{D}=\{(t,s): t\ge s\ge t_0\}$. Assume that $H\in C(\mathbb{D},\mathbb{R})$ satisfies the following assumptions:

(A1) $H(t,t)=0$, $t\ge t_0$; $H(t,s)>0$, $(t,s)\in\mathbb{D}_0$;

(A2) H has a non-positive continuous partial derivative with respect to the second variable in $\mathbb{D}_0$.

Then the function H has the property P.

Remark 1.2. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

Remark 1.3. Without loss of generality we can deal only with the positive solutions of equation (1.1).

2. Main results

The Kiguradze's lemma is stated below, the readers may find this result in [14, 15], which plays an important role in the oscillation of higher-order differential equations.

Lemma 2.1 (Kiguradze's lemma). *Let $f\in C^n([t_0,\infty),\mathbb{R})$ and its derivatives up to order $(n-1)$ are of constant sign in $[t_0,\infty)$. If $f^{(n)}$ is of constant sign and not identically zero on a sub-ray of $[t_0,\infty)$, then there exist $m\in\mathbb{Z}$ and $t_1\in[t_0,\infty)$ such that $0\le m\le n-1$, and $(-1)^{n+m}ff^{(n)}\ge 0$,*

$$ff^{(j)}>0 \quad \text{for } j=0,1,\dots,m-1 \text{ when } m\ge 1$$

and

$$(-1)^{m+j}ff^{(j)}>0 \quad \text{for } j=m,m+1,\dots,n-1 \text{ when } m\le n-1$$

hold on $[t_1,\infty)$.

Lemma 2.2 ([1, Lemma 2.2.3]). *Let f be a function as in Lemma 2.11. If $\lim_{t\to\infty} f(t)\ne 0$, then for every $\lambda\in(0,1)$, there exists $t_\lambda\in[t_1,\infty)$ such that*

$$|f|\ge\frac{\lambda}{(n-1)!}t^{n-1}|f^{(n-1)}|$$

holds on $[t_\lambda,\infty)$.

Lemma 2.3 ([22]). *Let f be a function as in Lemma 2.11. If*

$$f^{(n-1)}(t)f^{(n)}(t)\le 0,$$

then for any constant $\theta \in (0,1)$ and sufficiently large t, there exists a constant $M > 0$, satisfying

$$|f'(\theta t)| \geq M t^{n-2} |f^{(n-1)}(t)|.$$

Lemma 2.4. *If x is a positive solution of (1.1), then the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ satisfies*

$$z(t) > 0, \quad z^{(n-1)}(t) > 0, \quad z^{(n)}(t) < 0 \tag{2.1}$$

eventually.

Due to Lemma 2.1, the proof of the above lemma is simple and so is omitted.

Lemma 2.5 ([13, Lemma 3]). *Let f and $g \in C([t_0,\infty),\mathbb{R})$ and $\alpha \in C([t_0,\infty),\mathbb{R})$ satisfies $\lim_{t\to\infty} \alpha(t) = \infty$ and $\alpha(t) \leq t$ for all $t \in [t_0,\infty)$; further suppose that there exists $h \in C([t_{-1},\infty),\mathbb{R}^+)$, where $t_{-1} := \min_{t\in[t_0,\infty)}\{\alpha(t)\}$, such that $f(t) = h(t) + g(t)h(\alpha(t))$ holds for all $t \in [t_0,\infty)$. Suppose that $\lim_{t\to\infty} f(t)$ exists and $\liminf_{t\to\infty} g(t) > -1$. Then $\limsup_{t\to\infty} h(t) > 0$ implies $\lim_{t\to\infty} f(t) > 0$.*

Lemma 2.6. *Assume that $\alpha \geq 1$, $c, d \in \mathbb{R}$. If $c \geq 0$ and $d \geq 0$, then*

$$c^\alpha + d^\alpha \geq \frac{1}{2^{\alpha-1}}(c+d)^\alpha. \tag{2.2}$$

Proof. (i) Suppose that $c = 0$ or $d = 0$. Then we have (2.2). (ii) Suppose that $c > 0$ and $d > 0$. Define the function f by $f(x) = x^\alpha$, $x \in (0,\infty)$. Then $f''(x) = \alpha(\alpha-1)x^{\alpha-2} \geq 0$ for $x > 0$. Thus, f is a convex function. By the definition of convex function, we have

$$f\big(\frac{c+d}{2}\big) \leq \frac{f(c)+f(d)}{2};$$

that is,

$$c^\alpha + d^\alpha \geq \frac{1}{2^{\alpha-1}}(c+d)^\alpha.$$

This completes the proof. □

Next, we establish our main results. For the sake of convenience, let

$$Q(t) = \min\{q(t), q(\tau(t))\}. \tag{2.3}$$

Theorem 2.7. *Assume that*

$$\int_{t_0}^\infty t^{n-1} Q(t) \mathrm{d}t = \infty. \tag{2.4}$$

Further, assume that the first-order neutral differential inequality

$$\Big(y(t) + \frac{{p_0}^\alpha}{b}\, y(\tau(t))\Big)' + \frac{Q(t)}{2^{\alpha-1}}\Big(\frac{\lambda}{(n-1)!}\sigma^{n-1}(t)\Big)^\alpha y^\alpha(\sigma(t)) \leq 0 \tag{2.5}$$

has no positive solution for some $\lambda \in (0,1)$. Then (1.1) is almost oscillatory.

Proof. Assume that x is a positive solution of (1.1), which does not tend to zero asymptotically. Then the corresponding function z satisfies

$$\begin{aligned} z(\sigma(t)) &= x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t))) \\ &\leq x(\sigma(t)) + p_0 x(\sigma(\tau(t))), \end{aligned} \tag{2.6}$$

where we have used the hypothesis (H1). On the other hand, it follows from (1.1) that

$$z^{(n)}(t) + q(t)x^\alpha(\sigma(t)) = 0 \tag{2.7}$$

and moreover taking (H1) and (H2) into account, we have

$$\begin{aligned} 0 &= \frac{{p_0}^\alpha}{\tau'(t)}(z^{(n-1)}(\tau(t)))' + {p_0}^\alpha q(\tau(t))x^\alpha(\sigma(\tau(t))) \\ &= \frac{{p_0}^\alpha}{b}(z^{(n-1)}(\tau(t)))' + {p_0}^\alpha q(\tau(t))x^\alpha(\sigma(\tau(t))). \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we are led to

$$[z^{(n-1)}(t) + \frac{{p_0}^\alpha}{b} z^{(n-1)}(\tau(t))]' + q(t)x^\alpha(\sigma(t)) + {p_0}^\alpha q(\tau(t))x^\alpha(\sigma(\tau(t))) \le 0, \tag{2.9}$$

which in view of (2.2), (2.3) and (2.6) implies

$$[z^{(n-1)}(t) + \frac{{p_0}^\alpha}{b} z^{(n-1)}(\tau(t))]' + \frac{1}{2^{\alpha-1}} Q(t) z^\alpha(\sigma(t)) \le 0. \tag{2.10}$$

Next, we claim that $z'(t) > 0$ eventually. If not, then $\lim_{t\to\infty} z(t) = a > 0$ (a is finite) due to Lemma 2.5. From (2.1), we obtain $\lim_{t\to\infty} z^{(k)}(t) = 0$ for $k = 1, 2, \dots, n-1$. Integrating (2.10) from t to ∞ for a total of $(n-1)$ times and integrating the resulting inequality from t_1 (t_1 is large enough) to ∞, we obtain

$$\int_{t_1}^\infty \frac{(s-t_1)^{n-1}}{(n-1)!} Q(s) z^\alpha(\sigma(s)) \mathrm{d}s < \infty,$$

which yields

$$\int_{t_1}^\infty s^{n-1} Q(s) \mathrm{d}s < \infty.$$

This contradicts (2.4). Hence by Lemma 2.2 and Lemma 2.4, we obtain

$$z(t) \ge \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t) \text{ for every } \lambda \in (0,1).$$

Thus, it follows from (2.10) that

$$[z^{(n-1)}(t) + \frac{{p_0}^\alpha}{b} z^{(n-1)}(\tau(t))]' + \frac{Q(t)}{2^{\alpha-1}} \Big(\frac{\lambda}{(n-1)!} \sigma^{n-1}(t) z^{(n-1)}(\sigma(t)) \Big)^\alpha \le 0. \tag{2.11}$$

Therefore, setting $z^{(n-1)}(t) = y(t)$ in (2.11), one can see that y is a positive solution of (2.5). This contradicts our assumptions and the proof is complete. $\square$

Remark 2.8. In the comparison principle in Theorem 2.7 we do not assume that the deviating arguments is either delay or advanced type, and hence this result is applicable to all types of equations. Further, the comparison principle established in Theorem 2.7 reduces oscillation of equation (1.1) to find conditions for the first-order neutral differential inequality (2.5) has no positive solution. Therefore, applying the conditions for equation (2.5) to have no positive solution, one can immediately get oscillation criteria for equation (1.1).

Theorem 2.9. *Assume that* (2.4) *holds. If the first-order differential inequality*

$$w'(t) + \frac{Q(t)}{2^{\alpha-1}(1 + \frac{{p_0}^\alpha}{b})} \Big(\frac{\lambda}{(n-1)!} \sigma^{n-1}(t) \Big)^\alpha w^\alpha(\tau^{-1}(\sigma(t))) \le 0 \tag{2.12}$$

has no positive solution for some $0 < \lambda < 1$*, then* (1.1) *is almost oscillatory.*

Proof. Assume that x is a positive solution of (1.1), which does not tend to zero asymptotically. Then $y(t) = z^{(n-1)}(t) > 0$ is a decreasing solution of (2.5). We denote

$$w(t) = y(t) + \frac{{p_0}^{\alpha}}{b}\, y(\tau(t)).$$

It follows from $\tau(t) \le t$ that

$$w(t) \le y(\tau(t))\Big(1 + \frac{{p_0}^{\alpha}}{b}\Big).$$

Substituting this into (2.5), we obtain that w is a positive solution of (2.12). A contradiction. This completes the proof. $\square$

Corollary 2.10. *Assume that* (2.4) *holds, and* $\alpha = 1$ *and* $\sigma(t) < \tau(t)$. *If*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \sigma^{n-1}(s) Q(s)\, \mathrm{d}s > \frac{\left(1 + \frac{p_0}{b}\right)(n-1)!}{\mathrm{e}}, \tag{2.13}$$

then (1.1) *is almost oscillatory.*

Proof. According to [17, Theorem 2.1.1], the condition (2.13) guarantees that (2.12) with $\alpha = 1$ has no positive solution. Hence by Theorem 2.9, equation (1.1) is almost oscillatory. This completes the proof of Corollary 2.10. $\square$

Now, we shall establish some Philos-type oscillation criteria for the oscillation of (1.1).

Theorem 2.11. *Assume that* (2.4) *holds and* $\sigma(t) \ge \tau(t)/2$. *Further, assume that the function* $H \in C(\mathbb{D}, \mathbb{R})$ *has the property* P *and there exist functions* $h \in C(\mathbb{D}_0, \mathbb{R})$ *and* $\rho \in C^1([t_0, \infty), (0, \infty))$ *such that*

$$-\frac{\partial}{\partial s} H(t,s) - H(t,s)\frac{\rho'(s)}{\rho(s)} = h(t,s), \quad (t,s) \in \mathbb{D}_0. \tag{2.14}$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} K_1(t,s)\mathrm{d}s = \infty \tag{2.15}$$

for all constants $M > 0$, $L > 0$ *and for some* $\beta \ge 1$, *where*

$$K_1(t,s) := \big(\frac{L}{2}\big)^{\alpha-1} H(t,s)\rho(s)Q(s) - \Big(1 + \frac{{p_0}^{\alpha}}{b}\Big)\frac{\beta\rho(s)h^2(t,s)}{2bMH(t,s)\tau^{n-2}(s)},$$

then (1.1) *is almost oscillatory.*

Proof. Assume that x is a positive solution of (1.1), which does not tend to zero asymptotically. Proceeding as in the proof of Theorem 2.7, we obtain (2.10) and $z'(t) > 0$. Define

$$w(t) = \rho(t)\frac{z^{(n-1)}(t)}{z\big(\frac{\tau(t)}{2}\big)}, \tag{2.16}$$

then $w(t) > 0$, and

$$w'(t) = \rho'(t)\frac{z^{(n-1)}(t)}{z\big(\tau(t)/2\big)} + \rho(t)\frac{z^{(n)}(t)z\big(\tau(t)/2\big) - \frac{b}{2}z^{(n-1)}(t)z'\big(\tau(t)/2\big)}{z^2\big(\tau(t)/2\big)}. \tag{2.17}$$

It follows from Lemma 2.3 and Lemma 2.4 that there exists a constant $M > 0$, such that

$$z'\big(\tau(t)/2\big) \ge M\tau^{n-2}(t)z^{(n-1)}(\tau(t)) \ge M\tau^{n-2}(t)z^{(n-1)}(t), \tag{2.18}$$

which in view of (2.16) and (2.17) yields

$$w'(t) \leq \rho(t)\frac{z^{(n)}(t)}{z\big(\tau(t)/2\big)} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{bM}{2}\frac{\tau^{n-2}(t)}{\rho(t)}w^2(t), \tag{2.19}$$

Define another function

$$v(t) = \rho(t)\frac{z^{(n-1)}(\tau(t))}{z\big(\tau(t)/2\big)}, \tag{2.20}$$

then $v(t) > 0$, and

$$\begin{aligned} v'(t) &= \rho'(t)\frac{z^{(n-1)}(\tau(t))}{z\big(\tau(t)/2\big)} \\ &\quad + \rho(t)\frac{bz^{(n)}(\tau(t))z\big(\tau(t)/2\big) - \frac{b}{2}z^{(n-1)}(\tau(t))z'\big(\tau(t)/2\big)}{z^2\big(\tau(t)/2\big)}. \end{aligned} \tag{2.21}$$

It follows from (2.18), (2.20) and (2.21) that

$$v'(t) \leq \rho(t)\frac{z^{(n)}(\tau(t))}{z\big(\tau(t)/2\big)} + \frac{\rho'(t)}{\rho(t)}v(t) - \frac{bM}{2}\frac{\tau^{n-2}(t)}{\rho(t)}v^2(t). \tag{2.22}$$

In view of (2.19) and (2.22), we obtain

$$\begin{aligned} w'(t) + \frac{{p_0}^{\alpha}}{b}v'(t) \leq \rho(t)&\frac{z^{(n)}(t) + {p_0}^{\alpha}z^{(n)}(\tau(t))}{z\big(\tau(t)/2\big)} + \frac{\rho'(t)}{\rho(t)}w(t) \\ &- \frac{bM}{2}\frac{\tau^{n-2}(t)}{\rho(t)}w^2(t) + \frac{{p_0}^{\alpha}}{b}[\frac{\rho'(t)}{\rho(t)}v(t) - \frac{bM}{2}\frac{\tau^{n-2}(t)}{\rho(t)}v^2(t)]. \end{aligned}$$

It follows from (2.10) that there exists a constant $L > 0$, such that

$$\begin{aligned} w'(t) + \frac{{p_0}^{\alpha}}{b}v'(t) \leq -\big(\frac{L}{2}\big)^{\alpha-1}\rho(t)Q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{bM}{2}\frac{\tau^{n-2}(t)}{\rho(t)}w^2(t) \\ + \frac{{p_0}^{\alpha}}{b}[\frac{\rho'(t)}{\rho(t)}v(t) - \frac{bM}{2}\frac{\tau^{n-2}(t)}{\rho(t)}v^2(t)]. \end{aligned} \tag{2.23}$$

Multiplying (2.23), with t replaced by s, by $H(t,s)$ and integrating from T to t ,with $T \geq t_1$, we have

$$\begin{aligned} &\int_T^t \big(\frac{L}{2}\big)^{\alpha-1}H(t,s)\rho(s)Q(s)\mathrm{d}s \\ &\leq -\int_T^t H(t,s)w'(s)\mathrm{d}s + \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)\mathrm{d}s - \int_T^t \frac{bM}{2}H(t,s)\frac{\tau^{n-2}(s)}{\rho(s)}w^2(s)\mathrm{d}s \\ &\quad - \frac{{p_0}^{\alpha}}{b}\int_T^t H(t,s)v'(s)\mathrm{d}s + \frac{{p_0}^{\alpha}}{b}\int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}v(s)\mathrm{d}s \\ &\quad - \frac{{p_0}^{\alpha}}{b}\int_T^t \frac{bM}{2}H(t,s)\frac{\tau^{n-2}(s)}{\rho(s)}v^2(s)\mathrm{d}s. \end{aligned}$$

It follows from the above inequality and (2.14) that

$$\begin{aligned} &\int_T^t \big(\frac{L}{2}\big)^{\alpha-1}H(t,s)\rho(s)Q(s)\mathrm{d}s \\ &\leq H(t,T)w(T) - \int_T^t h(t,s)w(s)\mathrm{d}s - \int_T^t \frac{bM}{2}H(t,s)\frac{\tau^{n-2}(s)}{\rho(s)}w^2(s)\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\frac{p_0{}^{\alpha}}{b}H(t,T)v(T)-\frac{p_0{}^{\alpha}}{b}\int_T^t h(t,s)v(s)\mathrm{d}s\\
&-\frac{p_0{}^{\alpha}}{b}\int_T^t \frac{bM}{2}H(t,s)\frac{\tau^{n-2}(s)}{\rho(s)}v^2(s)\mathrm{d}s.
\end{aligned}
$$

Thus, for any $\beta\ge 1$,

$$
\begin{aligned}
&\int_T^t \big(\frac{L}{2}\big)^{\alpha-1}H(t,s)\rho(s)Q(s)\mathrm{d}s\\
&\le H(t,T)w(T)+\int_T^t \frac{\beta\rho(s)h^2(t,s)}{2bM\tau^{n-2}(s)H(t,s)}\mathrm{d}s\\
&\quad-\int_T^t\Big[\sqrt{\frac{bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)}}w(s)+\sqrt{\frac{2\beta\rho(s)}{4bM\tau^{n-2}(s)H(t,s)}}h(t,s)\Big]^2\mathrm{d}s\\
&\quad-\int_T^t \frac{(\beta-1)bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)}w^2(s)\mathrm{d}s\\
&\quad+\frac{p_0{}^{\alpha}}{b}H(t,T)v(T)+\frac{p_0{}^{\alpha}}{b}\int_T^t \frac{\beta\rho(s)h^2(t,s)}{2bM\tau^{n-2}(s)H(t,s)}\mathrm{d}s\\
&\quad-\frac{p_0{}^{\alpha}}{b}\int_T^t\Big[\sqrt{\frac{bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)}}v(s)+\sqrt{\frac{2\beta\rho(s)}{4bM\tau^{n-2}(s)H(t,s)}}h(t,s)\Big]^2\mathrm{d}s\\
&\quad-\frac{p_0{}^{\alpha}}{b}\int_T^t \frac{(\beta-1)bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)}v^2(s)\mathrm{d}s.
\end{aligned}\tag{2.24}
$$

From the above inequality, we obtain

$$
\begin{aligned}
&\int_T^t\Big[\big(\frac{L}{2}\big)^{\alpha-1}H(t,s)\rho(s)Q(s)-\big(1+\frac{p_0{}^{\alpha}}{b}\big)\frac{\beta\rho(s)h^2(t,s)}{2bMH(t,s)\tau^{n-2}(s)}\Big]\mathrm{d}s\\
&\le H(t,T)\Big(w(T)+\frac{p_0{}^{\alpha}}{b}v(T)\Big)\\
&\le H(t,t_0)\Big(w(T)+\frac{p_0{}^{\alpha}}{b}v(T)\Big),
\end{aligned}
$$

which yields

$$
\frac{1}{H(t,t_0)}\int_{t_0}^t\Big[\big(\frac{L}{2}\big)^{\alpha-1}H(t,s)\rho(s)Q(s)-\big(1+\frac{p_0{}^{\alpha}}{b}\big)\frac{\beta\rho(s)h^2(t,s)}{2bMH(t,s)\tau^{n-2}(s)}\Big]\mathrm{d}s<\infty.
$$

This contradicts condition (2.15). The proof is complete. □

As a consequence of Theorem 2.11, we obtain the following corollary.

Corollary 2.12. *Let condition* (2.15) *in Theorem 2.11 be replaced by*

$$
\begin{gathered}
\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t H(t,s)\rho(s)Q(s)\mathrm{d}s=\infty,\\
\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \frac{\rho(s)h^2(t,s)}{H(t,s)\tau^{n-2}(s)}\mathrm{d}s<\infty.
\end{gathered}
$$

Then (1.1) *is almost oscillatory.*

It may happen that assumption (2.15) in Theorem 2.11 fails to hold. The following result provide an essentially new oscillation criterion for (1.1).

Theorem 2.13. *Assume that (2.4) holds and $\sigma(t) \geq \tau(t)/2$. Let H, h, ρ be as in Theorem 2.11 and*

$$0 < \inf_{s \geq t_0} \Big[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \Big] \leq \infty. \tag{2.25}$$

Moreover, suppose that there exists a function $m \in C([t_0,\infty),\mathbb{R})$ such that for all $T \geq t_0$ and for some $\beta > 1$, one has

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t K_1(t,s)\mathrm{d}s \geq m(T) \tag{2.26}$$

for all constants $M > 0$ and $L > 0$, where K_1 is defined as in Theorem 2.11. If

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\tau^{n-2}(s) m_+^2(s)}{\rho(s)} \mathrm{d}s = \infty, \tag{2.27}$$

where $m_+(t) := \max\{m(t), 0\}$, then (1.1) is almost oscillatory.

Proof. Assume that x is a positive solution of (1.1), which does not tend to zero asymptotically. Proceeding as in the proof of Theorem 2.11, we obtain (2.24), which implies

$$\begin{aligned}
&\frac{1}{H(t,T)} \int_T^t \Big[\big(\frac{L}{2}\big)^{\alpha-1} H(t,s)\rho(s)Q(s) - \big(1 + \frac{{p_0}^\alpha}{b}\big) \frac{\beta\rho(s)h^2(t,s)}{2bMH(t,s)\tau^{n-2}(s)} \Big] \mathrm{d}s \\
&\leq w(T) - \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)} w^2(s)\mathrm{d}s \\
&\quad + \frac{{p_0}^\alpha}{b} v(T) - \frac{{p_0}^\alpha}{b} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)} v^2(s)\mathrm{d}s.
\end{aligned}$$

Therefore, for $t > T \geq t_1$, sufficiently large,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[\big(\frac{L}{2}\big)^{\alpha-1} H(t,s)\rho(s)Q(s) - \big(1 + \frac{{p_0}^\alpha}{b}\big) \frac{\beta\rho(s)h^2(t,s)}{2bMH(t,s)\tau^{n-2}(s)} \Big] \mathrm{d}s \\
&\leq w(T) + \frac{{p_0}^\alpha}{b} v(T) \\
&\quad - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)} \Big(w^2(s) + \frac{{p_0}^\alpha}{b} v^2(s) \Big) \mathrm{d}s.
\end{aligned}$$

It follows from (2.26) that

$$\begin{aligned}
&w(T) + \frac{{p_0}^\alpha}{b} v(T) \\
&\geq m(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)bM\tau^{n-2}(s)H(t,s)}{2\beta\rho(s)} \Big(w^2(s) + \frac{{p_0}^\alpha}{b} v^2(s) \Big) \mathrm{d}s,
\end{aligned}$$

for all $T \geq t_1$ and for any $\beta > 1$. Consequently, for all $T \geq t_1$, we obtain

$$w(T) + \frac{{p_0}^\alpha}{b} v(T) \geq m(T), \tag{2.28}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^t \frac{H(t,s)\tau^{n-2}(s)}{\rho(s)} \Big(w^2(s) + \frac{{p_0}^\alpha}{b} v^2(s) \Big) \mathrm{d}s \\
&\leq \frac{2\beta}{(\beta-1)bM} \Big(w(t_1) + \frac{{p_0}^\alpha}{b} v(t_1) - m(t_1) \Big) < \infty.
\end{aligned} \tag{2.29}$$

Now we claim that

$$\int_{t_1}^{\infty} \frac{\tau^{n-2}(s)\big(w^2(s)+\frac{p_0{}^{\alpha}}{b}v^2(s)\big)}{\rho(s)}\mathrm{d}s<\infty. \tag{2.30}$$

Suppose to the contrary that

$$\int_{t_1}^{\infty} \frac{\tau^{n-2}(s)\Big(w^2(s)+\frac{p_0{}^{\alpha}}{b}v^2(s)\Big)}{\rho(s)}\mathrm{d}s=\infty. \tag{2.31}$$

By (2.31), for any positive number κ, there exists a $T_1\ge t_1$ such that, for all $t\ge T_1$,

$$\int_{t_1}^{t} \frac{\tau^{n-2}(s)\big(w^2(s)+\frac{p_0{}^{\alpha}}{b}v^2(s)\big)}{\rho(s)}\mathrm{d}s\ge\frac{\kappa}{\rho}.$$

Assumption (2.25) implies the existence of a $\rho>0$ such that

$$\inf_{s\ge t_0}[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}]>\rho. \tag{2.32}$$

From (2.32), we have

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}>\rho>0,$$

and there exists a $T_2\ge T_1$ such that $H(t,T_1)/H(t,t_0)\ge\rho$, for all $t\ge T_2$. Using integration by parts, we conclude that, for all $t\ge T_2$,

$$\begin{aligned}
&\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\tau^{n-2}(s)}{\rho(s)}\Big(w^2(s)+\frac{p_0{}^{\alpha}}{b}v^2(s)\Big)\mathrm{d}s\\
&=\frac{1}{H(t,t_1)}\int_{t_1}^{t}[-\frac{\partial H(t,s)}{\partial s}]\Big[\int_{t_1}^{s}\frac{\tau^{n-2}(u)\big(w^2(u)+\frac{p_0{}^{\alpha}}{b}v^2(u)\big)}{\rho(u)}\mathrm{d}u\Big]\mathrm{d}s\\
&\ge\frac{\kappa}{\rho}\frac{1}{H(t,t_1)}\int_{T_1}^{t}[-\frac{\partial H(t,s)}{\partial s}]\mathrm{d}s=\frac{\kappa H(t,T_1)}{\rho H(t,t_1)}.
\end{aligned} \tag{2.33}$$

It follows from (2.33) that, for all $t\ge T_2$,

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\tau^{n-2}(s)}{\rho(s)}\Big(w^2(s)+\frac{p_0{}^{\alpha}}{b}v^2(s)\Big)\mathrm{d}s\ge\kappa.$$

Since κ is an arbitrary positive constant, we obtain

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\tau^{n-2}(s)}{\rho(s)}\Big(w^2(s)+\frac{p_0{}^{\alpha}}{b}v^2(s)\Big)\mathrm{d}s=\infty,$$

which contradicts (2.29). Consequently, (2.30) holds. Thus, we obtain

$$\int_{t_1}^{\infty}\frac{\tau^{n-2}(s)w^2(s)}{\rho(s)}\mathrm{d}s<\infty,\quad \int_{t_1}^{\infty}\frac{\tau^{n-2}(s)v^2(s)}{\rho(s)}\mathrm{d}s<\infty,$$

and, by (2.28),

$$\begin{aligned}
&\int_{t_1}^{\infty}\frac{\tau^{n-2}(s)m_+^2(s)}{\rho(s)}\mathrm{d}s\\
&\le\int_{t_1}^{\infty}\frac{\tau^{n-2}(s)w^2(s)+\big(\frac{p_0{}^{\alpha}}{b}\big)^2\tau^{n-2}(s)v^2(s)+\frac{2p_0{}^{\alpha}}{b}\tau^{n-2}(s)w(s)v(s)}{\rho(s)}\mathrm{d}s\\
&\le\int_{t_1}^{\infty}\frac{\tau^{n-2}(s)w^2(s)+\big(\frac{p_0{}^{\alpha}}{b}\big)^2\tau^{n-2}(s)v^2(s)+\frac{p_0{}^{\alpha}}{b}\tau^{n-2}(s)[w^2(s)+v^2(s)]}{\rho(s)}\mathrm{d}s<\infty,
\end{aligned}$$

which contracts (2.27). This completes the proof. □

Now, we establish some oscillation criteria for equation (1.1) when $\sigma(t) \le \tau(t)$.

Theorem 2.14. *Let $\sigma(t) \in C^1([t_0,\infty))$ and $\sigma'(t) > 0$. Assume that (2.4) holds and $\sigma(t) \le \tau(t)$. Furthermore, assume that the function $H \in C(\mathbb{D},\mathbb{R})$ has the property P and there exist functions $h \in C(\mathbb{D}_0,\mathbb{R})$ and $\rho \in C^1([t_0,\infty),(0,\infty))$ such that (2.14) holds. If*
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} K_2(t,s)\mathrm{d}s = \infty \tag{2.34}$$
for all constants $M > 0$ and $L > 0$ and for some $\beta \ge 1$, where
$$K_2(t,s) := \big(\frac{L}{2}\big)^{\alpha-1} H(t,s)\rho(s)Q(s) - \big(1 + \frac{{p_0}^{\alpha}}{b}\big) \frac{\beta\rho(s)h^2(t,s)}{2\sigma'(s)MH(t,s)\sigma^{n-2}(s)},$$
then (1.1) is almost oscillatory.

Proof. Define w and v by
$$w(t) = \rho(t)\frac{z^{(n-1)}(t)}{z\big(\sigma(t)/2\big)}, \quad v(t) = \rho(t)\frac{z^{(n-1)}(\tau(t))}{z\big(\sigma(t)/2\big)},$$
respectively. The rest of the proof is similar to that of Theorem 2.11 and so is omitted. □

From Theorem 2.14, wiht a proof similar to the one of Theorem 2.13, we obtain the following result.

Theorem 2.15. *Let $\sigma(t) \in C^1([t_0,\infty))$ and $\sigma'(t) > 0$. Assume that (2.4) holds and $\sigma(t) \le \tau(t)$. Let H, h, ρ be as in Theorem 2.11 such that (2.25) holds. Further, suppose that there exists a function $m \in C([t_0,\infty),\mathbb{R})$ such that for all $T \ge t_0$ and for some $\beta > 1$,*
$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} K_2(t,s)\mathrm{d}s \ge m(T) \tag{2.35}$$
for all constants $M > 0$ and $L > 0$, where K_2 is defined as in Theorem 2.14. If
$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\sigma'(s)\sigma^{n-2}(s)m_+^2(s)}{\rho(s)}\mathrm{d}s = \infty, \tag{2.36}$$
where $m_+(t) := \max\{m(t),0\}$, then (1.1) is almost oscillatory.

Remark 2.16. From Theorems 2.11–2.15, we can derive different conditions for the oscillation of equation (1.1) with different choices of ρ, H and m.

For an application of our results, we give the following example.

Example 2.17. Consider the odd-order delay differential equation
$$[x(t) + p_0 x\big(t/\tau\big)]^{(n)} + \frac{q_0}{t^n} x\big(t/\sigma\big) = 0, \quad t \ge 1, \tag{2.37}$$
where $p_0 \in [0,\infty)$, $q_0 \in (0,\infty)$ and $\sigma > \tau \ge 1$.

Let $q(t) = q_0/t^n$ and $v(t) = 0$. Then $Q(t) = q_0/t^n$. Moreover, we have
$$\int_{t_0}^{\infty} s^{n-1}Q(s)\mathrm{d}s = q_0 \int_{1}^{\infty} \frac{1}{s}\mathrm{d}s = \infty.$$

Hence by Corollary 2.10, equation (2.37) is almost oscillatory if

$$q_0 > \frac{(n-1)!(1+\tau p_0)\sigma^{n-1}}{\mathrm{e}\ln(\sigma/\tau)}.$$

If $p_0 \in [0,1)$, then by [13, Example 1], equation (2.37) is almost oscillatory provided that

$$q_0 > \frac{(n-1)!\sigma^{n-1}}{\mathrm{e}(1-p_0)\ln\sigma}.$$

We find that our results improve that of in [13] for some cases. For example, we let $\sigma = \mathrm{e}^2$ and $\tau = \mathrm{e}$. If we set $p_0 = 7/8$ or $15/16$, we see that

$$\frac{1}{2(1-p_0)} > 1 + \mathrm{e}p_0.$$

Further our results hold for $p_0 \geq 1$.

One can construct examples easily to illustrate other results, and the details are left to the reader.

Summary. We have established criteria for the oscillation of solutions to (1.1). Our technique permits us to relax restrictions usually imposed on the coefficients of equation (1.1). So our results are of high generality, and are easily applicable as illustrated with a suitable example.

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