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CONTINUOUS SPECTRUM OF A FOURTH ORDER NONHOMOGENEOUS ELLIPTIC EQUATION WITH VARIABLE EXPONENT

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Abstract. In this article, we consider the nonlinear eigenvalue problem

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $p, q : \overline{\Omega} \to (1, +\infty)$ are continuous functions. Considering different situations concerning the growth rates involved in the above quoted problem, we prove the existence of a continuous family of eigenvalues. The proofs of the main results are based on the mountain pass lemma and Ekelands variational principle.

1. Introduction

Nonlinear eigenvalue problems associated with differential operators with variable exponents have received a lot of attention in recent years; see e.g. [1, 4, 5, 8, 9]. The reason of such interest starts from the study of the role played by their applications in mathematical modelling of non-Newtonian fluids, in particular, the electrorheological fluids, see [10], and of other phenomena related to image processing, elasticity and the flow in porous media.

The aim of this article is to analyze the existence of solutions of the nonhomogeneous eigenvalue problem

$$\begin{gathered} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, λ is a positive number, and p, q are continuous functions on $\overline{\Omega}$.

In [1], authors have considered the case $p(x) = q(x)$. Using the Ljusternik-Schnirelmann critical point theory, they established the existence of a sequence of eigenvalues. Denoting by Λ the set of all nonnegative eigenvalues, they showed that $\sup \Lambda = +\infty$ and they pointed out that only under additional assumptions we have $\inf \Lambda = 0$. We remark that for the p-biharmonic operator (corresponding to $p(x) = p$) we always have $\inf \Lambda > 0$.

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As far as we are aware, nonlinear eigenvalue problems like (1.1) involving the iterated $p(x)$-Laplacian operator have not yet been studied. That is why, at our best knowledge, the present paper is a first contribution in this direction.

Here, problem (1.1) is stated in the framework of the generalized Sobolev space $X := W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ for which some elementary properties are stated below.

By a weak solution for (1.1) we understand a function $u \in X$ such that

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v\,dx - \lambda\int_\Omega |u|^{q(x)-2}uv\,dx = 0, \quad \forall v \in X.$$

We point out that in the case when u is nontrivial, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of (1.1) and u is called an associated eigenfunction.

Inspired by the works of Mihăilescu and Rădulescu [8, 9], we study (1.1) in three distinct situations.

This article consists of three sections. Section 2 contains some preliminary properties concerning the generalized Lebesgue-Sobolev spaces and an embedding result. The main results and their proofs are given in Section 3.

2. Preliminaries

To guarantee completeness of this paper, we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For details, see [2, 3]. Set

$$C_+(\Omega) = \{h; h \in C(\bar\Omega) \text{ and } h(x) > 1 \text{ for all } x \in \bar\Omega\}.$$

For any $h \in C(\bar\Omega)$, we denote

$$h^+ = \max_{\bar\Omega} h(x), \quad h^- = \min_{\bar\Omega} h(x).$$

For $p \in C_+(\Omega)$, define the space

$$L^{p(x)}(\Omega) = \{u;\ \text{measurable real-valued function and } \int |u(x)|^{p(x)}dx < \infty\}.$$

Equipped with the so-called Luxemburg norm

$$|u|_{p(x)} := \inf\{\mu > 0 : \quad \int |\frac{u(x)}{\mu}|^{p(x)}dx \le 1\},$$

$L^{p(x)}(\Omega)$ becomes a separable, reflexive and Banach space. An important role in manipulating the generalized Lebesgue spaces is played by the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$, called the *modular* of the $L^{p(x)}(\Omega)$ space, defined by

$$\rho(u) = \int_\Omega |u|^{p(x)}dx.$$

We recall the following

Proposition 2.1 ([2])**.** *. For all $u_n, u \in L^{p(x)}(\Omega)$, we have*

1. $|u|_{p(x)} = a \Leftrightarrow \rho(\frac{u}{a}) = 1$, *for $u \neq 0$ and $a > 0$.*
2. $|u|_{p(x)} > 1$ *($= 1; < 1$)* $\Leftrightarrow \rho(u) > 1$ *($= 1; < 1$).*
3. $|u|_{p(x)} \to 0$ *(resp. $\to +\infty$)* $\Leftrightarrow \rho(u) \to 0$ *(resp. $\to +\infty$).*
4. *The following statements are equivalent:*
   (i) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$,
   (ii) $\lim_{n\to\infty} \rho(u_n - u) = 0$,
   (iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

As in the constant exponent case, for any positive integer k, set

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \le k\},.$$

We define a norm on $W^{k,p(x)}(\Omega)$ by

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)},$$

then $W^{k,p(x)}(\Omega)$ also becomes a separable, reflexive and Banach space. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Definition 2.2. Assume that spaces E, F are Banach spaces, we define the norm on the space $E \cap F$ as $\|u\| = \|u\|_E + \|u\|_F$.

From the above definition, we can know that for any $u \in X$, $\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$, thus $\|u\|_X = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}$.

In Zanga and Fu [11], the equivalence of the norms was proved, and it was even proved that the norm $|\Delta u|_{p(x)}$ is equivalent to the norm $\|u\|_X$ (see [11, Theorem 4.4]).

Let us choose on X the norm defined by

$$\|u\| = |\Delta u|_{p(x)}.$$

Note that, $(X, \|.\|)$ is also a separable and reflexive Banach space. Similar to Proposition 2.1, we have the following.

Proposition 2.3. *For all $u \in X$, denote $I(u) = \int |\Delta u(x)|^{p(x)} dx$ then,*

1. *For $u \in X$ and $\|u\| = a$, we have*
   (i) *$a < 1$ $(= 1, > 1)$ $\Leftrightarrow I(u) < 1$ $(= 1 > 1)$;*
   (ii) *$a \ge 1 \Rightarrow a^{p^-} \le I(u) \le a^{p^+}$;*
   (iii) *$a \le 1 \Rightarrow a^{p^+} \le I(u) \le a^{p^-}$.*
2. *If $u, u_n \in X, n = 1, 2, \dots$, then the following statements are equivalent:*
   (i) *$\lim_{n\to\infty} \|u_n - u\| = 0$;*
   (ii) *$\lim_{n\to\infty} I(u_n - u) = 0$;*
   (iii) *$u_n \to u$ in measure in Ω and $\lim_{n\to\infty} I(u_n) = I(u)$.*

For $x \in \Omega$, let us define

$$p_2^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < N/2, \\ +\infty & \text{if } p(x) \ge N/2 \end{cases}$$

The following result [1, Theorem 3.2], which will be used later, is an embedding result between the spaces X and $L^{q(x)}(\Omega)$.

Theorem 2.4. *Let $p, q \in C_+(\Omega)$. Assume that $p(x) < \frac{N}{2}$ and $q(x) < p_2^*(x)$. Then there is a continuous and compact embedding X into $L^{q(x)}(\Omega)$.*

The Euler-Lagrange functional associated with (1.1) is defined as $\Phi_\lambda : X \to \mathbb{R}$,

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx.$$

Standard arguments imply that $\Phi_\lambda \in C^1(X, \mathbb{R})$ and

$$\langle \Phi_\lambda'(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\, dx - \lambda \int_\Omega |u|^{q(x)-2} uv dx,$$

for all $u, v \in X$. Thus the weak solutions of (1.1) coincide with the critical points of Φ_λ. If such a weak solution exists and is nontrivial, then the corresponding λ is an eigenvalue of problem (1.1).

Next, we write Φ'_λ as

$$\Phi'_\lambda = A - \lambda B,$$

where $A, B : X \to X'$ are defined by

$$\langle A(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v \, dx,$$
$$\langle B(u), v\rangle = \int_\Omega |u|^{q(x)-2} uv \, dx.$$

Proposition 2.5. (i) *B is completely continuous, namely, $u_n \rightharpoonup u$ in X implies $B'(u_n) \to B'(u)$ in X'.*

(ii) *A satisfies condition (S^+), namely, $u_n \rightharpoonup u$, in X and $\limsup\langle A(u_n), u_n - u\rangle \le 0$, imply $u_n \to u$ in X.*

Proof. First, recall the following elementary inequalities

$$(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta) \ge \frac{1}{2^p}|\xi - \zeta|^p \quad \text{if } p \ge 2, \tag{2.1}$$
$$(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p} \ge (p-1)|\xi - \zeta|^2 \quad \text{if } 1 < p < 2, \tag{2.2}$$

for any $\xi, \eta \in \mathbb{R}^N$.

(i) Let $u_n \rightharpoonup u$ in X. For any $v \in X$, by Hölder's inequality in X and continuous embedding of X into $L^{q(x)}(\Omega)$, it follows that

$$\begin{aligned}
|\langle B(u_n) - B(u), v\rangle| &= \Big| \int_\Omega (|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u) v \, dx\Big| \\
&\le d_3 \||u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\|_{r(x)} \|v\|_{q(x)}, \quad d_3 > 0, \\
&\le d_4 \||u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\|_{r(x)} \|v\|, \quad d_4 > 0,
\end{aligned}$$

where $r(x) = \frac{q(x)}{q(x)-1}$.

On the other hand, using the compact embedding of X into $L^{q(x)}(\Omega)$, we have $u_n \to u$ in $L^{q(x)}(\Omega)$. Thus,

$$|u_n|^{q(x)-2}u_n \to |u|^{q(x)-2}u \quad \text{in } L^{q(x)}(\Omega).$$

Therefore, from the above inequality, the first assertion is proved.

(ii) Let (u_n) be a sequence of X such that $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0.$$

Using again (2.1) and (2.2), we deduce

$$\langle A(u_n) - A(u), u_n - u\rangle \ge 0.$$

Since $u_n \rightharpoonup u$ in X, we have

$$\limsup_{n\to+\infty}\langle A(u_n) - A(u), u_n - u\rangle = 0. \tag{2.3}$$

Put

$$\mathcal{U}_p = \{x \in \Omega : p(x) \ge 2\}, \quad \mathcal{V}_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

Thus, from (2.1) and (2.2), we have

$$\int_{\mathcal{U}_p} |\Delta u_n - \Delta u|^{p(x)} dx \le c_1 \int_\Omega D(u_n, u) dx, \tag{2.4}$$

$$\int_{\mathcal{V}_p} |\Delta u_n - \Delta u|^{p(x)} dx \le c_2 \int_\Omega (D(u_n, u))^{p(x)/2} (C(u_n, u))^{(2-p(x))\frac{p(x)}{2}} dx, \tag{2.5}$$

where

$$\begin{gathered} D(u_n, u) = (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u)(\Delta u_n - \Delta u), \\ C(u_n, u) = (|\Delta u_n| + |\Delta u_n|)^{2-p(x)}, \quad c_i > 0, i = 1, 2. \end{gathered}$$

On the other hand, by (2.3) and since $\int_\Omega D(u_n, u)dx = \langle A(u_n) - A(u), u_n - u\rangle$, we can consider

$$0 \le \int_\Omega D(u_n, u) dx < 1.$$

If $\int_\Omega D(u_n, u)dx = 0$, then since $D(u_n, u) \ge 0$ in Ω, $D(u_n, u) = 0$.

If $0 < \int_\Omega D(u_n, u)dx < 1$, then thanks to Young's inequality, we have

$$\begin{aligned} &\int_{\mathcal{V}_p} (D(u_n, u))^{p(x)/2} \Big(\int_{\mathcal{V}_p} D(u_n, u)dx\Big)^{-p(x)/2} C(u_n, u))^{(2-p(x))\frac{p(x)}{2}} dx \\ &\le \int_{\mathcal{V}_p} (D(u_n, u)(\int_{\mathcal{V}_p} D(u_n, u)dx)^{-p(x)/2} + (C(u_n, u))^{p(x)}) dx \\ &\le 1 + \int_\Omega (C(u_n, u))^{p(x)} dx. \end{aligned}$$

Hence,

$$\int_{\mathcal{V}_p} |\Delta u_n - \Delta u|^{p(x)} dx \le \Big(\int_{\mathcal{V}_p} D(u_n, u)dx\Big)^{1/2} (1 + \int_\Omega (C(u_n, u))^{p(x)} dx).$$

The proof of the second assertion is complete. □

Remark 2.6. Noting that Φ'_λ is still of type (S^+). Hence, any bounded (PS) sequence of Φ_λ in the reflexive Banach space X has a convergent subsequence,

3. Main results and proofs

In what follows, we assume that the functions $p, q \in C_+(\overline{\Omega})$.

Theorem 3.1. *If*

$$q^+ < p^-, \tag{3.1}$$

then any $\lambda > 0$ is an eigenvalue for problem (1.1). Moreover, for any $\lambda > 0$ there exists a sequence (u_n) of nontrivial weak solutions for problem (1.1) such that $u_n \to 0$ in X.

We want to apply the symmetric mountain pass lemma in [6].

Theorem 3.2. *(Symmetric mountain pass lemma) Let E be an infinite dimensional Banach space and $I \in C^1(E, R)$ satisfy the following two assumptions:*

- (A1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS), namely, any sequence u_n in E such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ in E as $n \to \infty$ has a convergent subsequence.*
- (A2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

Then, $I(u)$ admits a sequence of critical points u_k such that

$$I(u_k) < 0, u_k \neq 0 \text{ and } \lim_k u_k = 0,$$

where Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$ with $\gamma(A)$ is the genus of A, i.e.,

$$\gamma(K) = \inf\{k \in \mathbb{N} : \exists h : K \to \mathbb{R}^k \backslash \{0\} \text{ such that } h \text{ is continuous and odd } \}.$$

We start with two auxiliary results.

Lemma 3.3. *The functional Φ_λ is even, bounded from below and satisfies the (PS) condition; $\Phi_\lambda(0) = 0$.*

Proof. It is clear that Φ_λ is even and $\Phi_\lambda(0) = 0$. Since $q^+ < p^-$ and X is continuously embedded both in $L^{q^\pm}(\Omega)$, there exist two positive constants $d_1, d_2 > 0$ such that

$$\int_\Omega |u|^{q^+} dx \leq d_1 \|u\|^{q^+}, \quad \int_\Omega |u|^{q^-} dx \leq d_2 \|u\|^{q^-}, \quad \forall u \in X.$$

According to the fact that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-}, \quad \forall x \in \overline{\Omega}, \tag{3.2}$$

for all $u \in X$, we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega |\Delta u|^{p(x)} - \frac{\lambda d_1}{q^-} \|u\|^{q^+} - \frac{\lambda d_2}{q^-} \|u\|^{q^-} \\ &\geq \frac{1}{p^+} \alpha(\|u\|) - \frac{\lambda d_1}{q^-} \|u\|^{q^+} - \frac{\lambda d_2}{q^-} \|u\|^{q^-}, \end{aligned}$$

where $\alpha : [0, +\infty[\to \mathbb{R}$ is defined by

$$\alpha(t) = \begin{cases} t^{p^+}, & \text{if } t \leq 1, \\ t^{p^-}, & \text{if } t > 1. \end{cases} \tag{3.3}$$

As $q^+ < p^-$, Φ_λ is bounded from below and coercive because, that is, $\Phi_\lambda(u) \to \infty$ as $\|u\| \to \infty$.

It remains to show that the functional $\Phi_{\lambda,k}$ satisfies the (PS) condition to complete the proof. Let $(u_n) \subset X$ be a (PS) sequence of Φ_λ in X; that is,

$$\Phi_\lambda(u_n) \text{ is bounded and } \Phi'_\lambda(u_n) \to 0 \text{ in } X'. \tag{3.4}$$

Then, by the coercivity of Φ_λ, the sequence (u_n) is bounded in X. By the reflexivity of X, for a subsequence still denoted (u_n), we have

$$u_n \rightharpoonup u \quad \text{in } X.$$

Since $q^+ < p^-$, it follows from theorem 3.2 that $u_n \rightharpoonup u$ in $L^{q(x)}(\Omega)$. Using the properties of Nemytskii operator $N_q(x)$ defined by

$$N_{q(x)}(v)(x) = \begin{cases} |v(x)|^{q(x)-2} v(x) & \text{if } v(x) \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

we deduce that

$$\langle B(u_n), u_n - u \rangle = \int_\Omega |u_n(x)|^{q(x)-2} u_n(x)(u_n(x) - u)\, dx \to 0. \tag{3.5}$$

In view of (3.4) and (3.5), we obtain

$$\Phi'_\lambda(u_n) + \lambda \langle B(u_n), u_n - u \rangle = \langle A(u_n), u_n - u \rangle \to 0 \quad \text{as } n \to \infty.$$

According to the fact that A satisfies condition (S^+), we have $u_n \to u$ in X. The proof is complete. $\square$

Lemma 3.4. *For each $n \in \mathbb{N}^*$, there exists an $H_n \in \Gamma_n$ such that*

$$\sup_{u \in H_n} \Phi_\lambda(u) < 0.$$

Proof. Let $v_1, v_2, \dots, v_n \in C_0^\infty(\Omega)$ such that $\operatorname{supp}(v_i) \cap \operatorname{supp}(v_j) = \emptyset$ if $i \neq j$ and $\operatorname{meas}(\operatorname{supp}(v_j)) > 0$ for $i, j \in \{1, 2, \dots, n\}$. Take $F_n = \operatorname{span}\{v_1, v_2, \dots, v_n\}$, it is clear that $\dim F_n = n$ and

$$\int_\Omega |v(x)|^{q(x)} dx > 0 \quad \text{for all } v \in F_n \setminus \{0\}.$$

Denote $S = \{v \in X : \|v\| = 1\}$ and $H_n(t) = t(S \cap F_n)$ for $0 < t \leq 1$. Obviously, $\gamma(H_n(t)) = n$, for all $t \in]0, 1]$.

Now, we show that, for any $n \in \mathbb{N}^*$, there exist $t_n \in]0, 1]$ such that

$$\sup_{u \in H_n(t_n)} \Phi_\lambda(u) < 0.$$

Indeed, for $0 < t \leq 1$, we have

$$\begin{aligned}
\sup_{u \in H_n(t)} \Phi_\lambda(u) &\leq \sup_{v \in S \cap F_n} \Phi_\lambda(tv) \\
&= \sup_{v \in S \cap F_n} \Big\{ \int_\Omega \frac{t^{p(x)}}{p(x)} |\Delta v(x)|^{p(x)} dx - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)} |v(x)|^{q(x)}\, dx \Big\} \\
&\leq \sup_{v \in S \cap F_n} \Big\{ \frac{t^{p^-}}{p^-} \int_\Omega \big|\Delta v(x)|^{p(x)} dx - \frac{\lambda t^{q^+}}{q^+} \int_\Omega |v(x)|^{q(x)}\, dx \Big\} \\
&= \sup_{v \in S \cap F_n} \Big\{ t^{p^-} \Big(\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t^{p^- - q^+}} \int_\Omega |v(x)|^{q(x)}\, dx \Big) \Big\}.
\end{aligned}$$

Since $m := \min_{v \in S \cap F_n} \int_\Omega |v(x)|^{q(x)}\, dx > 0$, we may choose $t_n \in]0, 1]$ which is small enough such that

$$\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t_n^{p^- - q^+}} m < 0.$$

This completes the proof. $\square$

Proof of Theorem 3.1. By lemmas 3.3, 3.4 and theorem 3.2, Φ_λ admits a sequence of nontrivial weak solutions $(u_n)_n$ such that for any n, we have

$$u_n \neq 0, \quad \Phi'_\lambda(u_n) = 0, \quad \Phi_\lambda(u_n) \leq 0, \quad \lim_n u_n = 0. \tag{3.6}$$

$\square$

Theorem 3.5. *If*

$$q^- < p^- \quad \text{and} \quad q^+ < p_2^*(x) \quad \text{for all } x \in \overline{\Omega}, \tag{3.7}$$

then there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem (1.1).

For applying Ekeland's variational principle. We start with two auxiliary results.

Lemma 3.6. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho, a > 0$ such that $\Phi_\lambda(u) \geq a > 0$ for any $u \in X$ with $\|u\| = \rho$.*

Proof. Since $q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, it follows that X is continuously embedded in $L^{q(x)}(\Omega)$. So, there exists a positive constant c_1 such that

$$|u|_{q(x)} \leq c_1 \|u\|, \quad \text{for all } u \in X. \tag{3.8}$$

Let us fix $\rho \in]0,1[$ such that $\rho < \frac{1}{c_1}$. Then relation (3.8) implies $|u|_{q(x)} < 1$, for all $u \in X$ with $\|u\| = \rho$. Thus,

$$\int_\Omega |u|^{q(x)} dx \leq |u|_{q(x)}^{q^-}, \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\int_\Omega |u|^{q(x)} dx \leq c_1^{q^-} \|u\|^{q^-}, \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{3.10}$$

Hence, from (3.10) we deduce that for any $u \in X$ with $\|u\|_k = \rho$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega |\Delta u|^{p(x)} dx - \frac{\lambda}{q^-} \int_\Omega |u|^{q(x)} dx \\
&\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} c_1^{q^-} \|u\|^{q^-} \\
&= \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{q^-} c_1^{q^-} \rho^{q^-} \\
&= \rho^{q^-} (\frac{1}{p^+} \rho^{p^+ - q^-} - \frac{\lambda}{q^-} c_1^{q^-}).
\end{aligned}$$

Putting

$$\lambda_* = \frac{\rho^{p^+ - q^-}}{2p^+} \frac{q^-}{c_1^{q^-}}, \tag{3.11}$$

for any $u \in X$ with $\|u\| = \rho$, there exist $a = \rho^{p^+}/(2p^+)$ such that

$$\Phi_\lambda(u) \geq a > 0.$$

This completes the proof. $\square$

Lemma 3.7. *There exists $\psi \in X$ such that $\psi \geq 0$, $\psi \neq 0$ and $\Phi_\lambda(t\psi) < 0$, for $t > 0$ small enough.*

Proof. Since $q^- < p^-$, there exist $\varepsilon_0 > 0$ such that

$$q^- + \varepsilon_0 < p^-.$$

Since $q \in C(\overline{\Omega})$, there exist an open set $\Omega_0 \subset \Omega$ such that

$$|q(x) - q^-| < \varepsilon_0, \quad \text{for all } \ x \in \Omega_0.$$

Thus, we deduce

$$q(x) \leq q^- + \varepsilon_0 < p^-, \quad \text{for all } x \in \Omega_0. \tag{3.12}$$

Take $\psi \in C_0^\infty(\Omega)$ such that $\overline{\Omega}_0 \subset \operatorname{supp} \psi$, $\psi(x) = 1$ for $x \in \overline{\Omega}_0$ and $0 \leq \psi \leq 1$ in Ω. Without loss of generality, we may assume $\|\psi\| = 1$, that is

$$\int_\Omega |\Delta \psi|^{p(x)} \, dx = 1. \tag{3.13}$$

By using (3.12), (3.13) and the fact

$$\int_{\Omega_0} |\psi|^{q(x)} dx = \operatorname{meas}(\Omega_0)$$

for all $t \in]0,1[$, we obtain

$$\begin{aligned}
\Phi_\lambda(t\psi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\Delta\psi|^{p(x)}\,dx - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\psi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-}\int_\Omega |\Delta\psi|^{p(x)}\,dx - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\psi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+}\int_{\Omega_0} t^{q(x)}|\psi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^-+\varepsilon_0}}{q^+}\operatorname{meas}(\Omega_0).
\end{aligned}$$

Then, for any $t < \delta^{\frac{1}{p^- - q^- - \varepsilon_0}}$, with $0 < \delta < \min\{1, \lambda p^- \operatorname{meas}(\Omega_0)/q^+\}$, we conclude that

$$\Phi_\lambda(t\psi) < 0.$$

The proof is complete. □

Proof of theorem 3.5. By lemma 3.6, we have

$$\inf_{\partial B_\rho(0)} \Phi_\lambda > 0. \tag{3.14}$$

On the other hand, from lemma 3.7, there exist $\psi \in X$ such that $\Phi_\lambda(t\psi) < 0$ for $t > 0$ small enough. Using (3.10), it follows that

$$\Phi_\lambda(u) \ge \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}c_1^{q^-}\|u\|^{q^-} \quad \text{for } u \in B_\rho(0).$$

Thus,

$$-\infty < \underline{c}_\lambda := \inf_{\overline{B_\rho(o)}} \Phi_\lambda < 0,$$

Let

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda - \inf_{\overline{B_\rho(0)}} \Phi_\lambda.$$

Then, by applying Ekelands variational principle to the functional

$$\Phi_\lambda : \overline{B_\rho(0)} \to \mathbb{R},$$

there exist $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\Phi_\lambda(u_\varepsilon) \le \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon,$$

$$\Phi_\lambda(u_\varepsilon) < \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\| \ \text{ for } u \ne u_\varepsilon.$$

Since $\Phi_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda$, we deduce $u_\varepsilon \in B_\rho(0)$.

Now, define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|.$$

It is clear that u_ε is an minimum of I_λ. Therefore, for $t > 0$ and $v \in B_1(0)$, we have

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} \ge 0$$

for $t > 0$ small enough and $v \in B_1(0)$; that is,

$$\frac{\Phi_\lambda(u_\varepsilon + tv) - \Phi_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \ge 0$$

for t positive and small enough, and $v \in B_1(0)$. As $t \to 0$, we obtain

$$\langle \Phi'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| \geq 0 \quad \text{for all } v \in B_1(0).$$

Hence, $\|\Phi'_\lambda(u_\varepsilon)\|_{X'} \leq \varepsilon$. We deduce that there exists a sequence $(u_n)_n \subset B_\rho(0)$ such that

$$\Phi_\lambda(u_n) \to \underline{c}_\lambda \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0. \tag{3.15}$$

It is clear that (u_n) is bounded in X. By a standard arguments and the fact A is type of (S^+), for a subsequence we obtain $u_n \to u$ in X as $n \to +\infty$. Thus, by (3.15) we have

$$\Phi_\lambda(u) = \underline{c}_\lambda < 0 \quad \text{and} \quad \Phi'_\lambda(u) = 0 \quad \text{as } n \to \infty. \tag{3.16}$$

The proof is complete. □

Theorem 3.8. *If*

$$p^+ < q^- \leq q^+ < p_2^*(x) \quad \text{for all } x \in \overline{\Omega}, \tag{3.17}$$

then for any $\lambda > 0$, problem (1.1) possesses a nontrivial weak solution.

We want to construct a mountain geometry, and first need two lemmas.

Lemma 3.9. *There exist $\eta, b > 0$ such that $\Phi_\lambda(u) \geq b$, for $u \in X$ with $\|u\| = \eta$.*

Proof. Since $q^+ < p_2^*$, in view the Theorem 3.2, there exist $d_1, d_2 > 0$ such that

$$|u|_{q^+} \leq d_1\|u\| \quad \text{and} \quad |u|_{q^+} \leq d_2\|u\|.$$

Thus, from (3.2) we obtain

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |\Delta u(x)|^{p(x)}dx - \frac{\lambda}{q^-}\big[(d_1\|u\|)^{q^+} + (d_2\|u\|)^{q^-}\big] \\
&\geq \frac{1}{p^+}\alpha(\|u\|) - \frac{\lambda d_1^{q^+}}{q^-}\|u\|^{q^+} - \frac{\lambda d_2^{q^-}}{q^-}\|u\|^{q^-} \\
&= \begin{cases} (\frac{1}{p^+} - \frac{d_1^{q^+}}{q^-}\|u\|^{q^+-p^+} - \frac{\lambda d_2^{q^-}}{q^-}\|u\|^{q^--p^+})\|u\|^{p^+} & \text{if } \|u\| \leq 1, \\ (\frac{1}{p^+} - \frac{d_1^{q^+}}{q^-}\|u\|^{q^+-p^-} - \frac{\lambda d_2^{q^-}}{q^-}\|u\|^{q^--p^-})\|u\|^{p^-} & \text{if } \|u\| > 1. \end{cases}
\end{aligned}$$

Since $p^+ < q^- \leq q^+$, the functional $g : [0, 1] \to \mathbb{R}$ defined by

$$g(s) = \frac{1}{p^+} - \frac{d_1^{q^+}}{q^-}s^{q^+-p^+} - \frac{\lambda d_2^{q^-}}{q^-}s^{q^--p^+}$$

is positive on neighborhood of the origin. So, the result of lemma 3.9 follows. □

Lemma 3.10. *There exists $e \in X$ with $\|e\| \geq \eta$ such that $\Phi_\lambda(e) < 0$, where η is given in lemma 3.9.*

Proof. Choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \neq 0$. For $t > 1$, we have

$$\Phi_\lambda(t\varphi) \leq \frac{t^{p^+}}{p^-}\int_\Omega \big|\Delta\varphi(x)|^{p(x)}dx - \frac{\lambda t^{q^-}}{q^+}\int_\Omega |\varphi(x)|^{q(x)}dx.$$

Then, since $p^+ < q^-$, we deduce that

$$\lim_{t\to\infty} \Phi_\lambda(t\varphi) = -\infty.$$

Therefore, for $t > 1$ large enough, there is $e = t\varphi$ such that $\|e\| \geq \eta$ and $\Phi_\lambda(e) < 0$. This completes the proof. □

Lemma 3.11. *The functional Φ_λ satisfies the condition (PS).*

Proof. Let $(u_n) \subset X$ be a sequence such that $d := \sup_n \Phi_\lambda(u_n) < \infty$ and $\Phi'_\lambda(u_n) \to 0$ in X'. By contradiction suppose that

$$\|u_n\| \to +\infty \text{ as } n \to \infty \quad \text{and} \quad \|u_n\| > 1 \quad \text{for any } n.$$

Thus,

$$\begin{aligned}
& d + 1 + \|u_n\| \\
& \geq \Phi_\lambda(u_n) - \frac{1}{q^-}\langle \Phi'_\lambda(u_n), u_n\rangle \\
& = \int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx - \frac{\lambda}{q^-}\int_\Omega |\Delta u_n|^{p(x)}dx + \lambda\int_\Omega (\frac{1}{q^-} - \frac{1}{q(x)})|u_n|^{q(x)}\,dx \\
& \geq (\frac{1}{p^+} - \frac{1}{q^-})\int_\Omega |\Delta u_n|^{p(x)}dx \\
& \geq (\frac{1}{p^+} - \frac{1}{q^-})\|u_n\|^{p^-}.
\end{aligned}$$

This contradicts the fact that $p^- > 1$. So, the sequence (u_n) is bounded in X and similar arguments as those used in the proof of lemma 3.4 completes the proof. □

Proof of theorem 3.8. From Lemmas 3.9 and 3.10, we deduce

$$\max(\Phi_\lambda(0), \Phi_\lambda(e)) = \Phi_\lambda(0) < \inf_{\|u\|=\eta} \Phi_\lambda(u) =: \beta.$$

By lemma 3.11 and the mountain pass theorem, we deduce the existence of critical points u of Φ_λ associated of the critical value given by

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi_\lambda(\gamma(t)) \geq \beta, \tag{3.18}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. This completes the proof. □

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