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MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR MULTI-POINT BOUNDARY-VALUE PROBLEM WITH GENERAL GROWTH ON THE POSITIVE HALF LINE

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Abstract. This work is devoted to the existence of nontrivial positive solutions for a class of second-order nonlinear multi-point boundary-value problems on the positive half-line. The novelty of this work is that the nonlinearity may exhibit a singularity at the origin simultaneously with respect to the solution and its derivative; moreover it satisfies quite general growth conditions far from the origin, including polynomial growth. New existence results of single, twin and triple solutions are proved using the fixed point index theory on appropriate cones in weighted Banach spaces together with two-functional and three-functional fixed point theorems. The singularity is treated by means of approximation and compactness arguments. The proofs of the existence results rely heavily on several sharp estimates and useful properties of the corresponding Green's function.

1. Introduction

This article concerns the existence of positive solutions to the multi-point boundary value problem posed on the positive half-line:

$$\begin{gathered} -y'' + cy' + \lambda y = \Phi(t) f(t, y(t), e^{-ct} y'(t)), \quad t \in I \\ y(0) = \sum_{i=1}^{n} k_i y(\xi_i), \quad \lim_{t\to\infty} e^{-ct} y'(t) = 0, \end{gathered} \tag{1.1}$$

where, for $i \in \{1, \dots, n\}$, $k_i \geq 0$ and the multi-points $0 < \xi_1 < \xi_2 < \dots < \xi_n < \infty$ satisfy

$$\sum_{i=1}^{n} k_i e^{r_2 \xi_i} < 1, \tag{1.2}$$

and where

$$r_2 = \frac{c - \sqrt{c^2 + 4\lambda}}{2} < 0 < r_1 = \frac{c + \sqrt{c^2 + 4\lambda}}{2}$$

are the roots of the algebraic equation $-r^2 + cr + \lambda = 0$. The parameters c and λ are real positive constants while the function $f = f(t, y, z) : I^2 \times \mathbb{R}^* \to \mathbb{R}^+$ is continuous and is allowed to have space singularities at $y = 0$ and/or $z = 0$,

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and $\Phi : I \to I$ is a continuous function. Recall that f is said to be singular at $y = 0$ if $\lim_{y\to 0} f(t, y, z) = +\infty$ uniformly in $(t, z) \in I \times \mathbb{R}^*$. Here and hereafter $I := (0, +\infty)$ denotes the set of positive real numbers, $\mathbb{R}^+ : = [0, +\infty)$, and $\mathbb{R}^* : = \mathbb{R} \setminus \{0\}$.

Throughout this paper, by positive solution we mean $y \in C^1([0, \infty))$ such that y'' exists and y satisfies (1.1) with $y(t) \geq 0$ on $(0, \infty)$.

Many problems in physics, chemistry and biology are governed by boundary value problems on the half-line, e.g., the flow of a premixed mixture inducing the propagation of a nonadiabatic flame in a long tube. For instance, the equation

$$-y''(t) + cy'(t) + \lambda y(t) = f(t, y(t))$$

subject to the boundary conditions

$$y(0) = y(+\infty) = 0$$

extends the classical Fisher-Kolmogorov model equation (see [20]) with no heat exchange, i.e. $\lambda = 0$. The positive nonlinear term is governed by classical physical laws. In combustion theory, the source term in the energy equation obeys Arrhenius' Law where $f = f(y)$ behaves as $y^n e^{-y}$ near positive infinity (see e.g., [3, 5, 7]). This motivates the general growth of the nonlinearity considered in this work, extending polynomials. In epidemiology, the propagation of epidemics through given populations is governed by the generalized Fisher autonomous equation $-y''+ cy' + \lambda y = yh(y)$ (see [11, 14] for a mathematical investigation). Here the positive constant c is the velocity of the travelling wave and the real parameter λ is a removal rate [30]. The function y represents a density of infectives. Thus, only positive solutions corresponding to a density, a temperature,... are useful from a physical point of view.

Moreover, various physiological processes in non-Newtonian fluid theory, boundary layer theory and nonlinear phenomena (see e.g., [31]) are modelled by singular equations such that the Emden-Fowler equation $y'' = -\varphi(t)y^{-\gamma}$ ($\gamma > 0$). Also, the boundary value problem for the electrical potential in an isolated neutral atom was derived in 1927 independently by Thomas [34] and Fermi [19]; it can be written as

$$\begin{gathered} y'' = \sqrt{y^3/t} \\ y(0) = 1, \quad y(+\infty) = 0. \end{gathered}$$

Another example is provided by the boundary layer equation for steady flow over a semi-infinite plate (see [8]):

$$\begin{gathered} y'' = -\frac{t}{2y^2} \\ y(0) = y(+\infty) = 0. \end{gathered}$$

These behaviors of the nonlinearities have motivated our investigation of problem (1.1) with a nonlinearity allowed to have a singularity not only in y but also in y'.

There have been recently so much work devoted to the investigation of existence of positive solutions for boundary value problems on infinite intervals of the real line and where the nonlinearity satisfies either superlinear or sublinear growth assumptions (see [11, 12, 13, 21, 35, 36] and the references therein). A few methods have been employed to deal with such problems which lack compactness; we cite upper and lower solution techniques [32], fixed point theorems in special Banach spaces and index fixed point theory on cones of special Banach spaces [4, 26, 35] as well as diagonalization processes. Existence of single or multiple solutions have been proved for two-point boundary value problems, three-point and even multi-point BVPs in [26, 29, 35, 36]. We point out that several existence results for general problems posed on unbounded intervals may be found in the book by Agarwal and O'Regan [2].

In [28], the authors have recently considered the generalized Fisher equation $-y'' + py' + qy = h(t)f(t,y)$ with h singular in time while the nonlinearity f may change sign. When f further depends on the first derivative, existence of multiple solutions is given in [15] and the nonlinearity includes sublinear and superlinear growth conditions; fixed point theory in cones of special Banach spaces is employed. In [17, 18], the authors combine the fixed point index theory with the upper and lower solution method to prove existence of solutions when the nonlinearity satisfies various growth assumptions.

The second-order differential equation $(p(t)y'(t))' + \lambda\phi(t)f(t,y(t)) = 0$ with $\lim_{t\to+\infty} p(t)y'(t) = 0$ as a boundary condition is studied in [26, 39] while the same equation where f also depends on y' is considered in [36] with Dirichlet condition at positive infinity; fixed point theorems in cones are used to prove existence of positive solutions; the condition $\int_0^{+\infty} \frac{dt}{p(t)} < \infty$ is assumed. A discussion along with the smallness of the parameter λ is also given in [37] for a nonlinearity of the form $\lambda(f(t,y) - k^2 y)$.

A three-point boundary value problem associated with the Sturm-Liouville differential equation

$$\big(\frac{1}{p(t)}(p(t)y'(t)\big)' + q(t)f(t,y(t),p(t)y'(t)) = 0$$

is discussed in [23] and [33] with $\lim_{t\to+\infty} p(t)y'(t) = b \geq 0$; the technique of upper and lower solutions and the theory of fixed point theory are employed to get existence of multiple solutions. The same technique is employed in [27] when f does not depend on the first derivative. Notice that this equation is also investigated in [38] and existence of multiple solutions is proved when f may be singular at $y = 0$ and $py' = 0$. We point out that in all of these works, the conditions $\int_0^{+\infty} \frac{dt}{p(t)} < \infty$ is assumed which is not the case in the present work since $p(t) = e^{-ct}$.

Our aim in this work is further to extend some of these works to the case in which a positive nonlinearity does also depend on the first derivative and is allowed to be singular at the origin in both its second and third arguments; in addition it satisfies general growth far from the singular origin, extending the classical polynomial growth. We prove existence and multiplicity of nontrivial positive solutions in a weighted Banach space. The singularity of the nonlinearity is treated by approximating a fixed point operator with the help of some compactness arguments.

The proofs of our existence theorems rely on recent fixed point theorems of two or three functionals [4, 25] together with the fixed point index theory in cones of Banach spaces [22]. Some preliminaries needed to transform problem (1.1) into a fixed point theorem are presented in Section 2 together with appropriate compactness criteria. In particular, essential properties of the Green's function are given and the main assumptions are enunciated. Then, we construct a special cone in a weighted Banach space. The properties of a fixed point operator denoted T are studies in detail in Section 3. Section 4 is devoted to proving three existence results successively of a single, twin and triple solutions. The existence theorems obtained in this paper extend similar results available in the literature in case the nonlinearity f is either nonsingular or does not depend on the first derivative (see e.g., [11, 12, 13, 16, 26, 29, 35, 36]). We end the paper with an example of application in Section 5 and some concluding remarks in Section 6.

2. Functional framework

In this section, we present some definitions and lemmas which will be needed in the proofs of the main results. Let

$$C_l([0,\infty),\mathbb{R}) = \{y \in C([0,\infty),\mathbb{R}) : \lim_{t\to\infty} y(t) \text{ exists}\}.$$

It is easy to see that C_l is a Banach space with the norm $\|y\|_l = \sup_{t\in[0,\infty)} |y(t)|$. For a real parameter $\theta > r_1$, consider the Banach space of Bielecki type [6] defined by

$$X = C^1_\infty([0,\infty),\mathbb{R}) = \big\{y \in C^1([0,\infty),\mathbb{R}) : \lim_{t\to+\infty} \frac{y(t)}{e^{\theta t}} \text{ and } \lim_{t\to+\infty} \frac{y'(t)}{e^{\theta t}} \text{ exist}\big\}$$

with norm

$$\|y\|_\theta = \max\{\|y\|_1, \|y\|_2\},$$

where

$$\|y\|_1 = \sup_{t\in[0,\infty)} \frac{|y(t)|}{e^{\theta t}}, \quad \|y\|_2 = \sup_{t\in[0,\infty)} \frac{|y'(t)|}{e^{\theta t}}.$$

Lemma 2.1 ([15, Lemma 2.1])**.** *$X = C^1_\infty$ is a Banach space.*

For some $0 < \gamma < \delta$, let

$$0 < \Lambda_0 := \min\{e^{r_2\delta}, e^{r_1\gamma} - e^{r_2\gamma}\}, \quad \Lambda = \Lambda_0 \max_{t\in[\gamma,\delta]} \sigma(t). \tag{2.1}$$

Since $r_2 < 0$, we have $0 < \Lambda_0 < 1$. Here

$$\sigma(t) = \begin{cases} \min\Big(\frac{1-e^{(r_2-r_1)t}}{2r_1(1-\sum_{i=1}^n k_i e^{r_2\xi_i})e^{(r_1-\theta)t}}, \frac{1}{|r_2|}\Big), & t < \xi_1; \\ \min\Big(\frac{1-\sum_{i=1}^j k_i e^{r_2\xi_i} - e^{(r_2-r_1)t}(1-\sum_{i=1}^j k_i e^{r_1\xi_i})}{2r_1(1-\sum_{i=1}^n k_i e^{r_2\xi_i})e^{(r_1-\theta)t}}, \frac{1}{|r_2|}\Big), \\ \quad 0 < \xi_j \le t \le \xi_{j+1},\ j = 1,2,\dots,n-1; \\ \min\Big(\frac{1-\sum_{i=1}^j k_i e^{r_2\xi_i} - e^{(r_2-r_1)t}(1-\sum_{i=1}^n k_i e^{r_1\xi_i})}{2r_1(1-\sum_{i=1}^n k_i e^{r_2\xi_i})e^{(r_1-\theta)t}}, \frac{1}{|r_2|}\Big), & t \ge \xi_n. \end{cases}$$

Then define the positive cone

$$\mathcal{P} = \big\{y \in X : y(t) \ge 0 \text{ on } \mathbb{R}^+,\ y(t) \ge \Lambda\|y\|_2,\ \forall\, t \in [\gamma,\delta] \text{ and } y(0) = \sum_{i=1}^n k_i y(\xi_i)\big\}. \tag{2.2}$$

Lemma 2.2. *Let $\rho = \frac{1}{\theta(1-\sum_{i=1}^n k_i e^{\theta\xi_i})}$. Then $\|y\|_1 \le \rho\|y\|_2$ for all $y \in \mathcal{P}$.*

Proof. Since $y(0) = \sum_{i=1}^n k_i y(\xi_i)$, then for every $t \in \mathbb{R}^+$, we have

$$\begin{aligned} \frac{y(t)}{e^{\theta t}} &= e^{-\theta t}\Big\{\int_0^t y'(s)ds + y(0)\Big\} \\ &= e^{-\theta t}\Big\{\int_0^t y'(s)ds + \sum_{i=1}^n k_i y(\xi_i)\Big\} \end{aligned}$$

$$
\begin{aligned}
&= e^{-\theta t}\Big\{\int_0^t e^{\theta s}\frac{y'(s)}{e^{\theta s}}ds + \sum_{i=1}^n k_i e^{\theta\xi_i}\frac{y(\xi_i)}{e^{\theta\xi_i}}\Big\}\\
&\le e^{-\theta t}\Big\{\frac{1}{\theta}(e^{\theta t}-1)\|y\|_2 + \sum_{i=1}^n k_i e^{\theta\xi_i}\|y\|_1\Big\}\\
&\le \frac{1}{\theta}(1-e^{-\theta t})\|y\|_2 + \sum_{i=1}^n k_i e^{\theta\xi_i}\|y\|_1.
\end{aligned}
$$

Passing to the supremum over $t \ge 0$, we complete the proof. □

Arguing as in [15, Lemma 2.2], we deduce the following result.

Lemma 2.3. *Let $y \in \mathcal{P}$. Then, for any $t \in [\gamma, \delta]$, we have $y(t) \ge \Gamma\|y\|_\theta$, where $\Gamma = \Lambda/\max(1, \rho)$.*

2.1. **Construction of the Green's function.** In the following lemma which generalizes [15, Lemma 2.4], we determine the Green's function for problem (1.1).

Lemma 2.4. *Let v be a continuous function such that $\int_0^\infty e^{-r_1 s}v(s)ds < \infty$ and $\lim_{s\to+\infty} e^{-cs}v(s) = 0$. Then $y \in C^1(I)$ is a solution of*

$$
\begin{gathered}
-y'' + cy' + \lambda y = v(t), \quad t \in I\\
y(0) = \sum_{i=1}^n k_i y(\xi_i), \quad \lim_{t\to\infty}\frac{y'(t)}{e^{ct}} = 0,
\end{gathered} \tag{2.3}
$$

if and only if it may be expressed in the form

$$
y(t) = \int_0^\infty G(t,s)v(s)ds, \quad t \in I. \tag{2.4}
$$

Hereafter the positive Green's function G is defined on $I \times I$ by $G(t,s) = \frac{1}{\Delta}G^1(t,s)$ with $\Delta = (r_1 - r_2)(1 - \sum_{i=1}^n k_i e^{r_2\xi_i})$ and

$$
G^1(t,s) = \begin{cases}
e^{r_2 t}(e^{-r_2 s} - e^{-r_1 s}), \quad \text{if } 0 < s \le \min(t,\xi_1) < \infty;\\
e^{r_1(t-s)}(1-\sum_{i=1}^n k_i e^{r_2\xi_i}) - e^{r_2 t}\Big(e^{-r_1 s} - \sum_{i=1}^n k_i e^{r_2(\xi_i - s)}\Big),\\
\quad \text{if } 0 < t \le s \le \xi_1 < \infty;\\
e^{r_2 t}\Big(e^{-r_2 s}(1-\sum_{i=1}^j k_i e^{r_2\xi_i}) - e^{-r_1 s}(1-\sum_{i=1}^j k_i e^{r_1\xi_i})\Big),\\
\quad \text{if } 0 < \xi_j \le s \le \xi_{j+1},\ s \le t,\ j = 1,2,\dots,n-1;\\
e^{-r_1 s}\Big(e^{r_1 t}(1-\sum_{i=1}^n k_i e^{r_2\xi_i}) - e^{r_2 t}(1-\sum_{i=1}^j k_i e^{r_1\xi_i}\\
-\sum_{i=j+1}^n k_i e^{r_2(\xi_i - s)+r_1 s})\Big),\\
\quad \text{if } 0 < \xi_j \le s \le \xi_{j+1},\ t \le s,\ j = 1,2,\dots,n-1;\\
e^{r_2 t}\Big(e^{-r_2 s}(1-\sum_{i=1}^n k_i e^{r_2\xi_i}) - e^{-r_1 s}(1-\sum_{i=1}^n k_i e^{r_1\xi_i})\Big),\\
\quad \text{if } 0 < \xi_n \le s \le t < \infty;\\
e^{-r_1 s}\Big(e^{r_1 t}(1-\sum_{i=1}^n k_i e^{r_2\xi_i})\\
-e^{r_2 t}(1-\sum_{i=1}^n k_i e^{r_1\xi_i})\Big),\\
\quad \text{if } 0 < \max(\xi_n, t) \le s < \infty.
\end{cases}
$$

Proof. (a) It is easy to show that the general solution of the equation in the boundary value problem (2.3) reads

$$y(t)=\frac{1}{r_1-r_2}\Big(Ae^{r_1t}+Be^{r_2t}+\int_0^t\Big(e^{r_2(t-s)}-e^{r_1(t-s)}\Big)v(s)ds\Big) \tag{2.5}$$

where $A=y'(0)-r_2y(0)$ and $B=r_1y(0)-y'(0)$. Differentiating (2.5) yields

$$y'(t)=\frac{1}{r_1-r_2}\Big(Ar_1e^{r_1t}+Br_2e^{r_2t}+\int_0^t(r_2e^{r_2(t-s)}-r_1e^{r_1(t-s)})v(s)ds\Big). \tag{2.6}$$

From (2.3) and (2.5), we obtain

$$\begin{aligned}
0&=y(0)-\sum_{i=1}^n k_iy(\xi_i)\\
&=\frac{1}{r_1-r_2}\Big(A+B-\sum_{i=1}^n k_i(Ae^{r_1\xi_i}+Be^{r_2\xi_i})\\
&\quad+\int_0^{\xi_i}\Big(e^{r_2(\xi_i-s)}-e^{r_1(\xi_i-s)}\Big)v(s)ds\Big);
\end{aligned}$$

that is,

$$(1-\sum_{i=1}^n k_ie^{r_1\xi_i})A+(1-\sum_{i=1}^n k_ie^{r_2\xi_i})B=\int_0^{\xi_i}\Big(e^{r_2(\xi_i-s)}-e^{r_1(\xi_i-s)}\Big)v(s)ds. \tag{2.7}$$

Moreover, (2.6) yields

$$\frac{y'(t)}{e^{ct}}=\frac{\Sigma(t)}{r_1-r_2}$$

where

$$\begin{aligned}
\Sigma(t)&=Ar_1e^{(r_1-c)t}+Br_2e^{(r_2-c)t}\\
&\quad+r_2e^{(r_2-c)t}\int_0^t e^{-r_2s}v(s)ds-r_1e^{(r_1-c)t}\int_0^t e^{-r_1s}v(s)ds.
\end{aligned}$$

We claim that

$$\lim_{t\to\infty}e^{(r_2-c)t}\int_0^t e^{-r_2s}v(s)ds=0. \tag{2.8}$$

Indeed, if $\int_0^\infty e^{-r_2s}v(s)ds<\infty$, then (2.8) holds. Now assume $\int_0^\infty e^{-r_2s}v(s)ds=\infty$. Since $\lim_{s\to\infty}e^{-cs}v(s)=0$, L'Hospital's rule yields

$$\begin{aligned}
\lim_{t\to\infty}e^{(r_2-c)t}\int_0^t e^{-r_2s}v(s)ds&=\lim_{t\to\infty}\frac{\int_0^t e^{-r_2s}v(s)ds}{e^{(c-r_2)t}}\\
&=\lim_{t\to\infty}\frac{e^{-r_2t}v(t)}{(c-r_2)e^{(c-r_2)t}}\\
&=\lim_{t\to\infty}\frac{e^{-ct}v(t)}{c-r_2}=0.
\end{aligned}$$

From (2.7), (2.8) and the boundary conditions, we find the values

$$A=\int_0^\infty e^{-r_1s}v(s)ds,$$

$$
\begin{aligned}
B = (1 - \sum_{i=1}^{n} k_i e^{r_1 \xi_2})^{-1} \Big\{ & \sum_{i=1}^{n} \int_0^{\xi_i} (e^{r_2(\xi_i - s)} - e^{r_1(\xi_i - s)}) v(s) ds \\
& - (1 - \sum_{i=1}^{n} k_i e^{r_1 \xi_i}) \int_0^{\infty} e^{-r_1 s} v(s) ds \Big\}.
\end{aligned}
$$

By substitution in (2.5), we obtain

$$
\begin{aligned}
y(t) = \frac{1}{r_1 - r_2} \Big(& \int_0^{\infty} e^{r_1(t-s)} v(s) ds \\
& + \big(1 - \sum_{i=1}^{n} k_i e^{r_2 \xi_i}\big)^{-1} \sum_{i=1}^{n} \int_0^{\xi_i} (e^{r_2(t+\xi_i - s)} - e^{r_1(\xi_i - s) + r_2 t}) v(s) ds \Big) \\
& - \Big(\big(1 - \sum_{i=1}^{n} k_i e^{r_2 \xi_i}\big)^{-1} \big(1 - \sum_{i=1}^{n} k_i e^{r_1 \xi_i}\big) \int_0^{\infty} e^{r_2 t - r_1 s} v(s) ds \\
& - \int_0^{t} (e^{r_2(t-s)} - e^{r_1(t-s)}) v(s) ds \Big) := \frac{1}{\Delta} y_1(t),
\end{aligned}
$$

with

$$
y_1(t) = \begin{cases}
\int_0^t e^{r_2 t}(e^{-r_2 s} - e^{-r_1 s}) v(s) ds + \int_t^{\xi_1} \Big((1 - \sum_{i=1}^n k_i e^{r_2 \xi_i}) e^{r_1(t-s)} \\
- e^{r_2 t}(e^{-r_1 s} - \sum_{i=1}^n k_i e^{r_2(\xi_i - s)}) \Big) v(s) ds \\
+ \sum_{i=1}^n k_i \int_{\xi_1}^{\xi_i} (e^{r_2(\xi_i - s)} - e^{r_1(\xi_i - s)}) v(s) ds \\
+ \int_{\xi_1}^{+\infty} e^{-r_1 s} \Big((1 - \sum_{i=1}^n k_i e^{r_2 \xi_i}) e^{r_1 t} - (1 - \sum_{i=1}^n k_i e^{r_1 \xi_i}) e^{r_2 t} \Big) v(s) ds, \\
\qquad \text{if } t \le \xi_1; \\
(1 - \sum_{i=1}^n k_i e^{r_2 \xi_i}) \int_{\xi_j}^{\xi_{j+1}} e^{r_1(t-s)} v(s) \\
-(1 - \sum_{i=1}^n k_i e^{r_1 \xi_i}) \int_{\xi_j}^{\xi_{j+1}} e^{-r_1 s} v(s) ds \\
+ e^{r_2 t} \Big(\sum_{i=j+1}^n k_i \int_0^{\xi_i} (e^{r_2(\xi_i - s)} - e^{r_1(\xi_i - s)}) v(s) \Big), \\
\qquad \text{if } \xi_j \le t \le \xi_{j+1},\ j = 1, 2, \dots, n-1; \\
\int_0^{\xi_n} e^{r_2 t}(e^{-r_2 s} - e^{-r_1 s}) v(s) ds \\
+ \int_{\xi_n}^t e^{r_2 t} \Big((1 - \sum_{i=1}^n k_i e^{r_2 \xi_i}) e^{-r_2 s} - (1 - \sum_{i=1}^n k_i e^{r_1 \xi_i}) e^{-r_1 s} \Big) v(s) ds \\
+ \int_t^{+\infty} e^{-r_1 s} \Big((1 - \sum_{i=1}^n k_i e^{r_2 \xi_i}) e^{r_1 t} - (1 - \sum_{i=1}^n k_i e^{r_1 \xi_i}) e^{r_2 t} \Big) v(s) ds, \\
\qquad \text{if } t \ge \xi_n
\end{cases}
$$

whence the form of the Green's function G.

(b) Conversely, let $y \in C^1(I)$ be as defined by (2.4). A direct differentiation of (2.4) yields

$$
y'(t) = \int_0^{\infty} G_t(t, s) v(s) ds, \quad t \in I, \tag{2.9}
$$

where $G_t(t,s) = \frac{1}{\Delta} G_t^1(t,s)$ is the partial derivative of $G(t,s)$ with respect to t and

$$G_t^1(t,s) = \begin{cases} r_2 e^{r_2 t}(e^{-r_2 s} - e^{-r_1 s}), \quad \text{if } 0 < s \le \min(t,\xi_1) < \infty; \\ r_1 e^{r_1(t-s)}(1 - \sum_{i=1}^n k_i e^{r_2\xi_i}) - r_2 e^{r_2 t}\Big(e^{-r_1 s} - \sum_{i=1}^n k_i e^{r_2(\xi_i - s)}\Big), \\ \qquad \text{if } 0 < t \le s \le \xi_1 < \infty; \\ r_2 e^{r_2 t}\Big(e^{-r_2 s}(1 - \sum_{i=1}^j k_i e^{r_2\xi_i}) - e^{-r_1 s}(1 - \sum_{i=1}^j k_i e^{r_1\xi_i})\Big), \\ \qquad \text{if } 0 < \xi_j \le s \le \xi_{j+1},\ s \le t,\ j = 1,2,\dots,n-1; \\ e^{-r_1 s}\Big(r_1 e^{r_1 t}(1 - \sum_{i=1}^n k_i e^{r_2\xi_i}) - r_2 e^{r_2 t}(1 - \sum_{i=1}^j k_i e^{r_1\xi_i} \\ - \sum_{i=j+1}^n k_i e^{r_2(\xi_i - s) + r_1 s})\Big), \\ \qquad \text{if } 0 < \xi_j \le s \le \xi_{j+1},\ t \le s,\ j = 1,2,\dots,n-1; \\ r_2 e^{r_2 t}\Big(e^{-r_2 s}(1 - \sum_{i=1}^n k_i e^{r_2\xi_i}) - e^{-r_1 s}(1 - \sum_{i=1}^n k_i e^{r_1\xi_i})\Big), \\ \qquad \text{if } 0 < \xi_n \le s \le t < \infty; \\ e^{-r_1 s}\Big(r_1 e^{r_1 t}(1 - \sum_{i=1}^n k_i e^{r_2\xi_i}) - r_2 e^{r_2 t}(1 - \sum_{i=1}^n k_i e^{r_1\xi_i})\Big), \\ \qquad \text{if } 0 < \max(\xi_n, t) \le s < \infty. \end{cases}$$

Differentiating again (2.9) yields

$$\begin{aligned} y''(t) &= -v(t) + c\int_0^\infty G_t(t,s)v(s)ds + \lambda\int_0^\infty G(t,s)v(s)ds \\ &= -v(t) + cy'(t) + \lambda y(t), \quad t \in I. \end{aligned}$$

Hence $y \in C^1(I)$ and y satisfies (2.3). □

The following two lemmas are crucial; the proofs are lengthy; so we only prove the second one.

Lemma 2.5. *The function $G(t,s)$ given by Lemma 2.4 satisfies*

(a) *$G(t,s) \ge 0$ for all $t,s \in I$*
(b) *$e^{-\mu t}G(t,s) \le e^{-r_1 s}G(s,s)$, for all $t,s \in I$ and all $\mu \ge r_1$.*
(c) *$G(t,s) \ge \Lambda_0 G(s,s)e^{-r_1 s}$ for all $t \in [\gamma,\delta]$ and all $s \in I$,*

where Λ_0 is as defined by (2.1).

Lemma 2.6. *Assume that*

$$\begin{gathered} 1 - \sum_{i=1}^n k_i e^{r_1\xi_i} > 0, \\ 1 - \sum_{i=1}^n k_i e^{r_2(\xi_i - s) + r_1 s} > 0, \quad 0 < s \le \xi_1 \\ 1 - \sum_{i=1}^j k_i e^{r_1\xi_i} - \sum_{i=j+1}^n k_i e^{r_2(\xi_i - s) + r_1 s} > 0, \quad 0 < \xi_j \le s \le \xi_{j+1},\ 1 \le j \le n-1. \end{gathered} \tag{2.10}$$

Then, we have the estimates

$$e^{-\mu t}|G_t(t,s)| \le e^{-r_1 s}\overline{G}(s), \quad \forall\, t,s \in I,\ \mu \ge r_1, \tag{2.11}$$

where

$$\overline{G}(s)=\begin{cases}\max\Big(|r_2|G(s,s),\frac{r_1}{\Delta}\Big(2-\sum_{i=1}^n k_ie^{r_2\xi_i}-\sum_{i=1}^n k_ie^{r_2(\xi_i-s)+r_1s}\Big)\Big),\\ \qquad \text{if } s\le\xi_1;\\ \max\Big(|r_2|G(s,s),\frac{r_1}{\Delta}\Big(2-\sum_{i=1}^j k_ie^{r_1\xi_i}-\sum_{i=j+1}^n k_ie^{r_2(\xi_i-s)+r_1s}\\ -\sum_{i=1}^n k_ie^{r_2\xi_i}\Big)\Big),\\ \qquad \text{if } \xi_j\le s\le\xi_{j+1},\ 1\le j\le n-1;\\ \max\Big(|r_2|G(s,s),\frac{r_1}{\Delta}\Big(2-\sum_{i=1}^n k_ie^{r_1\xi_i}-\sum_{i=1}^n k_ie^{r_2\xi_i}\Big)\Big), \quad \text{if } s\ge\xi_n.\end{cases}$$

and

$$(b)\quad e^{-\theta t}\sigma(t)|G_t(t,s)|\le e^{-r_1s}G(s,s),\quad \forall\, t,s\in I. \tag{2.12}$$

Proof. (a) For any $s\in I$, we have

$$G^1(s,s)=\begin{cases}1-e^{(r_2-r_1)s}, & 0\le s\le\xi_1\\ 1-\sum_{i=1}^j k_ie^{r_2\xi_i}-e^{(r_2-r_1)s}(1-\sum_{i=1}^j k_ie^{r_1\xi_i}),\\ \qquad \text{if } \xi_j\le s\le\xi_{j+1}, j=1,2,\dots,n-1\\ 1-\sum_{i=1}^n k_ie^{r_2\xi_i}-e^{(r_2-r_1)s}(1-\sum_{i=1}^n k_ie^{r_1\xi_i}), & s\ge\xi_n.\end{cases}$$

We distinguish four cases.

(1) If either $0<s\le\min(t,\xi_1)<\infty$ or $0<\xi_j\le s\le\xi_{j+1}$, $s\le t$, $j=1,2,\dots,n-1$ or $\xi_n\le s\le t$, then for any $\mu\ge r_1$,

$$e^{-\mu t}|G_t(t,s)|=e^{-\mu t}|r_2G(t,s)|\le|r_2|e^{-r_1s}G(s,s),\quad \forall\mu\ge r_1.$$

(2) If $0<t<s\le\xi_1<\infty$, then for any $\mu\ge r_1$,

$$\begin{aligned}&e^{-\mu t}|G_t^1(t,s)|\\&=\Big|r_1e^{-\mu t}e^{r_1(t-s)}\Big(1-\sum_{i=1}^n k_ie^{r_2\xi_i}\Big)-r_2e^{(r_2-\mu)t}\Big(e^{-r_1s}-\sum_{i=1}^n k_ie^{r_2(\xi_i-s)}\Big)\Big|\\&\le r_1e^{-r_1s}\Big(e^{(r_1-\mu)t}\Big(1-\sum_{i=1}^n k_ie^{r_2\xi_i}\Big)+\frac{|r_2|}{r_1}e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^n k_ie^{r_2(\xi_i-s)+r_1s}\Big)\Big)\\&\le r_1e^{-r_1s}\Big(\Big(1-\sum_{i=1}^n k_ie^{r_2\xi_i}\Big)+\Big(1-\sum_{i=1}^n k_ie^{r_2(\xi_i-s)+r_1s}\Big)\Big)\\&=r_1e^{-r_1s}\Big(2-\sum_{i=1}^n k_ie^{r_2\xi_i}-\sum_{i=1}^n k_ie^{r_2(\xi_i-s)+r_1s}\Big).\end{aligned}$$

(3) If $0<\xi_j\le s\le\xi_{j+1}$, $t\le s$, $j=1,2,\dots,n-1$, then for any $\mu\ge r_1$,

$$\begin{aligned}&e^{-\mu t}|G_t^1(t,s)|\\&=e^{-r_1s}\Big|r_1e^{(r_1-\mu)t}\Big(1-\sum_{i=1}^n k_ie^{r_2\xi_i}\Big)\\&\quad -r_2e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^j k_ie^{r_1\xi_i}-\sum_{i=j+1}^n k_ie^{r_2(\xi_i-s)+r_1s}\Big)\Big|\end{aligned}$$

$$
\begin{aligned}
&\le r_1 e^{-r_1 s}\Big(e^{(r_1-\mu)t}\Big(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\Big)+\frac{|r_2|}{r_1}e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^{j}k_i e^{r_1\xi_i}\\
&\quad -\sum_{i=j+1}^{n}k_i e^{r_2(\xi_i-s)+r_1 s}\Big)\Big)\\
&\le r_1 e^{-r_1 s}\Big(\Big(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\Big)+\Big(1-\sum_{i=1}^{j}k_i e^{r_1\xi_i}-\sum_{i=j+1}^{n}k_i e^{r_2(\xi_i-s)+r_1 s}\Big)\Big)\\
&= r_1 e^{-r_1 s}\Big(2-\sum_{i=1}^{j}k_i e^{r_1\xi_i}-\sum_{i=j+1}^{n}k_i e^{r_2(\xi_i-s)+r_1 s}-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\Big).
\end{aligned}
$$

(4) If $0<\max(\xi_n,t)\le s<\infty$, then for any $\mu\ge r_1$,

$$
\begin{aligned}
&e^{-\mu t}|G_t^1(t,s)|\\
&= e^{-r_1 s}\Big|r_1 e^{(r_1-\mu)t}\Big(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\Big)-r_2 e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^{n}k_i e^{r_1\xi_i}\Big)\Big|\\
&\le r_1 e^{-r_1 s}\Big(e^{(r_1-\mu)t}\Big(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\Big)+\frac{|r_2|}{r_1}e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^{n}k_i e^{r_1\xi_i}\Big)\Big)\\
&\le r_1 e^{-r_1 s}\Big(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i}+(1-\sum_{i=1}^{n}k_i e^{r_1\xi_i})\Big)\\
&= r_1 e^{-r_1 s}\Big(2-\sum_{i=1}^{n}k_i e^{r_1\xi_i}-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\Big).
\end{aligned}
$$

Hence

$$
e^{-\mu t}|G_t(t,s)|\le e^{-r_1 s}\overline{G}(s),\quad \forall t,s\in I;\ \forall\,\mu\ge r_1.
$$

(b) For any $s\in I$, we have the discussion

(1) If either $0<s\le\min(t,\xi_1)<\infty$ or $0<\xi_j\le s\le\xi_{j+1}$, $s\le t$, $j=1,2,\dots,n-1$ or $\xi_n\le s\le t$, for any $\mu\ge r_1$, then we have

$$
e^{-\mu t}|G_t(t,s)|=e^{-\mu t}|r_2 G(t,s)|\le |r_2|e^{-r_1 s}G(s,s).
$$

Hence

$$
\frac{e^{-r_1 s}G(s,s)}{e^{-\mu t}|r_2 G(t,s)|}\ge\frac{1}{|r_2|},\quad \forall\,\mu\ge r_1.
$$

(2) If $0<t<s\le\xi_1<\infty$, then for any $\mu\ge r_1$,

$$
\begin{aligned}
&\frac{e^{-r_1 s}G(s,s)}{e^{-\mu t}|G_t(t,s)|}\\
&=\frac{e^{-r_1 s}\big(1-e^{(r_2-r_1)s}\big)}{\big|r_1 e^{-\mu t}e^{r_1(t-s)}\big(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\big)-r_2 e^{(r_2-\mu)t}\big(e^{-r_1 s}-\sum_{i=1}^{n}k_i e^{r_2(\xi_i-s)}\big)\big|}\\
&\ge\frac{e^{-r_1 s}(1-e^{(r_2-r_1)s})}{r_1 e^{-r_1 s}\Big(e^{(r_1-\mu)t}\big(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i}\big)+\frac{|r_2|}{r_1}e^{(r_2-\mu)t}\big(1-\sum_{i=1}^{n}k_i e^{r_2(\xi_i-s)+r_1 s}\big)\Big)}\\
&\ge\frac{1-e^{(r_2-r_1)t}}{2r_1(1-\sum_{i=1}^{n}k_i e^{r_2\xi_i})e^{(r_1-\mu)t}}\,.
\end{aligned}
$$

(3) If $0<\xi_j\le s\le \xi_{j+1}$, $t\le s$, $j=1,2,\dots,n-1$, then for any $\mu\ge r_1$,

$$\begin{aligned}
\frac{e^{-r_1 s}G(s,s)}{e^{-\mu t}|G_t(t,s)|} &= \frac{e^{-r_1 s}\Big(1-\sum_{i=1}^{j}k_ie^{r_2\xi_i}-e^{(r_2-r_1)s}\big(1-\sum_{i=1}^{j}k_ie^{r_1\xi_i}\big)\Big)}{e^{-r_1 s}} \\
&\quad\times\Big|r_1e^{(r_1-\mu)t}\big(1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}\big)-r_2e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^{j}k_ie^{r_1\xi_i} \\
&\quad-\sum_{i=j+1}^{n}k_ie^{r_2(\xi_i-s)+r_1 s}\Big)\Big|^{-1} \\
&\ge \frac{\Big(1-\sum_{i=1}^{j}k_ie^{r_2\xi_i}-e^{(r_2-r_1)s}\big(1-\sum_{i=1}^{j}k_ie^{r_1\xi_i}\big)\Big)}{r_1} \\
&\quad\times\Big(e^{(r_1-\mu)t}\big(1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}\big)+\frac{|r_2|}{r_1}e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^{j}k_ie^{r_1\xi_i} \\
&\quad-\sum_{i=j+1}^{n}k_ie^{r_2(\xi_i-s)+r_1 s}\Big)\Big)^{-1} \\
&\ge \frac{1-\sum_{i=1}^{j}k_ie^{r_2\xi_i}-e^{(r_2-r_1)t}\big(1-\sum_{i=1}^{j}k_ie^{r_1\xi_i}\big)}{2r_1\big(1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}\big)e^{(r_1-\mu)t}}\,.
\end{aligned}$$

(4) If $0<\max(\xi_n,t)\le s<\infty$, then for any $\mu\ge r_1$,

$$\begin{aligned}
&\frac{e^{-r_1 s}G(s,s)}{e^{-\mu t}|G_t(t,s)|} \\
&= \frac{e^{-r_1 s}\Big(1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}-e^{(r_2-r_1)s}\Big(1-\sum_{i=1}^{n}k_ie^{r_1\xi_i}\Big)\Big)}{e^{-r_1 s}\Big|r_1e^{(r_1-\mu)t}\Big(1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}\Big)-r_2e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^{n}k_ie^{r_1\xi_i}\Big)\Big|} \\
&\ge \frac{1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}-e^{(r_2-r_1)s}\Big(1-\sum_{i=1}^{n}k_ie^{r_1\xi_i}\Big)}{r_1\Big(e^{(r_1-\mu)t}\Big(1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}\Big)+\frac{|r_2|}{r_1}e^{(r_2-\mu)t}\Big(1-\sum_{i=1}^{n}k_ie^{r_1\xi_i}\Big)\Big)} \\
&\ge \frac{1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}-e^{(r_2-r_1)t}\Big(1-\sum_{i=1}^{n}k_ie^{r_1\xi_i}\Big)}{2r_1\Big(1-\sum_{i=1}^{n}k_ie^{r_2\xi_i}\Big)e^{(r_1-\mu)t}}\,.
\end{aligned}$$

Hence

$$e^{-\mu t}\sigma(t)|G_t(t,s)|\le e^{-r_1 s}G(s,s),\ \forall\, t,s\in I,\ \forall\,\mu\ge r_1.$$

$\square$

2.2. **A compact fixed point operator.** On the space X, define the mapping T by

$$Ty(t)=\int_0^\infty G(t,s)\Phi(s)f(s,y(s),e^{-cs}y'(s))ds,\quad t\in I. \tag{2.13}$$

Remark 2.7. Let $y\in X$ be a fixed point of T in X. Then it is a solution of problem (1.1) provided the integral in (2.13) converges.

Recall that an operator is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Lemma 2.8 ([9, p. 62]). *Let $M \subseteq C_l(\mathbb{R}^+, \mathbb{R})$. Then M is relatively compact in $C_l(\mathbb{R}^+, \mathbb{R})$ if the following conditions hold:*

(a) *M is uniformly bounded in $C_l(\mathbb{R}^+, \mathbb{R})$;*
(b) *Functions belonging to M are almost equicontinuous on $\mathbb{R}^+$; i.e., equicontinuous on every compact interval of $\mathbb{R}^+$.*
(c) *The functions from M are equiconvergent; that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - l| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$,*

From the above lemma we easily deduce the following result (see e.g., [15]).

Lemma 2.9. *Let $M \subseteq C^1_\infty(\mathbb{R}^+, \mathbb{R})$. Then M is relatively compact in $C^1_\infty(\mathbb{R}^+, \mathbb{R})$ if the following conditions hold:*

(a) *M is uniformly bounded in $C^1_\infty(\mathbb{R}^+, \mathbb{R})$.*
(b) *The functions belonging to the sets $\{y : y(t) = x(t)/e^{\theta t}, x \in M\}$ and $\{z : z(t) = x'(t)/e^{\theta t}, x \in M\}$ are locally equicontinuous on $\mathbb{R}^+$.*
(c) *The functions from the sets $\{y : y(t) = x(t)/e^{\theta t}, x \in M\}$ and $\{z|z(t) = x'(t)/e^{\theta t}, x \in M\}$ are equiconvergent at $+\infty$.*

2.3. **General assumptions.** Regarding the growth of the function $F(t, u, v) = f(t, ue^{\theta t}, ve^{\theta t})$, we first enunciate the main assumptions to be considered in this paper:

(H1) $F : I^2 \times \mathbb{R}^* \to \mathbb{R}^+$ is a continuous function and there exist functions g, $w \in C(I, I)$ and h, $k \in C(\mathbb{R}^*, I)$ such that

$$0 \leq F(t, u, v) \leq (g(u) + w(u))(h(v) + k(v)), \quad \forall\, (t, u, v) \in I^2 \times \mathbb{R}^*$$

where g, h are non-increasing functions, w/g, k/h are nondecreasing functions and for all positive number R

$$\Pi(R) = \int_0^{+\infty} e^{-r_1 s} \max\{G(s, s), \overline{G}(s)\}\Phi(s) g(e^{-\theta s}\Gamma R) h(-e^{-cs} R) ds < \infty.$$

(H2) There exits $R_0 > 0$ such that

$$\Big(1 + \frac{w(R_0)}{g(R_0)}\Big)\Big(1 + \frac{k(R_0)}{h(R_0)}\Big)\Pi(R_0) < R_0. \tag{2.14}$$

3. Properties of the operator T

In the subsequent two lemmas, we study the properties of the operator T including its compactness when the nonlinearity f is assumed to have no singularities.

Lemma 3.1. *Under Assumptions* (H1), (H2)*, the operator T maps $\mathcal{P}$ into itself, where the cone $\mathcal{P}$ is as defined by* (2.2).

Proof. **Claim 1.** $T(\mathcal{P}) \subset X$. Indeed, from Assumptions (H1) and (H2) and with Lemma 2.5(a), (b) and Lemma 2.6(a) with $\mu = \theta$, we obtain, for any $y \in \mathcal{P}$, and $t \in \mathbb{R}^+$ the following estimates:

$$\begin{aligned}
|Ty(t)|e^{-\theta t} &= \int_0^{+\infty} e^{-\theta t} G(t, s)\Phi(s) f(s, y(s), e^{-cs} y'(s)) ds \\
&= \int_0^{+\infty} e^{-\theta t} G(t, s)\Phi(s) f\Big(s, \frac{e^{\theta s}}{e^{\theta s}}\, y(s), \frac{e^{\theta s}}{e^{\theta s}}\, e^{-cs} y'(s)\Big) ds \\
&\leq \int_0^{+\infty} e^{-r_1 s} G(s, s)\Phi(s) F(s, e^{-\theta s} y(s), e^{-(c+\theta)s} y'(s)) ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^{+\infty} e^{-r_1 s} G(s,s)\Phi(s)\left(g(e^{-\theta s}y(s)) + w(e^{-\theta s}y(s))\right) \\
&\quad \times \Big(h(e^{-(c+\theta)s}y'(s)) + k(e^{-(c+\theta)s}y'(s))\Big)ds \\
&= \int_0^{+\infty} e^{-r_1 s} G(s,s)\Phi(s)\Big(1 + \frac{w(e^{-\theta s}y(s))}{g(e^{-\theta s}y(s))}\Big) \\
&\quad \times \Big(1 + \frac{k(e^{-(c+\theta)s}y'(s))}{h(e^{-(c+\theta)s}y'(s))}\Big) g(e^{-\theta s}y(s)) h(e^{-(c+\theta)s}y'(s))ds \\
&\le \Big(1 + \frac{w(\|y\|_\theta)}{g(\|y\|_\theta)}\Big)\Big(1 + \frac{k(\|y\|_\theta)}{h(\|y\|_\theta)}\Big)\Pi(\|y\|_\theta) < \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
|(Ty)'(t)|e^{-\theta t} &= \int_0^{+\infty} e^{-\theta t} G_t(t,s)\Phi(s) f(s, y(s), e^{-cs}y'(s))ds \\
&\le \int_0^{+\infty} e^{-r_1 s}\overline{G}(s)\Phi(s)\left(g(e^{-\theta s}y(s)) + w(e^{-\theta s}y(s))\right) \\
&\quad \times \Big(h(e^{-(c+\theta)s}y'(s)) + k(e^{-(c+\theta)s}y'(s))\Big)ds \\
&\le \Big(1 + \frac{w(\|y\|_\theta)}{g(\|y\|_\theta)}\Big)\Big(1 + \frac{k(\|y\|_\theta)}{h(\|y\|_\theta)}\Big)\Pi(\|y\|_\theta) < \infty.
\end{aligned}
$$

Claim 2. $T(\mathcal{P}) \subset \mathcal{P}$. Let $y \in \mathcal{P}$. Clearly, $Ty(t) \ge 0$ for all $t \in I$. Moreover, by Lemma 2.5(c) and Lemma 2.6(b), for $t \in [\gamma, \delta]$, we have

$$
\begin{aligned}
Ty(t) &= \int_0^{+\infty} G(t,s)\Phi(s) f(s, y(s), e^{-cs}y'(s))ds, \\
&\ge \int_0^{\infty} \min_{t\in[\gamma,\delta]} G(t,s)\Phi(s) f(s, y(s), e^{-cs}y'(s))ds \\
&\ge \int_0^{\infty} e^{-r_1 s}\Lambda_0 G(s,s)\Phi(s) f(s, y(s), e^{-cs}y'(s))ds \\
&\ge \int_0^{\infty} \Lambda_0 \sigma(\tau) e^{-\theta\tau} G_t(\tau,s)\Phi(s) f(s, y(s), e^{-cs}y'(s))ds.
\end{aligned}
$$

Passing to the supremum over $\tau \in \mathbb{R}^+$, we obtain

$$
\begin{aligned}
Ty(t) &\ge \Lambda_0 \sup_{\tau\in\mathbb{R}^+}\Big(\sigma(\tau)\frac{(Ty)'(\tau)}{e^{\theta\tau}}\Big) \\
&= \Lambda_0 \sup_{\tau\in\mathbb{R}^+}\sigma(\tau) \sup_{\tau\in\mathbb{R}^+}\frac{(Ty)'(\tau)}{e^{\theta\tau}} \\
&\ge \Lambda_0 \sup_{\tau\in[\gamma,\delta]}\sigma(\tau) \sup_{\tau\in\mathbb{R}^+}\frac{(Ty)'(\tau)}{e^{\theta\tau}}.
\end{aligned}
$$

Hence

$$
Ty(t) \ge \Lambda\|Ty\|_2, \quad \forall\, t \in [\gamma, \delta].
$$

Finally, by the property of the Green's function

$$
Ty(0) = \sum_{i=1}^{n} k_i Ty(\xi_i).
$$

$\square$

Lemma 3.2. *Under Assumptions* (H1), (H2), *the mapping* $T:\mathcal{P}\to\mathcal{P}$ *is completely continuous.*

Proof. **Claim 1.** $T:\mathcal{P}\to\mathcal{P}$ is continuous. Let a sequence $\{y_n\}_{n\geq 1}\subseteq\mathcal{P}$ and $y_0\in\mathcal{P}$ with $\lim_{n\to+\infty}y_n\to y_0$ in $\mathcal{P}$. Then, there exists an $M>0$ such that $\max\{\|y_n\|_\theta,\|y_0\|_\theta\}\leq M$ for all $n\in\{1,2,\dots\}$. Thus, arguing as in Lemma 3.1, Claim 1 and using Assumptions (H1) and (H2), we arrive at the estimates

$$\begin{aligned}&\int_0^{+\infty}e^{-\theta t}G(t,s)\Phi(s)f(s,y_n(s),e^{-cs}y_n'(s))ds\\&\leq\Big(1+\frac{w(M)}{g(M)}\Big)\Big(1+\frac{k(M)}{h(M)}\Big)\Pi(\|y_n\|_\theta)<\infty\end{aligned}$$

and

$$\begin{aligned}&\int_0^{+\infty}e^{-\theta t}|G_t(t,s)\Phi(s)|f(s,y_n(s),e^{-cs}y_n'(s))ds\\&\leq\Big(1+\frac{w(M)}{g(M)}\Big)\Big(1+\frac{k(M)}{h(M)}\Big)\Pi(\|y_n\|_\theta)<\infty.\end{aligned}$$

By continuity of f, we obtain

$$\lim_{n\to+\infty}f(t,y_n(t),e^{-ct}y_n'(t))=f(t,y_0(t),e^{-ct}y_0'(t)),\quad t\in I.$$

Then the Lebesgue Dominated Convergence Theorem implies

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\{|Ty_n(t)-Ty_0(t)|e^{-\theta t}\}\\&=\sup_{t\in\mathbb{R}^+}\Big|\int_0^\infty e^{-\theta t}G(t,s)\Phi(s)\left(f(s,y_n(s),e^{-cs}y_n'(s))-f(s,y_0(s),e^{-cs}y_0'(s)\right)ds\Big|\\&\leq\sup_{t\in\mathbb{R}^+}\int_0^\infty G(s,s)\Phi(s)e^{-r_1 s}\left|f(s,y_n(s),e^{-cs}y_n'(s))-f(s,y_0(s),e^{-cs}y_0'(s))\right|ds\\&\to 0,\quad\text{as } n\to+\infty\end{aligned}$$

and

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\{|(Ty_n)'(t)-(Ty_0)'(t)|e^{-\theta t}\}\\&=\sup_{t\in\mathbb{R}^+}\Big|\int_0^\infty e^{-\theta t}G_t(t,s)\Phi(s)\left(f(s,y_n(s),e^{-cs}y_n'(s))-f(s,y_0(s),e^{-cs}y_0'(s)\right)ds\Big|\\&\leq\sup_{t\in\mathbb{R}^+}\int_0^\infty \overline{G}(s)\Phi(s)e^{-r_1 s}\left|f(s,y_n(s),e^{-cs}y_n'(s))-f(s,y_0(s),e^{-cs}y_0'(s))\right|ds\\&\to 0,\quad\text{as } n\to+\infty.\end{aligned}$$

As a result

$$\|Ty_n-Ty_0\|_\theta\to 0,\quad n\to+\infty.$$

Claim 2. Let $\Omega\subset X$ be a bounded subset, say $\Omega=\{y\in X:\|y\|_\theta\leq r\}$. We prove that $T(\Omega\cap\mathcal{P})$ is relatively compact.

(a) For some $y\in\Omega\cap\mathcal{P}$, we have

$$\|Ty\|_\theta\leq\Big(1+\frac{w(r)}{g(r)}\Big)\Big(1+\frac{k(r)}{h(r)}\Big)\Pi(\|y\|_\theta),$$

yielding that $T(\Omega\cap\mathcal{P})$ is uniformly bounded.

(b) $T(\Omega\cap\mathcal{P})$ is locally equicontinuous on I. The functions in $\{Ty(t)/e^{\theta t}, y\in\Omega\cap\mathcal{P}\}$ and the functions belonging to $\{(Ty)'(t)/e^{\theta t}, y\in\Omega\cap\mathcal{P}\}$ are locally equicontinuous on I. Indeed, $G(t,s)$ is continuously differentiable in t on $[0,\infty)$ except for $t=s$; so the Lebesgue dominated convergence theorem yields

$$\begin{aligned}|Ty(t_1)-Ty(t_2)|e^{-\theta t} &\le \int_0^\infty e^{-\theta t}|G(t_1,s)-G(t_2,s)|\Phi(s)f(s,y(s),e^{-cs}y'(s))ds\\ &\to 0,\quad \text{as } t_1\to t_2,\end{aligned}$$

as well as

$$\begin{aligned}&|(Ty)'(t_1)-(Ty)'(t_2)|e^{-\theta t}\\ &\le \int_0^\infty e^{-\theta t}|G_t(t_1,s)-G_t(t_2,s)|\Phi(s)f(s,y(s),e^{-cs}y'(s))ds\\ &\to 0,\quad \text{as } t_1\to t_2,\end{aligned}$$

(c) $T(\Omega\cap\mathcal{P})$ is locally equiconvergent at $+\infty$. Let $y\in\Omega\cap\mathcal{P}$. From the expression of the Green's function G in Lemmas 2.5, 2.6, we infer that

$$\lim_{t\to+\infty}\frac{G(t,s)}{e^{\theta t}}=0,\quad \lim_{t\to+\infty}\frac{G_t(t,s)}{e^{\theta t}}=0,\quad s\in[0,+\infty). \tag{3.1}$$

With the estimates in Lemma 3.1, Claim 1 and the Lebesgue dominated convergence theorem, we finally obtain

$$\begin{aligned}&\lim_{t\to+\infty}\big|e^{-\theta t}Ty(t)-\lim_{s\to+\infty}e^{-\theta s}Ty(s)\big|\\ &=\lim_{t\to+\infty}\Big|\int_0^\infty e^{-\theta t}G(t,s)\Phi(s)f(s,y(s),e^{-cs}y'(s))ds\Big|\\ &\le \int_0^\infty \lim_{t\to+\infty}|e^{-\theta t}G(t,s)\Phi(s)f(s,y(s),e^{-cs}y'(s))ds|=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{t\to+\infty}\big|e^{-\theta t}(Ty)'(t)-\lim_{t\to+\infty}e^{-\theta s}(Ty)'(s)\big|\\ &=\lim_{t\to+\infty}\Big|\int_0^\infty e^{-\theta t}G_t(t,s)\Phi(s)f(s,y(s),e^{-cs}y'(s))ds\Big|=0.\end{aligned}$$

By Lemma 2.9, $T(\Omega\cap P)$ is relatively compact. $\square$

4. Main existence results

4.1. **Single solution.** The following Lemmas are needed in this section. The proofs and more details on the index fixed point theory in cones can be found in [1, 10, 22, 24, 40].

Lemma 4.1. *Let Ω be a bounded open set in a real Banach space $E,\mathcal{P}$ be a cone of $E,\theta\in\Omega$ and $A:\overline{\Omega}\cap\mathcal{P}\to\mathcal{P}$ be a completely continuous operator. Assume that*

$$Ax\ne\lambda x,\quad \forall\, x\in\partial\Omega\cap\mathcal{P},\ \lambda\ge 1.$$

Then $i(A,\Omega\cap\mathcal{P},\mathcal{P})=1$.

Lemma 4.2. *Let Ω be a bounded open set in a real Banach space $E,\mathcal{P}$ be a cone of $E,\theta\in\Omega$ and $A:\overline{\Omega}\cap\mathcal{P}\to\mathcal{P}$ be a completely continuous operator. Assume that*

$$Ax\not\le x,\quad \forall\, x\in\partial\Omega\cap\mathcal{P}.$$

Then $i(A,\Omega\cap\mathcal{P},\mathcal{P})=0$.

We are now in position to prove our first existence result. Let

$$\ell := \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s)ds.$$

Theorem 4.3. *Assume* (H1), (H2) *hold together with*

(H3) $f(t,y,z) \geq \varphi(t,y)$ *for all* $t \in [\gamma,\delta]$ *and all* $(y,z) \in (0,+\infty) \times \mathbb{R}^*$, *where* $\varphi \in C([\gamma,\delta] \times (0,+\infty))$ *satisfies*

$$\liminf_{y\to 0} \min_{t\in[\gamma,\delta]} \frac{\varphi(t,y)}{y} > \frac{1}{\Lambda_0 \ell}\,.$$

Then problem (1.1) *has at least one positive solution* y *such that*

$$\|y\|_\theta \leq R_0, \quad y(t) \geq \Gamma\|y\|_\theta, \quad \forall\, t \in [\gamma,\delta].$$

Proof. For each $n \in \{1,2,\dots\}$, define a sequence of functions by

$$f_n(t,y,z) = f\left(t, \max\{e^{\theta t}/n, y(t)\}, \max\{e^{\theta t}/n, z(t)\}\right). \tag{4.1}$$

Then, for $y \in \mathcal{P}$, define a sequence of operators by

$$T_n y(t) = \int_0^{+\infty} G(t,s)\Phi(s) f_n(s,y(s),e^{-cs}y'(s))ds,\ t \in I. \tag{4.2}$$

Lemma 3.2 guarantees that $T_n : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator. By the inequality of (H3), there exist an $r > 0$ and $\varepsilon > 0$ such that

$$\varphi(t,y) \geq \Big(\frac{1}{\Lambda_0\ell} + \varepsilon\Big)y, \quad \text{for each } y \in [0,r] \text{ and } t \in [\gamma,\delta]. \tag{4.3}$$

Let R_0 be as defined by Assumption (H2) and $\widetilde{R} = \min(R_0/2, r/e^{\theta\delta})$ and consider the open sets

$$\Omega_1 := \{y \in X : \|y\|_\theta < R_0\}, \quad \Omega_2 := \{y \in X : \|y\|_\theta < \widetilde{R}\}.$$

Claim 1. $T_n y \neq \lambda y$ for any $y \in \partial\Omega_1 \cap P$, $\lambda \geq 1$ and $n \geq n_0 > \frac{1}{R_0}$. Let $y \in \partial\Omega_1 \cap \mathcal{P}$. By Assumptions (H1) and (H2), we obtain successively the following estimates

$$\begin{aligned}
&e^{-\theta t}|T_n y(t)| \\
&= \int_0^{+\infty} e^{-\theta t} G(t,s)\Phi(s) f_n(s,y(s),e^{-cs}y'(s))ds \\
&= \int_0^{+\infty} e^{-\theta t} G(t,s)\Phi(s) f\left(s, \max\{e^{\theta s}/n, y(s)\}, \max\{e^{\theta s}/n, e^{-cs}y'(s)\}\right) ds \\
&= \int_0^{+\infty} e^{-\theta t} G(t,s)\Phi(s) F(s, \max\{1/n, e^{-\theta s}y(s)\}, \max\{1/n, e^{-(c+\theta)s}y'(s)\})ds \\
&\leq \int_0^{+\infty} e^{-r_1 s} G(s,s)\Phi(s) \left(g(\max\{1/n, e^{-\theta s}y(s)\}) + w(\max\{1/n, e^{-\theta s}y(s)\})\right) \\
&\quad \times \Big(h(\max\{1/n, e^{-(c+\theta)s}y'(s)\}) + k(\max\{1/n, e^{-(c+\theta)s}y'(s)\})\Big)\, ds \\
&= \int_0^{+\infty} \Big(1 + \frac{w(\max\{1/n, e^{-\theta s}y(s)\})}{g(\max\{1/n, e^{-\theta s}y(s)\})}\Big) \Big(1 + \frac{k(\max\{1/n, e^{-(c+\theta)s}y'(s)\})}{h(\max\{1/n, e^{-(c+\theta)s}y'(s)\})}\Big) \\
&\quad \times e^{-r_1 s} \max\{G(s,s), \overline{G}(s)\}\Phi(s) g(e^{-\theta s}y(s)) h(e^{-(c+\theta)s}y'(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\le \Big(1+\frac{w(\max\{1/n,\|y\|_\theta\})}{g(\max\{1/n,\|y\|_\theta\})}\Big)\Big(1+\frac{k(\max\{1/n,\|y\|_\theta\})}{h(\max\{1/n,\|y\|_\theta\})}\Big)\\
&\quad\times\int_0^{+\infty} e^{-r_1 s}\max\{G(s,s),\overline{G}(s)\}\Phi(s)g(e^{-\theta s}\Gamma\|y\|_\theta)h(-e^{-cs}\|y\|_\theta)ds\\
&\le \Big(1+\frac{w(R_0)}{g(R_0)}\Big)\Big(1+\frac{k(R_0)}{h(R_0)}\Big)\Pi(R_0)<R_0
\end{aligned}
$$

and

$$
\begin{aligned}
&e^{-\theta t}|(T_n y)'(t)|\\
&=\int_0^{+\infty} e^{-\theta t}G_t(t,s)\Phi(s)F(s,\max\{1/n,e^{-\theta s}y(s)\},\max\{1/n,e^{-(c+\theta)s}y'(s)\})ds\\
&\le \int_0^{+\infty} e^{-r_1 s}\overline{G}(s)\Phi(s)\left(g(\max\{1/n,e^{-\theta s}y(s)\})+w(\max\{1/n,e^{-\theta s}y(s)\})\right)\\
&\quad\times\Big(h(\max\{1/n,e^{-(c+\theta)s}y'(s)\})+k(\max\{1/n,e^{-(c+\theta)s}y'(s)\})\Big)\,ds\\
&=\int_0^{+\infty}\Big(1+\frac{w(\max\{1/n,e^{-\theta s}y(s)\})}{g(\max\{1/n,e^{-\theta s}y(s)\})}\Big)\Big(1+\frac{k(\max\{1/n,e^{-(c+\theta)s}y'(s)\})}{h(\max\{1/n,e^{-(c+\theta)s}y'(s)\})}\Big)\\
&\quad\times e^{-r_1 s}\max\{G(s,s),\overline{G}(s)\}\Phi(s)g(e^{-\theta s}y(s))h(e^{-(c+\theta)s}y'(s))ds\\
&\le \Big(1+\frac{w(\max\{1/n,\|y\|_\theta\})}{g(\max\{1/n,\|y\|_\theta\})}\Big)\Big(1+\frac{k(\max\{1/n,\|y\|_\theta\})}{h(\max\{1/n,\|y\|_\theta\})}\Big)\Pi(\|y\|_\theta)\\
&\le \Big(1+\frac{w(R_0)}{g(R_0)}\Big)\Big(1+\frac{k(R_0)}{h(R_0)}\Big)\Pi(R_0)<R_0.
\end{aligned}
$$

Passing to the supremum over t, we infer that

$$
\|T_n y\|_\theta < R_0 = \|y\|_\theta,\quad \forall\, y\in\partial\Omega_1\cap\mathcal{P}. \tag{4.4}
$$

As a consequence, we may conclude that

$$
T_n y\ne \lambda y,\quad \forall\, y\in\partial\Omega_1\cap\mathcal{P},\ \forall\,\lambda\ge 1,\ n\ge n_0. \tag{4.5}
$$

Otherwise, for some $n_1\ge n_0$, there would exist $y_1\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda_1\ge 1$ such that $T_{n_1}y_1=\lambda_1 y_1$. Thus

$$
\|T_{n_1}y_1\|_\theta=\lambda_1\|y_1\|_\theta\ge\|y_1\|_\theta=R_0,
$$

contradicting (4.4). This implies that (4.5) holds. Therefore, Lemma 4.1 and (4.5) imply

$$
i(T_n,\Omega_1\cap\mathcal{P},\mathcal{P})=1,\quad \forall\, n\in\{n_0,n_0+1,\dots\}. \tag{4.6}
$$

Claim 2. $T_n y\not\le y$ for any $y\in\partial\Omega_2\cap\mathcal{P}$. Otherwise, let $y_2\in\partial\Omega_2\cap P$ and $n_2\ge n_0$ with

$$
T_{n_2}y_2\le y_2. \tag{4.7}
$$

From (4.3) and the fact that $\frac{|y_2(t)|}{e^{\theta t}}\le\|y_2\|_\theta=\widetilde{R}\le\frac{r}{e^{\theta\delta}}$, we infer that $y_2(t)\le r$, for each $t\in[\gamma,\delta]$. Then

$$
\varphi(t,y_2(t))\ge\Big(\frac{1}{\Lambda_0\ell}+\varepsilon\Big)y_2(t),\quad \forall\, t\in[\gamma,\delta]. \tag{4.8}
$$

By (4.7), (4.8) and Lemma 2.5, the following estimates are straightforward:

$$
y_2(t)\ge\int_0^{+\infty} e^{-\theta t}G(t,s)\Phi(s)f\left(s,\max\{e^{\theta s}/n_2,y_2(s)\},\max\{e^{\theta s}/n_2,e^{-cs}y_2'(s)\}\right)ds
$$

$$
\begin{aligned}
&\geq \Lambda_0 \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s)\varphi\left(s, \max\{e^{\theta s}/n_2, y_2(s)\}\right) ds \\
&\geq \Lambda_0 \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s)\Big(\frac{1}{\Lambda_0 \ell} + \varepsilon\Big) \max\{e^{\theta s}/n_2, y_2(s)\} ds \\
&\geq \Lambda_0 \left(\frac{1}{\Lambda_0 \ell} + \varepsilon\right) \min_{t \in [\gamma,\delta]} y_2(t) \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s) ds \\
&= \Lambda_0 \ell \left(\frac{1}{\Lambda_0 \ell} + \varepsilon\right) \min_{t \in [\gamma,\delta]} y_2(t) \\
&> \min_{t \in [\gamma,\delta]} y_2(t), \quad \forall\, t \in [\gamma,\delta],
\end{aligned}
$$

contradicting the continuity of the function y_2 on the compact interval $[\gamma,\delta]$; this implies that Claim 2 holds. Then, Lemma 4.2 yields

$$
i(T_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 0, \quad \forall\, n \in \{1, 2, \dots\}. \tag{4.9}
$$

Consequently, from (4.6), (4.9) and the fact that $\overline{\Omega}_1 \subset \Omega_2$, we find

$$
i(T_n, (\Omega_1 \setminus \overline{\Omega}_2) \cap \mathcal{P}, \mathcal{P}) = -1, \quad \forall\, n \in \{n_0, n_0+1, \dots\}. \tag{4.10}
$$

This equality and the solution property of the fixed point index imply that, for each $n \geq n_0$, there exists some $y_n \in (\Omega_1 \setminus \overline{\Omega}_2) \cap \mathcal{P}$ such that $T_n y_n = y_n$ with $0 < \widetilde{R} < \|y_n\|_\theta < R_0$. Consider the sequence of functions $\{y_n\}_{n \geq n_0}$. Clearly, the functions belonging to $\{\frac{y_n(t)}{e^{\theta t}},\ n \geq n_0\}$ and the functions belonging to $\{\frac{y_n'(t)}{e^{\theta t}},\ n \geq n_0\}$ are uniformly bounded on $\mathbb{R}^+$. Since $\widetilde{R} < \|y_n\|_\theta < R_0$, (H1) and (H2) imply that, for each $n \geq n_0$,

$$
\begin{aligned}
&\int_0^{+\infty} e^{-\theta t} G(t,s)\Phi(s) f\left(s, \max\{e^{\theta s}/n, y_n(s)\}, \max\{e^{\theta s}/n, e^{-cs} y_n'(s)\}\right) ds \\
&\leq \int_0^{+\infty} e^{-r_1 s} G(s,s)\Phi(s) \left(g(\max\{1/n, e^{-\theta s} y_n(s)\}) + w(\max\{1/n, e^{-\theta s} y_n(s)\})\right) \\
&\quad \times \Big(h(\max\{1/n, e^{-(c+\theta)s} y_n'(s)\}) + k(\max\{1/n, e^{-(c+\theta)s} y_n'(s)\})\Big) ds \\
&= \int_0^{+\infty} \Big(1 + \frac{w(\max\{1/n, e^{-\theta s} y_n(s)\})}{g(\max\{1/n, e^{-\theta s} y_n(s)\})}\Big)\Big(1 + \frac{k(\max\{1/n, e^{-(c+\theta)s} y_n'(s)\})}{h(\max\{1/n, e^{-(c+\theta)s} y_n'(s)\})}\Big) \\
&\quad \times e^{-r_1 s} \max\{G(s,s), \overline{G}(s)\}\Phi(s) g(e^{-\theta s} y_n(s)) h(e^{-(c+\theta)s} y_n'(s)) ds \\
&\leq \Big(1 + \frac{w(\max\{1/n, \|y_n\|_\theta\})}{g(\max\{1/n, \|y_n\|_\theta\})}\Big)\Big(1 + \frac{k(\max\{1/n, \|y_n\|_\theta\})}{h(\max\{1/n, \|y_n\|_\theta\})}\Big)\Pi(\|y_n\|_\theta) \\
&\leq \Big(1 + \frac{k(R_0)}{h(R_0)}\Big)\Big(1 + \frac{k(R_0)}{h(R_0)}\Big) \\
&\quad \times \int_0^{+\infty} e^{-r_1 s} \max\{G(s,s), \overline{G}(s)\}\Phi(s) g(e^{-\theta s}\Gamma \widetilde{R}) h(-e^{-cs} R_0) ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{+\infty} e^{-\theta t} G_t(t,s)\Phi(s) f\left(s, \max\{e^{\theta s}/n, y_n(s)\}, \max\{e^{\theta s}/n, e^{-cs} y_n'(s)\}\right) ds \\
&\leq \Big(1 + \frac{w(\max\{1/n, \|y_n\|_\theta\})}{g(\max\{1/n, \|y_n\|_\theta\})}\Big)\Big(1 + \frac{k(\max\{1/n, \|y_n\|_\theta\})}{h(\max\{1/n, \|y_n\|_\theta\})}\Big)\Pi(\|y_n\|_\theta)
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big(1+\frac{w(R_0)}{g(R_0)}\Big)\Big(1+\frac{k(R_0)}{h(R_0)}\Big)\\
&\quad\times \int_0^{+\infty} e^{-r_1 s}\max\{G(s,s),\overline{G}(s)\}\Phi(s)g(e^{-\theta s}\Gamma\widetilde{R})h(-e^{-cs}R_0)ds.
\end{aligned}
$$

Then, for some $a>0$ and $t_1,t_2\in[0,a]$, we have for $n\in\{n_0,n_0+1,\dots\}$,

$$
\begin{aligned}
|y_n(t_1)-y_n(t_2)|e^{-\theta t} \le \int_0^\infty & e^{-\theta t}|G(t_1,s)-G(t_2,s)|\Phi(s)\\
&\times f\left(s,\max\{e^{\theta s}/n,y_n(s)\},\max\{e^{\theta s}/n,e^{-cs}y_n'(s)\}\right)ds
\end{aligned}
$$

and

$$
\begin{aligned}
|y_n'(t_1)-y_n'(t_2)|e^{-\theta t} \le \int_0^\infty & e^{-\theta t}|G_t(t_1,s)-G_t(t_2,s)|\Phi(s)\\
&\times f\Big(s,\max\{e^{\theta s}/n,y_n(s)\},\max\{e^{(\theta-c)s}/n,e^{-cs}y_n'(s)\}\Big)\,ds.
\end{aligned}
$$

Consequently, the functions belonging to $\{\frac{y_n(t)}{e^{\theta t}},\ n\ge n_0\}$ and the functions belonging to $\{\frac{y_n'(t)}{e^{\theta t}},\ n\ge n_0\}$ are locally equicontinuous on $\mathbb{R}^+$. Similarly, we have

$$
\begin{aligned}
&\lim_{t\to+\infty}\sup_{n\ge n_0}\left|e^{-\theta t}y_n(t)-\lim_{s\to+\infty}e^{-\theta s}y_n(s)\right|\\
&=\lim_{t\to+\infty}\sup_{n\ge n_0}\Big|\int_0^\infty e^{-\theta t}G(t,s)\Phi(s)f_n(s,y_n(s),e^{-cs}y_n'(s))ds\Big|\\
&\le \int_0^\infty \lim_{t\to+\infty}\left|e^{-\theta t}G(t,s)\Phi(s)f_n(s,y_n(s),e^{-cs}y_n'(s))\right|ds=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{t\to+\infty}\sup_{n\ge n_0}\left|e^{-\theta t}y_n'(t)-\lim_{s\to+\infty}e^{-\theta s}y_n'(s)\right|\\
&\le \lim_{t\to+\infty}\int_0^\infty e^{-\theta t}\left|G_t(t,s)\Phi(s)f(s,y_n(s),e^{-cs}y_n'(s))\right|ds=0.
\end{aligned}
$$

Thus, the functions functions belonging to $\{\frac{y_n(t)}{e^{\theta t}},\ n\ge n_0\}$ and the functions belonging to $\{\frac{y_n'(t)}{e^{\theta t}},\ n\ge n_0\}$ are locally equiconvergent on $+\infty$. Consequently, Lemma 2.9 guarantees that there is a convergent subsequence $\{y_{n_j}\}_{j\ge1}$ of $\{y_n\}_{n\ge n_0}$ such that $\lim_{j\to+\infty}y_{n_j}=y$ strongly X. Moreover the continuity of f yields

$$
\begin{aligned}
\lim_{j\to+\infty}f_{n_j}(t,y_{n_j},y_{n_j}') &= \lim_{j\to+\infty}f\Big(t,\max\{e^{\theta t}/_{n_j},y_{n_j}\},\max\{e^{\theta t}/_{n_j},e^{-ct}y_{n_j}'\}\Big)\\
&= f(t,y(t),e^{-ct}y'(t)).
\end{aligned}
$$

Then the dominated convergence theorem guarantees that

$$
\begin{aligned}
y(t)&=\lim_{j\to+\infty}y_{n_j}(t)\\
&=\lim_{j\to+\infty}\int_0^{+\infty}G(t,s)\Phi(s)f\Big(t,\max\{e^{\theta t}/_{n_j},y_{n_j}\},\max\{e^{\theta t}/_{n_j},e^{-ct}y_{n_j}'\}\Big)\\
&=\int_0^{+\infty}G(t,s)\Phi(s)f(s,y(s),e^{-cs}y'(s))ds,\quad t\in I.
\end{aligned}
$$

Finally, $\widetilde{R} < |y_{n_j}\|_\theta < R_0$, for all $j \geq 1$ implies $\widetilde{R} \leq \|y\|_\theta \leq R_0$. Hence

$$0 < \widetilde{R} \leq \|y\|_\theta \leq R_0, \quad y(t) \geq \Gamma\|y\|_\theta, \quad \forall\, t \in [\gamma, \delta],$$

as claimed. ◻

The following result can be proved in an analogous manner. The proof is omitted.

Theorem 4.4. *Assume* (H1)–(H2) *hold and*

(H3') *$f(t, y, z) \geq \varphi'(t, y)$ for all $t \in [\gamma, \delta]$ and all $(y, z) \in (0, +\infty) \times \mathbb{R}^*$, where $\varphi' \in C([\gamma, \delta] \times (0, +\infty))$ satisfies*

$$\liminf_{y\to+\infty} \min_{t\in[\gamma,\delta]} \frac{\varphi'(t, y)}{y} > \frac{1}{\Lambda_0 \ell}\,.$$

Then problem (1.1) *has at least one nontrivial positive solution.*

4.2. **Twin solutions.** Let $\mathcal{P}$ be a cone of a real Banach space E. Let $0 < c < d$ be constants and β, α two continuous functionals on $\mathcal{P}$ convex and concave, respectively. Define the convex sets:

$$\begin{gathered} P_d = \{y \in \mathcal{P} : \|y\| < d\}, \\ P(\beta, d) = \{x \in \mathcal{P} : \beta(x) \leq d\}, \\ P(\beta, \alpha, c, d) = \{x \in \mathcal{P} : \alpha(x) \geq c,\ \beta(x) \leq d\}. \end{gathered}$$

We will apply the following fixed point theorem to prove the existence of two positive fixed points for the operator T.

Lemma 4.5 ([25])**.** *Let $A : \mathcal{P} \to \mathcal{P}$ be a completely continuous operator. Let β and α be continuous convex and concave functionals on $\mathcal{P}$, respectively. Let d and c be real numbers. Assume that*

(i) *$0 \in \{x \in \mathcal{P} : \beta(x) < d\}$ and the set $\{x \in \mathcal{P} : \beta(x) < d\}$ is bounded;*
(ii) *$\{x \in P(\beta, \alpha, c, d) : \beta(x) < d\} \neq \emptyset$ and $\beta(Ax) < d$ for all $x \in P(\beta, \alpha, c, d)$;*
(iii) *$\beta(Ax) < d$ for all $x \in P(\beta, d)$ with $\alpha(Ax) < c$;*
(iv) *$i(A, P_r, \mathcal{P}) = 0$ for sufficiently small positive number r, $i(A, P_R, \mathcal{P}) = 0$ for sufficiently large positive number L.*

Then A has at least two fixed points x_1, x_2 in $\mathcal{P}$ such that $\|x_1\| > r$ with $\beta(x_1) < d$, and $\|x_2\| < L$ with $\beta(x_2) > d$.

Our main result in this section is as follows.

Theorem 4.6. *Assume* (H1)–(H2) *and*

(H4) *$f(t, y, z) \geq \varrho(t, y)$ for all $t \in [\gamma, \delta]$ and all $(y, z) \in (0, +\infty) \times \mathbb{R}^*$, where the function $\varrho \in C([\gamma, \delta] \times (0, +\infty))$ satisfies*

$$\liminf_{y\to 0} \min_{t\in[\gamma,\delta]} \frac{\varrho(t, y)}{y} > \frac{1}{\Lambda_0 \ell}, \quad \liminf_{y\to+\infty} \min_{t\in[\gamma,\delta]} \frac{\varrho(t, y)}{y} > \frac{1}{\Lambda_0 \ell}\,.$$

Then, problem (1.1) *has at least two positive solutions y_1, y_2 such that*

$$0 < \|y_1\|_\theta \leq R_0 \leq \|y_2\|_\theta.$$

Proof. Define a sequence of operators T_n by (4.2) and then consider the nonnegative, continuous concave and convex functionals α, β defined respectively by

$$\alpha(y) = \min_{y\in[\gamma,\delta]} \frac{y(t)}{e^{\theta t}}, \quad \beta(y) = \|y\|_\theta.$$

Lemmas 3.1 and 3.2 guarantee that $T_n : \mathcal{P} \to \mathcal{P}$ is completely continuous. So, we only have to verify the conditions of Lemma 4.5.

Claim 1. $\beta(y) = \|y\|_\theta$. For R_0 given by the inequality (2.14) in Assumption (H2), it is clear that $0 \in \{\, y \in \mathcal{P} : \beta(y) < R_0\}$ and the set $\{y \in \mathcal{P} : \beta(y) < R_0\}$ is bounded.

Claim 2. The set $\{y \in P(\beta, \alpha, \frac{R_0}{2} e^{-\theta\delta}, R_0) : \beta(y) < R_0\}$ is nonempty since it contains the constant function $y_0 \equiv \frac{R_0}{2}$. Indeed, $\beta(y_0) = \frac{R_0}{2} \sup_{t\in\mathbb{R}^+} e^{-\theta t} < R_0$ and $\alpha(y_0) = \frac{R_0}{2} e^{-\theta\delta}$. Let $y \in P(\beta, \alpha, \frac{R_0}{2} e^{-\theta\delta}, R_0)$; then $\beta(y) = \|y\|_\theta \le R_0$. As in the proof of Theorem 4.3, Claim (a), for $n \ge n_0 > \frac{1}{R_0}$, we can check that $\beta(T_n y) = \|T_n y\|_\theta < R_0$. So the condition (ii) of Lemma 4.5 is satisfied.

Claim 3. Arguing as in Claim 2, we obtain

$$\beta(T_n y) = \|T_n y\|_\theta < R_0, \quad \forall\, y \in P(\beta, R_0),\ \forall\, n \in \{n_0, n_0 + 1, \dots\}.$$

So the condition (iii) of Lemma 4.5 is satisfied.

Claim 4. Since $\liminf_{y\to 0} \min_{t\in[\gamma,\delta]} \frac{\varrho(t,y)}{y} > \frac{1}{\Lambda_0 \ell}$, there exist ε_0 and $r_0 > 0$ such that

$$\varrho(t, y) \ge \big(\frac{1}{\Lambda_0 \ell} + \varepsilon_0\big) y, \quad \forall y \in [0, r_0] \text{ and } \forall\, t \in [\gamma, \delta].$$

We choose a sufficiently small $r = \min(R_0/2, r_0/e^{\theta\delta})$. Proceeding as in the proof of Theorem 4.3, Claim (b), we can prove that

$$T_n y \not\le y, \quad \text{for any } y \in \partial P_r.$$

According to Lemma 4.2, we infer that

$$i(T_n, P_r, \mathcal{P}) = 0.$$

Claim 5. Since $\liminf_{y\to+\infty} \min_{t\in[\gamma,\delta]} \frac{\varrho(t,y)}{y} > \frac{1}{\Lambda_0 \ell}$, there exist ε_1 and $\sigma > 0$ such that

$$\varrho(t, y) \ge \big(\frac{1}{\Lambda_0 \ell} + \varepsilon_1\big) y, \quad \text{for each } y \ge \sigma \text{ and } t \in [\gamma, \delta]. \tag{4.11}$$

Choose sufficiently large $L = \max(2R_0, \frac{\sigma}{\Gamma})$. So $y \in \partial P_L$ implies

$$y(t) \ge \Gamma \|y\|_\theta \ge L\Gamma \ge \frac{\sigma}{\Gamma} \Gamma = \sigma, \quad t \in [\gamma, \delta].$$

Then, using the inequality

$$\varrho(t, y(t)) \ge \big(\frac{1}{\Lambda_0 \ell} + \varepsilon\big) y(t), \quad \text{for any } t \in [\gamma, \delta]$$

and arguing as in the proof of Theorem 4.3, Claim (b), we can prove that

$$T_n y \not\le y, \ \text{ for any } y \in \partial P_L.$$

By Lemma 4.2, we deduce that

$$i(T_n, P_L, \mathcal{P}) = 0.$$

Thus, the condition (vi) of Lemma 4.5 is satisfied. According to this lemma with $c = \frac{R_0}{2} e^{-\theta\delta}$ and $d = R_0$, we infer that, for each $n \in \{n_0, n_0 + 1, \dots\}$, T_n has at least two positive fixed points $y_{n,1}, y_{n,2} \in \mathcal{P}$ such that $r < \|y_{n,1}\|_\theta < R_0 < \|y_{n,2}\|_\theta < L$. Now consider the sequence of functions $\{y_{n,i}\}_{n\ge n_0}$, $i = 1, 2$. Essentially the same argument used for $\{y_n\}_{n\ge n_0}$ in Theorem 4.3 guarantees that $\{y_{n,i}\}_{n\ge n_0}$, $i = 1, 2$ has a convergent subsequence $\{y_{n_j,i}\}_{j\ge 1}$ such that $\lim_{j\to+\infty} y_{n_j,i} = y_i$, $i = 1, 2$

for the norm topology of X. Consequently, y_1 and y_2 are two positive solutions of problem (1.1) with

$$r \leq \|y_1\|_\theta \leq R_0 \leq \|y_2\|_\theta \leq L.$$

□

4.3. **Triple nonnegative solutions.** Let $r > a > 0$, $L > 0$ be constants, ψ a nonnegative continuous concave functional and α, β nonnegative continuous convex functionals on a cone $\mathcal{P}$ of a Banach space $(E, \|\cdot\|)$. Define the convex sets:

$$\begin{aligned} P(\alpha, r; \beta, L) &= \{x \in \mathcal{P} : \alpha(x) < r,\ \beta(x) < L\}, \\ \overline{P}(\alpha, r; \beta, L) &= \{x \in \mathcal{P} : \alpha(x) \leq r,\ \beta(x) \leq L\}, \\ P(\alpha, r; \beta, L; \psi, a) &= \{x \in \mathcal{P} : \alpha(x) < r,\ \beta(x) < L,\ \psi(x) > a\}, \\ \overline{P}(\alpha, r; \beta, L; \psi, a) &= \{x \in \mathcal{P} : \alpha(x) \leq r,\ \beta(x) \leq L,\ \psi(x) \geq a\}. \end{aligned}$$

The following assumptions about the nonnegative continuous convex functionals α, β will be considered:

- (A1) there exists $M > 0$ such that $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$, for all $x \in \mathcal{P}$;
- (A2) $P(\alpha, r; \beta, L) \neq \emptyset$ for all $r > 0$, $L > 0$.

Lemma 4.7 ([4]). *Let E be a Banach space $\mathcal{P} \subset E$ a cone and $r_2 \geq d > c > r_1 > 0$, $L_2 \geq L_1 > 0$ be constants. Assume that α, β are nonnegative continuous convex functionals satisfying (A1) and (A2). Let ψ be a nonnegative continuous concave functional on $\mathcal{P}$ such that $\psi(x) \leq \alpha(x)$ for all $x \in \overline{P}(\alpha, r_2; \beta, L_2)$ and let $A : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$ be a completely continuous operator. Assume*

- (B1) *$\{x \in \overline{P}(\alpha, d; \beta, L_2; \psi, c) : \psi(x) > c\} \neq \emptyset$ and $\psi(Ax) > c$, for all x in $\overline{P}(\alpha, d; \beta, L_2; \psi, c)$;*
- (B2) *$\alpha(Ax) < r_1$, $\beta(Ax) < L_1$, for all $x \in \overline{P}(\alpha, r_1; \beta, L_1)$;*
- (B3) *$\psi(Ax) > c$ for all $x \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, c)$ with $\alpha(Ax) > d$.*

Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{P}(\alpha, r_2; \beta, L_2)$ with

$$\begin{gathered} x_1 \in P(\alpha, r_1; \beta, L_1), \\ x_2 \in \{x \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, c) : \psi(x) > c\}, \\ x_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus \overline{P}(\alpha, r_2; \beta, L_2; \psi, c) \cup \overline{P}(\alpha, r_1; \beta, L_1). \end{gathered}$$

Now we arrive at our final existence result in this paper.

Theorem 4.8. *Assume the following assumptions hold:*

- (H1') *$F : I^2 \times \mathbb{R}^* \to \mathbb{R}^+$ is a continuous function and there exist functions $g, w \in C((1, \infty), I)$ and $h, k \in C(\mathbb{R}^*, I)$ such that*
$$0 \leq F(t, u, v) \leq (g(u+1) + w(u+1))(h(v) + k(v)), \quad \forall\, (t, u, v) \in I^2 \times \mathbb{R}^*$$
where g, h are non-increasing functions, w/g and k/h are nondecreasing functions.
- (H2') *For all $\Re > 0$,*
$$\widetilde{\Pi}(\Re) = \int_0^{+\infty} e^{-r_1 s} \max\{G(s, s), \overline{G}(s)\} \Phi(s) g(e^{-\theta s}) h(-e^{-cs}\Re) ds < \infty$$
and there exists constants R_1, R_2 with $R_2 < \frac{\Lambda_0}{2e^{\theta\delta}} R_1$ such that for $i = 1, 2$
$$\Big(1 + \frac{w(R_i + 1 + \frac{1}{\Gamma})}{g(R_i + 1 + \frac{1}{\Gamma})}\Big)\Big(1 + \frac{k(R_i)}{h(R_i)}\Big)\widetilde{\Pi}(R_i) < R_i. \tag{4.12}$$

(H5) *$f(t,y,z) \geq \zeta(t,y)$ for all $t \in [\gamma,\delta]$ and all $(y,z) \in (0,+\infty) \times \mathbb{R}^*$, where $\zeta \in C([\gamma,\delta] \times (0,+\infty))$ satisfies*
$$\liminf_{y\to 0} \min_{t\in[\gamma,\delta]} \frac{\zeta(t,y)}{y} = +\infty.$$

Then (1.1) has at least three nonnegative solutions (two of which are positive) y_1, y_2 and y_3 in $\overline{P}(\alpha, R_1; \beta, R_1)$ such that for $t \in [0,\infty)$,
$$\begin{gathered}
e^{-\theta t}|y_1(t)| \leq R_2, \quad e^{-\theta t}|y_1'(t)| \leq R_2,\\
e^{-\theta t}|y_2(t)| \leq R_1, \quad e^{-\theta t}|y_2'(t)| \leq R_1,\\
R_2 \leq e^{-\theta t}|y_3(t)| \leq R_1, \quad R_2 \leq e^{-\theta t}|y_3'(t)| \leq R_1,
\end{gathered}$$
and for $t \in [\gamma,\delta]$,
$$|y_2(t)| \geq \frac{\Lambda_0}{2e^{\theta\delta}} R_1, \quad |y_3(t)| \leq \frac{\Lambda_0}{2e^{\theta\delta}} R_1.$$

Proof. Define an operator sequence by (4.2) and consider the functionals
$$\alpha(y) = \frac{1}{\Gamma} + \sup_{t\in\mathbb{R}^+} \frac{|y(t)|}{e^{\theta t}}, \quad \beta(y) = \sup_{t\in\mathbb{R}^+} \frac{|y'(t)|}{e^{\theta t}}, \quad \psi(y) = \frac{1}{\Gamma} + \min_{t\in[\gamma,\delta]} \frac{|y(t)|}{e^{\theta t}}.$$
Then α, β are nonnegative continuous convex functionals satisfying (A1) and (A2); ψ is a nonnegative continuous concave functional with $\psi(y) \leq \alpha(y)$ for all $y \in \mathcal{P}$. Here $\mathcal{P}$ is the cone defined in (2.2). For this, we will apply Theorem 4.4 to verify the existence of fixed points for the operator T_n. Lemmas 3.1 and 3.2 guarantee that $T_n : \mathcal{P} \to \mathcal{P}$, is completely continuous.

Claim 1. $T_n : \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1) \to \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1)$, for $n \geq n_0 > \frac{1}{R_1 + \frac{1}{\Gamma}}$. Indeed, if $y \in \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1)$, then $\alpha(y) \leq R_1 + \frac{1}{\Gamma}$ and $\beta(y) \leq R_1$. Arguing as in the proof of Theorem 4.3, Claim 1, we obtain, using Assumptions (H1') and (H2'), the following estimates valid for $t \in \mathbb{R}^+$:
$$\begin{aligned}
&\frac{1}{\Gamma} + e^{-\theta t}|T_n y(t)|\\
&= \frac{1}{\Gamma} + \int_0^{+\infty} e^{-\theta t} G(t,s)\Phi(s) f_n(s, y(s), e^{-cs}y'(s))ds\\
&= \frac{1}{\Gamma} + \int_0^{+\infty} e^{-\theta t} G(t,s)\Phi(s) f\left(s, \max\{e^{\theta s}/n, y(s)\}, \max\{e^{\theta s}/n, e^{-cs}y'(s)\}\right) ds\\
&\leq \frac{1}{\Gamma} + \int_0^{+\infty} e^{-r_1 s} G(s,s)\Phi(s) \Big(g(\max\{1/n, e^{-\theta s}y(s)\} + 1)\\
&\quad + w(\max\{1/n, e^{-\theta s}y(s)\} + 1)\Big)\Big(h(\max\{1/n, e^{-(c+\theta)s}y'(s)\})\\
&\quad + k(\max\{1/n, e^{-(c+\theta)s}y'(s)\})\Big)ds\\
&\leq \frac{1}{\Gamma} + \int_0^{+\infty} \left(1 + \frac{w(\max\{1/n, e^{-\theta s}y(s)\} + 1)}{g(\max\{1/n, e^{-\theta s}y(s)\} + 1)}\right)\\
&\quad \times \left(1 + \frac{k(\max\{1/n, e^{-(c+\theta)s}y'(s)\})}{h(\max\{1/n, e^{-(c+\theta)s}y'(s)\})}\right)\\
&\quad \times e^{-r_1 s} \max\{G(s,s), \overline{G}(s)\}\Phi(s) g(e^{-\theta s}y(s) + 1) h(e^{-(c+\theta)s}y'(s))ds.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\frac{1}{\Gamma}+e^{-\theta t}|T_ny(t)|\\
&\leq \frac{1}{\Gamma}+\Big(1+\frac{w(\max\{1/n,\alpha(y)\}+1)}{g(\max\{1/n,\alpha(y)\}+1)}\Big)\Big(1+\frac{k(\max\{1/n,\beta(y)\})}{h(\max\{1/n,\beta(y)\})}\Big)\\
&\quad\times\int_0^{+\infty}e^{-r_1s}\max\{G(s,s),\overline{G}(s)\}\Phi(s)g(e^{-\theta s}\Gamma\alpha(y))h(-e^{-cs}\beta(y))ds\\
&\leq \frac{1}{\Gamma}+\Big(1+\frac{w\left(R_1+1+\frac{1}{\Gamma}\right)}{g\left(R_1+1+\frac{1}{\Gamma}\right)}\Big)\Big(1+\frac{k(R_1)}{h(R_1)}\Big)\widetilde{\Pi}(R_1)\\
&< R_1<R_1+\frac{1}{\Gamma}.
\end{aligned}
$$

Therefore, $\alpha(T_ny)\leq R_1+\frac{1}{\Gamma}$, and

$$
\begin{aligned}
&e^{-\theta t}|(T_ny)'(t)|\\
&=\int_0^{+\infty}e^{-\theta t}G_t(t,s)\Phi(s)F(s,\max\{1/n,e^{-\theta s}y(s)\},\max\{1/n,e^{-(c+\theta)s}y'(s)\})ds\\
&\leq\int_0^{+\infty}e^{-r_1s}\overline{G}(s)\Phi(s)\left(g(\max\{e^{\theta s}/n,y(s)\}+1)+w(\max\{1/n,e^{-\theta s}y(s)\}+1\right)\\
&\quad\times\Big(h(\max\{1/n,e^{-(c+\theta)s}y'(s)\})+k(\max\{1/n,e^{-(c+\theta)s}y'(s)\})\Big)ds\\
&\leq\int_0^{+\infty}\Big(1+\frac{w(\max\{1/n,e^{-\theta s}y(s)\}+1)}{g(\max\{1/n,e^{-\theta s}y(s)\}+1)}\Big)\Big(1+\frac{k(\max\{1/n,e^{-(c+\theta)s}y'(s)\})}{h(\max\{1/n,e^{-(c+\theta)s}y'(s)\})}\Big)\\
&\quad\times e^{-r_1s}\max\{G(s,s),\overline{G}(s)\}\Phi(s)g(e^{-\theta s}y(s)+1)h(e^{-(c+\theta)s}y'(s))ds\\
&\leq\Big(1+\frac{w\left(R_1+1+\frac{1}{\Gamma}\right)}{g\left(R_1+1+\frac{1}{\Gamma}\right)}\Big)\Big(1+\frac{k(R_1)}{h(R_1)}\Big)\widetilde{\Pi}(R_1)<R_1.
\end{aligned}
$$

Consequently, $\beta(T_ny)\leq R_1$.

Claim 2. Condition (4.12) implies that $T_n:\overline{P}(\alpha,R_2+\frac{1}{\Gamma};\beta,R_2)\to\overline{P}(\alpha,R_2+\frac{1}{\Gamma};\beta,R_2)$ for $n\geq n_1>\frac{1}{R_2+\frac{1}{\Gamma}}$. The proof is identical to that in Claim 1. So the condition $(\mathfrak{B}2)$ of Lemma 4.7 is satisfied.

Claim 3. The set $\{y\in\overline{P}(\alpha,\frac{R_1}{2}+\frac{1}{\Gamma};\beta,R_1;\psi,\frac{\Lambda_0}{2e^{\theta\delta}}R_1):\psi(y)>\frac{\Lambda_0}{2e^{\theta\delta}}R_0)\}$ is nonempty. Notice that the constant function $y_0\equiv\frac{R_1}{2}$ lies in the set $\overline{P}(\alpha,\frac{R_1}{2}+\frac{1}{\Gamma};\beta,R_1;\psi,\frac{\Lambda_0}{2e^{\theta\delta}}R_1)$ and $\psi(y_0)>\frac{\Lambda_0}{2e^{\theta\delta}}R_1$. Indeed, $\alpha(y_0)=\frac{R_1}{2}\sup_{t\in[0,\infty)}e^{-\theta t}+\frac{1}{\Gamma}\leq\frac{R_1}{2}+\frac{1}{\Gamma}$, $\beta(y_0)=0$ and $\psi(y_0)=e^{-\theta\delta}\frac{R_1}{2}>\frac{\Lambda_0}{2e^{\theta\delta}}R_1$ for $\Lambda_0<1$.

Claim 4. We prove that $\psi(T_ny)>\frac{\Lambda_0}{2e^{\theta\delta}}R_1$, $\forall\,y\in\overline{P}(\alpha,\frac{R_1}{2}+\frac{1}{\Gamma};\beta,R_1;\psi,\frac{\Lambda_0}{2e^{\theta\delta}}R_1)$. If $y\in\overline{P}(\alpha,\frac{R_1}{2}+\frac{1}{\Gamma};\beta,R_1;\psi,\frac{\Lambda_0}{2e^{\theta\delta}}R_1)$, then $\alpha(y)\leq\frac{R_1}{2}+\frac{1}{\Gamma}$. Moreover, the condition (H5) tells us that, if $M_4=\frac{2e^{(\theta+r_1)\delta}}{\Lambda_0\Gamma\ell}$ then there exists some $\mu>\frac{R_1}{2}e^{\theta\delta}$ such that

$$
\zeta(t,y)\geq M_4y,\quad\forall\,y\in(0,\mu),\forall\,t\in[\gamma,\delta]. \tag{4.13}
$$

We can see that, for any $y\in\overline{P}(\alpha,\frac{R_1}{2};\beta,R_1;\psi,\frac{\Lambda}{2e^{\theta\delta}}R_1)$ and $t\in[\gamma,\delta]$, we have

$$
\alpha(y)\leq\frac{R_1}{2}+\frac{1}{\Gamma}\;\Rightarrow\;y(t)\leq\frac{R_1}{2}e^{\theta\delta}<\mu,\quad\forall\,t\in[\gamma,\delta].
$$

With Lemma 2.5 (c) and (4.13), we obtain the estimates:

$$
\begin{aligned}
\psi(T_n y) &> \min_{t\in[\gamma,\delta]} \int_0^{+\infty} e^{-\theta t} G(t,s) \\
&\quad \times \Phi(s) f\left(s, \max\{e^{\theta s}/n, y(s)\}, \max\{e^{\theta s}/n, e^{-cs} y'(s)\}\right) ds \\
&\geq \Lambda_0 e^{-\theta\delta} \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s)\zeta(s, \max\{e^{\theta s}/n, y(s)\}) ds \\
&\geq \Lambda_0 e^{-\theta\delta} \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s) M_4 \max\{e^{\theta s}/n, y(s)\} ds \\
&\geq \Lambda_0 e^{-\theta\delta} \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s) M_4 y(s) ds \\
&\geq \Lambda_0 M_4 e^{-\theta\delta} \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s)\Gamma \|y\|_\theta ds \\
&> \frac{1}{2} M_4 \Lambda_0 \Gamma\, e^{-(\theta+r_1)\delta} \|y\|_\theta \int_\gamma^\delta e^{-r_1 s} G(s,s)\Phi(s) ds \\
&= \|y\|_\theta \geq \psi(y) \geq \frac{\Lambda_0}{2e^{\theta\delta}} R_1.
\end{aligned}
$$

Claim 5. $\psi(T_n y) > \frac{\Lambda_0}{2e^{\theta\delta}} R_1$ for all $y \in \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1; \psi, \frac{\Lambda_0}{2e^{\theta\delta}} R_1)$ with $\alpha(T_n y) > \frac{R_1}{2} + \frac{1}{\Gamma}$. Let $y \in \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1; \psi, \frac{\Lambda_0}{2e^{\theta\delta}} R_1)$ be such that $\alpha(T_n y) > \frac{R_1}{2} + \frac{1}{\Gamma}$. For any $\sigma \in \mathbb{R}^+$, we know by Lemma 2.5(b),(c) that

$$
\begin{aligned}
\psi(T_n y) &= \frac{1}{\Gamma} + e^{-\theta\delta} \min_{t\in[\gamma,\delta]} \int_0^{+\infty} G(t,s)\Phi(s) f_n(s, y(s), e^{-cs} y'(s)) ds \\
&\geq \frac{1}{\Gamma} + e^{-\theta\delta} \int_0^{+\infty} \Lambda_0 e^{-r_1 s} G(s,s)\Phi(s) f_n(s, y(s), e^{-cs} y'(s)) ds \\
&\geq \frac{1}{\Gamma} + \Lambda_0 e^{-\theta\delta} \int_0^{+\infty} e^{-\theta\sigma} G(\sigma,s)\Phi(s) f_n(s, y(s), e^{-cs} y'(s)) ds \\
&= \frac{1}{\Gamma} + \Lambda_0 e^{-\theta\delta} e^{-\theta\sigma} \int_0^{+\infty} G(\sigma,s)\Phi(s) f_n(s, y(s), e^{-cs} y'(s)) ds \\
&= \Lambda_0 e^{-\theta\delta} (\frac{1}{\Gamma}\frac{e^{\theta\delta}}{\Lambda_0} + e^{-\theta\sigma} T_n y(\sigma)) \\
&\geq \Lambda_0 e^{-\theta\delta} (\frac{1}{\Gamma} + e^{-\theta\sigma} T_n y(\sigma)).
\end{aligned}
$$

Passing to the supremum over σ, we obtain that $y \in \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1; \psi, \frac{\Lambda_0}{2e^{\theta\delta}} R_1)$,

$$
\psi(T_n y) \geq \Lambda_0 e^{-\theta\delta} \alpha(T_n y) \geq \Lambda_0 e^{-\theta\delta} (R_1 + \frac{1}{\Gamma}) > \frac{\Lambda_0}{2e^{\theta\delta}} R_1.
$$

To sum up, all of the hypotheses of Lemma 4.7 are met if we take $L_2 = R_1$, $r_2 = R_1 + \frac{1}{\Gamma}$, $L_1 = R_2$, $r_1 = R_2 + \frac{1}{\Gamma}$ $d = \frac{R_1}{2} + \frac{1}{\Gamma}$ and $c = \frac{\Lambda_0}{2e^{\theta\delta}} R_1$. Hence, for each $n \in \{n_1, n_1+1, \dots\}$, T_n has at least three nonnegative fixed points $y_{n,i} \in \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1)$, $i = 1, 2, 3$, with

$$
y_{n,1} \in P(\alpha, R_2 + \frac{1}{\Gamma}; \beta, R_2),
$$

$$
\begin{gathered}
y_{n,2} \in \{\, y \in \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1; \psi, \frac{\Lambda_0}{2e^{\theta\delta}} R_1) : \psi(y) > \frac{\Lambda_0}{2e^{\theta\delta}} R_1 \},\\
y_{n,3} \in \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1) \setminus \overline{P}(\alpha, R_1 + \frac{1}{\Gamma}; \beta, R_1; \psi, \frac{\Lambda_0}{2e^{\theta\delta}} R_1) \cup \overline{P}(\alpha, R_2; \beta, R_2).
\end{gathered}
$$

Consider the sequence of functions $\{y_{n,i}\}_{n\geq n_1}$, $i = 1, 2, 3$. Arguing as in the proof as in Theorem 4.3, we can show that $\{y_{n,i}\}_{n\geq n_1}$, $i = 1, 2, 3$ has a convergent subsequence $\{y_{n_j,i}\}_{j\geq 1}$, such that $\lim_{j\to+\infty} y_{n_j,i} = y_i$, $i = 1, 2, 3$ for the strong topology of X. Consequently, y_1, y_2 and y_3 are three different nonnegative solutions of problem (1.1) and satisfy

$$
\begin{gathered}
e^{-\theta t}|y_1(t)| \leq R_2, \quad e^{-\theta t}|y_1'(t)| \leq R_2, \quad t \in [0, \infty),\\
e^{-\theta t}|y_2(t)| \leq R_1, \quad e^{-\theta t}|y_2'(t)| \leq R_1, \quad t \in [0, \infty),\\
R_2 < e^{-\theta t}|y_3(t)| \leq R_1, \quad R_2 < e^{-\theta t}|y_3'(t)| \leq R_1, \quad t \in [0, \infty),\\
|y_2(t)| \geq \frac{\Lambda_0}{2e^{\theta\delta}} R_1, \quad |y_3(t)| \leq \frac{\Lambda_0}{2e^{\theta\delta}} R_1, \quad t \in [\gamma, \delta].
\end{gathered}
$$

□

5. Examples

Let $\Phi(t) = e^{-\mu t}$ and consider the nonlinearity

$$
f(t, y, z) = \left(g(ye^{-\theta t}) + w(ye^{-\theta t})\right)\left(h(ze^{-\theta t}) + k(ze^{-\theta t})\right), \quad (t, y, z) \in I^2 \times \mathbb{R}^*
$$

where $g(u) = 1/u$, $w(u) = u^2$ and the functions h and k are defined by

$$
h(v) = \begin{cases} -v, & v \leq -1;\\ \frac{1}{\sqrt{-v}}, & -1 \leq v < 0;\\ \frac{1}{\sqrt{v}}, & v > 0. \end{cases} \quad k(v) = \begin{cases} -v, & v \leq -1;\\ \frac{1}{\sqrt{-v}}, & -1 \leq v < 0;\\ 1 + v, & v \geq 0. \end{cases}
$$

To check the inequality (2.14) in (H2), take $\gamma = 1/3$, $\delta = 1/2$, $c = 1/2$, $\lambda = 1/3$, $\eta = 2$, $\alpha = 1/8$ and $\mu = 100$; so we can choose $\theta = 1$ and $R_0 = 5$. In addition, we have

$$
G(s, s) = \begin{cases} \frac{1}{\Delta}\left(1 - e^{(r_2 - r_1)s}\right), & \text{if } s \leq \eta;\\ \frac{1}{\Delta}\left(1 - \alpha e^{r_2\eta} - e^{(r_2 - r_1)s}(1 - \alpha e^{r_1\eta})\right), & \text{if } s \geq \eta \end{cases}
$$

and

$$
\overline{G}(s) = \begin{cases} \frac{r_1}{\Delta}\left(2 - \alpha e^{r_2\eta} - \alpha e^{r_2(\eta - s) + r_1 s)}\right), & \text{if } s \leq \eta;\\ \frac{r_1}{\Delta}\left(2 - \alpha e^{r_2\eta} - \alpha e^{r_1\eta}\right), & \text{if } s \geq \eta. \end{cases}
$$

Using Matlab 7, we have found $\Pi(5) = 6.9589.10^{-4}$, whence

$$
\Big(1 + \frac{w(R_0)}{g(R_0)}\Big)\Big(1 + \frac{k(R_0)}{h(R_0)}\Big)\Pi(R_0) = 1.2641.
$$

Therefore Assumptions (H1) and (H2) are met. Also, Assumption (H3) in Theorem 4.3 is clearly satisfied. As a consequence, if

$$
f(t, y, y'e^{-t/2}) = \left(g(ye^{-t}) + w(ye^{-t})\right)\left(h(y'e^{-3t/2}) + k(y'e^{-3t/2})\right)
$$

then the singular boundary value problem

$$
\begin{gathered}
-y'' + 1/2y' + 1/3y = e^{-10^2 t} f(t, y, y'e^{-t/2}), \quad t > 0\\
y(0) = \alpha y(\eta), \quad \lim_{t\to\infty} e^{-t/2} y'(t) = 0,
\end{gathered} \tag{5.1}
$$

has at least one nontrivial positive solution. Moreover, we can check that Assumption $(\mathcal{H}_4)$ in Theorem 4.6 is fulfilled. Therefore, this problem has also two nontrivial positive solutions.

Let $\hat{g}, \hat{w}$ the functions defined by

$$\hat{g}(u) = \begin{cases} 1/(u-1), & u > 1; \\ 1/4, & 0 \le u < 1. \end{cases} \qquad \hat{w}(u) = (u-1)^2$$

The inequality (4.12) in Theorem 4.8 holds true for $R_1 = 3$ and $R_2 = 4/10$. Indeed $\widetilde{\Pi}(3) = 0.0020, \widetilde{\Pi}(4/10) = 0.0055$ and

$$\begin{gathered} \Big(1 + \frac{\hat{w}(R_1 + 1 + \frac{1}{\Gamma})}{\hat{g}(R_1 + 1 + \frac{1}{\Gamma})}\Big)\Big(1 + \frac{k(R_1)}{h(R_1)}\Big)\widetilde{\Pi}(R_1) = 0.6000 < 3, \\ \Big(1 + \frac{\hat{w}(R_2 + 1 + \frac{1}{\Gamma})}{\hat{g}(R_2 + 1 + \frac{1}{\Gamma})}\Big)\Big(1 + \frac{k(R_2)}{h(R_2)}\Big)\widetilde{\Pi}(R_2) = 0.1875 < 0.4. \end{gathered}$$

Therefore, the singular boundary value problem

$$\begin{gathered} -y'' + 1/2y' + 1/3y = e^{-10^2 t}\hat{f}(t, y, y'e^{-t/2}), \quad t > 0 \\ y(0) = \alpha y(\eta), \quad \lim_{t\to\infty} e^{-t/2}y'(t) = 0, \end{gathered} \tag{5.2}$$

where

$$\hat{f}(t, y, y'e^{-t/2}) = \big(\hat{g}(ye^{-t} + 1) + \hat{w}(ye^{-t} + 1)\big)\Big(h(y'e^{-3t/2}) + k(y'e^{-3t/2})\Big)$$

has in fact three nonnegative solutions, at least two of which are positive.

6. Concluding remarks

In this work, we have considered problem (1.1) when the nonlinearity may not only possess space-singularities in y and y' at the origin, but also takes quite general asymptotic behaviors near positive infinity, including polynomial growth as a special case. Indeed, we can consider the special cases in which F behaves in the first argument as $g(u) + w(u)$ with $g(u) = u^{-\sigma}$, $w(u) = u^m$ ($\sigma > 0$, $m \in \mathbb{N}^*$) and in the second argument as $h(v) + k(v)$ with $h(v) = v^{-\mu}$, $k(v) = v^n$ ($\mu > 0$, $n \in \mathbb{N}^*$). In this respect, the main assumptions are (H1) and (H2).

The existence results obtained in this paper have the advantages to allow working in a special cone of a Banach space such that most of solutions are positive hence nontrivial. With (H3) (or (H3')), we have proved in Theorem 4.3 and 4.4 existence of at least one positive solution with $y(t) \ge \Gamma\|y\|_\theta$ for $t \in [\gamma, \delta]$; that is $y \in \mathcal{P}$. At this step, notice that $[\gamma, \delta]$ is an arbitrary chosen interval which helps to get nontrivial solutions; this does not always hold true when one applies the Schauder fixed point theorem which rather provides solutions in a ball. In addition (H3) covers nonlinearities which are bounded below by sublinear functions near the origin while in (H3'), f may be superlinear at positive infinity.

Using a recent fixed point theorem of two functionals, we have obtained existence of a second solution in Theorem 4.6 satisfying $0 < \|y_1\|_\theta \le R \le \|y_2\|_\theta$. However, we may notice that (H4) combines Assumptions (H3) and (H3') and in counterpart yields precise information about solutions.

Finally, Assumption (H5) is of the form of (H2). However with a stronger assumption than (H3), we have even proved existence of three solutions by means of a three-functional fixed point theorem; notice however that one of them lies in a ball and thus could be a trivial solution.

The multi-point condition at 0 has given rise to a new and elaborated Green's function; its properties have enabled us to choose an appropriate cone to contain the desired solutions. The space singularities have been treated by approximation through the nonlinearity (4.1) and the operator (4.2) for which existence of fixed points has been proved under sharp estimates of the Green's function. Then, the solutions have been obtained as limits as $n \to +\infty$ via compactness sequential arguments.

The example of application shows that all of these hypotheses can be satisfied for quite simple and general nonlinearities. One of the novelty of this work is that we have considered a class of space-singular nonlinearities at the origin with general growth at positive infinity. We hope this work can provide improvements of the rich literature developed for multi-point boundary value problems on the positive half line.

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