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MIXED TWO-POINT BOUNDARY-VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this article, we prove the existence of solutions to mixed two-point boundary-value problem for impulsive differential equations by variational methods, in both resonant and the non resonant cases.

1. Introduction

In this article, we study the existence of solutions to the impulsive problem

$$\begin{gathered} -u'' + \lambda u = f(t,u), \quad \text{a.e. } t \in (0,T), \\ u(0) = 0, \quad u'(T) = 0, \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\dots p, \end{gathered} \tag{1.1}$$

where $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\dots,p$, are continuous functions and $f(t,u)$ satisfies the condition

- (A) $f(t,u)$ is measurable in t for each $u \in \mathbb{R}$, continuous in u for a.e. $t \in [0,T]$, and there exist functions $g \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $h \in L^1(0,T;\mathbb{R}^+)$ such that $|f(t,u)| \le g(|u|)h(t)$, for all $u \in \mathbb{R}$ and a.e. $t \in [0,T]$.

The theory of impulsive differential equations describes processes which experience a sudden change of their states at certain time. It can be successfully used for mathematical simulation in some problems from theoretical physics, chemistry, medicine, population dynamics, optimal control and in some other processes and phenomena in science and technology, see [9, 15] for the general aspects of the theory. Some classical tools such as upper and lower solutions, monotone iterative technique, fixed point theory, degree theory and so on have been widely used to such equations, we refer to [1, 2, 3, 4, 5, 7, 9, 11, 13] for some references. In recent years, the variational methods [10, 14] have been applied to such equations and are proved to be very effective, we refer to [12, 16, 17, 18, 19] for some results.

Let us recall some results related to the impulsive problem (1.1) obtained by the variational methods. Nieto and O'Regan ([12]) first noticed that the equation in (1.1) coupled with the Dirichlet boundary value condition and the impulsive conditions has a variational structure and obtained some existence results for the problem. The results are extended to more general nonlinearities in [19]. Recently,

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Tian and Ge [16, 17] investigated the equation with the more general (Sturm-Liouville) boundary value conditions. That is they considered the equation with the Sturm-Liouville boundary value conditions $\alpha u'(0)-\beta u(0)=0$, $\gamma u'(T)+\sigma u(T)=0$ but with a restriction $\alpha,\gamma>0$, $\beta,\sigma\geq 0$. The restriction excludes the boundary value conditions of the problem (1.1) investigated in this paper.

In this paper, we investigate the problem (1.1) still by the variational methods. The framework involved is different from those for the other kinds of boundary value conditions. We obtain some existence results both by the Ambrosetti-Rabinowitz type condition in the non-resonant case and the generalized Ahmad-Lazer-Paul type condition in the resonant case.

2. Preliminaries

Denote $H=\{u(t)|u(t)$ is absolutely continuous on $[0,T]$, $u(0)=0$, $u'(t)\in L^2(0,T;\mathbb{R})\}$. It is easy to see that $H_0^1(0,T)\subset H\subset H^1(0,T)$ and H is a closed subset of $H^1(0,T)$. So H is a Hilbert space with the usual inner product in $H^1(0,T)$.

Proposition 2.1. *If $u\in H$, then $\|u\|_c\leq\sqrt{T}\|u'\|_{L^2}$, where*

$$\|u\|_c=\max_{t\in[0,T]}|u(t)|.$$

Proof. For $u\in H$, we have

$$|u(t)|=|\int_0^t u'(s)ds|\leq\sqrt{t}\Big(\int_0^t|u'(s)|^2ds\Big)^{1/2}\leq\sqrt{T}\|u'\|_{L^2}.$$

□

For the linear problem

$$\begin{gathered}-u''(t)=\lambda u(t),\quad t\in(0,T),\\ u(0)=0,\quad u'(T)=0,\end{gathered}\tag{2.1}$$

there is a sequence of eigenvalues $\lambda_k=(2k+1)^2\pi^2/(4T^2)$ and the corresponding $L^2(0,T)$-normalized eigenfunctions $\phi_k(t)=\sqrt{(2/T)}\sin((2k+1)\pi t/(2T)$, $k=0,1,2\dots$.

If the equation in the problem (2.1) is coupled with the boundary value condition $u'(0)=0$, $u(T)=0$, the eigenvalues are as before and the corresponding $L^2(0,T)$-normalized eigenfunctions are $\psi_k(t)=\sqrt{(2/T)}\cos((2k+1)\pi t/(2T))$, $k=0,1,2\dots$.

According to the Sturm-Liouville theory, $\{\phi_k\}$ and $\{\psi_k\}$ are complete bases in the Hilbert space $L^2(0,T)$.

Claim If $u\in H$, then $\|u'\|^2_{L^2}\geq\lambda_0\|u\|^2_{L^2}$ (Poincaré inequality).

In fact, for any $u\in H$, set $u=\sum_{k=0}^\infty c_k\phi_k$, where

$$c_k=(u,\phi_k)=\frac{2}{T}\int_0^T u\phi_k dt.$$

By the Parseval identity, $\int_0^T u^2dt=\sum_{k=0}^\infty c_k^2$. Let $u'=\sum_{k=0}^\infty a_k\psi_k$, where $a_k=(u',\psi_k)=\frac{2}{T}\int_0^T u'\psi_k dt$. By an easy calculation,

$$a_k=\frac{2}{T}\int_0^T u'\psi_k dt=-\frac{2}{T}\int_0^T u\psi_k' dt=\frac{(2k+1)\pi}{2T}\cdot\frac{2}{T}\int_0^T u\phi_k dt=\frac{(2k+1)\pi}{2T}c_k.$$

Then we have

$$\int_0^T |u'|^2 dt = \sum_{k=0}^{\infty} a_k^2 = \sum_{k=0}^{\infty} \frac{(2k+1)^2\pi^2}{4T^2} c_k^2 \geq \frac{\pi^2}{4T^2} \sum_{k=0}^{\infty} c_k^2 = \lambda_0 \int_0^T |u|^2 dt.$$

Similarly, we can prove that $\|u'\|_{L^2}^2 \geq \lambda_1 \|u\|_{L^2}^2$, for all $u \in H \ominus \operatorname{span}\{\phi_0\}$, which will be used later.

According to the Poincaré inequality, we can define an inner product in H, $(u, v) = \int_0^T u'(t)v'(t)dt$. The induced norm $\|u\| = \sqrt{\int_0^T u'^2 dt}$ is equivalent to the old one. Throughout the paper, we will use this norm.

Proposition 2.2. *Assume that $\mu(t) \in L^1(0,T)$ satisfies*

$$0 \leq \mu(t) \leq \lambda_0, \tag{2.2}$$

for a.e. $t \in [0,T]$ and $\mu(t) < \lambda_0$ holds on a subset of $[0,T]$ with positive measure. Then there exists $\delta > 0$ such that for all $u \in H$, one has

$$\int_0^T [u'^2(t) - \mu(t)u^2(t)]dt \geq \delta \int_0^T u'^2(t)dt.$$

Proof. We use some known arguments [8] to prove it. By (2.2) and the Poincaré inequality, we see that for all $u \in H$,

$$\int_0^T [u'^2(t) - \mu(t)u^2(t)]dt \geq \int_0^T [u'^2(t) - \lambda_0 u^2(t)]dt \geq 0. \tag{2.3}$$

Now we assume that the proposition were false. Then we can find a sequence $\{u_n\}$ in H, $u_n \neq 0$ such that

$$\int_0^T [u_n'^2(t) - \mu(t)u_n^2(t)]dt < \frac{1}{n} \int_0^T u_n'^2(t)dt.$$

Let $v_n = u_n/\|u_n\|$. We obtain

$$\int_0^T [v_n'^2(t) - \mu(t)v_n^2(t)]dt < \frac{1}{n}.$$

We can assume that $v_n \rightharpoonup v$ in H and $v_n \to v$ in $C[0,T]$. We obtain

$$1 \leq \int_0^T \mu(t)v^2(t)dt. \tag{2.4}$$

From (2.4), we have

$$v \neq 0. \tag{2.5}$$

Since $v_n \rightharpoonup v$ in H, we have

$$\|v\|^2 \leq \liminf_{n\to\infty} \|v_n\|^2 = 1.$$

Hence, we have

$$\int_0^T [v'^2(t) - \mu(t)v^2(t)]dt \leq 0.$$

Therefore,

$$\int_0^T [v'^2(t) - \mu(t)v^2(t)]dt = 0. \tag{2.6}$$

From (2.3), we have

$$\int_0^T [v'^2(t) - \lambda_0 v^2(t)]dt = 0,$$

which shows that $v(t)$ is an eigenfunction corresponding to λ_0. So we set $v(t) = C\phi_0(t)$, where C is a constant. Substituting it to (2.6) and noticing that $\mu(t) < \lambda_0$ holds on a subset of $[0, T]$ with positive measure, we obtain $C = 0$, which contradicts to (2.5). This completes the proof. □

3. Non-resonant case

In this section, we study the existence of solutions for the non-resonant impulsive problem

$$\begin{gathered} -u'' + \lambda u = f(t, u), \quad \text{a.e. } t \in (0, T), \\ u(0) = 0, \quad u'(T) = 0, \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \dots p \end{gathered} \tag{3.1}$$

with $\lambda > -\lambda_0$, where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, $f(t, u)$ satisfies the condition (A) and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \dots, p$, are continuous functions. Here $u'(t_j^+)$ (respectively $u'(t_j^-)$) denotes the right limit (respectively left limit) of $u'(t)$ at $t = t_j$.

Definition 3.1. A function u is said to be a classical solution of (3.1) if u satisfies:

1. $u \in C[0, T]$;
2. For all $j = 0, 1, 2, \dots, p$, $u_j = u|_{(t_j, t_{j+1})} \in H^{2,1}(t_j, t_{j+1})$;
3. u satisfies the equation in (3.1) a.e. on $(0, T)$, the boundary value condition and the impulsive conditions.

Set $F(t, u) = \int_0^u f(t, \xi)d\xi$ and define the functional J on H by

$$J(u) = \frac{1}{2}\int_0^T u'^2 dt + \frac{\lambda}{2}\int_0^T u^2 dt + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t, u)dt.$$

It is easily verified that J is continuously differentiable on H and

$$(J'(u), v) = \int_0^T u'v'dt + \lambda \int_0^T uv\, dt + \sum_{j=1}^p I_j(u(t_j))v(t_j) - \int_0^T f(t, u)v\, dt$$

for all $u, v \in H$.

Definition 3.2. A weak solution for problem (3.1) is a function $u \in H$ satisfying $(J'(u), v) = 0$ for all $v \in H$.

Lemma 3.3. *If $u \in H$ is a weak solution of* (3.1)*, then it is a classical solution.*

Proof. Let $u \in H$ be a weak solution of (3.1). Then

$$0 = (J'(u), v) = \int_0^T u'v'dt + \lambda \int_0^T uv\, dt + \sum_{j=1}^p I_j(u(t_j))v(t_j) - \int_0^T f(t, u)v\, dt \tag{3.2}$$

for all $v \in H$. For $j \in \{0, 1, \dots, p\}$ and every $v \in H_0^1(t_j, t_{j+1})$, set

$$\widetilde{v}(t) = \begin{cases} v(t), & t_j \le t \le t_{j+1}, \\ 0, & t \in [0, T] \setminus [t_j, t_{j+1}]. \end{cases}$$

Then $\widetilde{v} \in H_0^1(0,T) \subset H$. Then replacing v in (3.2) by $\widetilde{v}$, we have

$$\int_{t_j}^{t_{j+1}} u'v'dt + \lambda \int_{t_j}^{t_{j+1}} uv\,dt - \int_{t_j}^{t_{j+1}} f(t,u)v\,dt = 0$$

for every $v \in H_0^1(t_j, t_{j+1})$. Hence by standard results, $u \in H^{2,1}(t_j, t_{j+1})$ and

$$-u'' + \lambda u = f(t,u), \quad \text{a.e. on } (t_j, t_{j+1}).$$

Hence u satisfies

$$-u'' + \lambda u = f(t,u), \quad \text{a. e. on } (0,T). \tag{3.3}$$

Fix a positive number δ such that $t_p + \delta < T$. Now, for every $v_1 \in H_0^1(0, t_p + \delta) \subset H_0^1(0,T) \subset H$. Multiplying (3.3) by v_1 and integrating it between 0 and T, we obtain

$$-\int_0^T u''v_1dt + \lambda \int_0^T uv_1dt = \int_0^T f(t,u)v_1dt.$$

That is,

$$\sum_{j=1}^p \Delta u'(t_j)v_1(t_j) + \int_0^T u'v_1'dt + \lambda \int_0^T uv_1dt = \int_0^T f(t,u)v_1dt.$$

According to (3.2), we have

$$\sum_{j=1}^p \Delta u'(t_j)v_1(t_j) = \sum_{j=1}^p I_j(u(t_j))v_1(t_j).$$

Hence we obtain $\Delta u'(t_j) = I_j(u(t_j))$ for all $j = 1, 2, \dots, p$.

Finally, we prove that u satisfies the condition $u'(T) = 0$. For all $v_2 \in H^1(t_p, T)$ with $v_2(t_p) = 0$, we define

$$v^*(t) = \begin{cases} v_2(t), & t \in [t_p, T], \\ v_2(t_p), & t \in [0, t_p]. \end{cases}$$

It is easy to see that $v^* \in H$. Hence,

$$\int_{t_p}^T u'v_2'dt + \lambda \int_{t_p}^T uv_2dt - \int_{t_p}^T f(t,u)v_2dt = 0.$$

Set $u^*(t) = \int_t^T(-\lambda u(s) + f(s,u(s)))ds$. By the Fubini theorem, we obtain

$$\begin{aligned}
\int_{t_p}^T u^*(t)v_2'(t)dt &= \int_{t_p}^T \Big[\int_t^T (-\lambda u(s) + f(s,u(s)))v_2'(t)ds\Big]dt \\
&= \int_{t_p}^T \Big[\int_{t_p}^s (-\lambda u(s) + f(s,u(s)))v_2'(t)dt\Big]ds \\
&= \int_{t_p}^T (-\lambda u(s) + f(s,u(s)))v_2(s)dt \\
&= \int_{t_p}^T u'(t)v_2'(t)dt.
\end{aligned}$$

Hence, for every $v_2 \in H^1(t_p, T)$ with $v_2(t_p) = 0$, we have

$$\int_{t_p}^T (u^*(t) - u'(t))v_2'(t)dt = 0.$$

In particular, we can choose $v_2(t) = \phi_k^*(t) = \sqrt{2/(T-t_p)}\sin(\frac{(2k+1)\pi}{2(T-t_p)}(t-t_p))$, $k = 0, 1, 2 \dots$, then

$$\int_{t_p}^{T}(u^*(t) - u'(t))\psi_k^*(t)dt = 0, \ \forall k = 0, 1, 2 \dots$$

where

$$\psi_k^*(t) = \sqrt{\frac{2}{(T-t_p)}}\frac{(2k+1)\pi}{2(T-t_p)}\cos(\frac{(2k+1)\pi}{2(T-t_p)}(t-t_p)).$$

By noticing that $\{\psi_k^*(t)\}$ is the sequence of the eigenfunctions of the eigenvalue problem

$$-u''(t) = \lambda u(t), \quad t \in (t_p, T)$$

with the boundary value condition $u(t_p) = 0$, $u'(T) = 0$, hence it is complete in $L^2(t_p, T)$. Thus, $u^*(t) = u'(t)$ a.e. on (t_p, T); that is,

$$u'(t) = \int_t^T (-\lambda u(s) + f(s, u(s)))ds.$$

Hence, we obtain $u'(T) = 0$ and $u \in H^{2,1}(t_p, T)$ by condition (A). So, u is a classical solution of (3.1). This completes the proof. □

Theorem 3.4. *Suppose that $f(t, u)$ satisfies condition* (A). *Moreover, the following conditions hold.*

- (F1) $F(t, u) \le a(t)|u|^2 + b(t)|u| + c(t)$, *where* $a(t), b(t), c(t) \in L^1(0, T)$;
- (I1) $|I_j(u)| \le a_j + b_j|u|, \forall u \in \mathbb{R}$, *where* $a_j, b_j \ge 0$, $j = 1, 2, \dots, p$, *and* $\frac{1}{2} - \frac{T}{2}(\sum_{j=1}^p b_j) > 0$;
- (I2) $a(t) \le \lambda_0[\frac{1}{2} - \frac{T}{2}(\sum_{j=1}^p b_j)] + \frac{\lambda}{2} = a^*$ *with* $a(t) < a^*$ *holds on a subset of* $[0, T]$ *with positive measure.*

Then problem (3.1) *has at least one solution.*

Proof. For all $u \in H$,

$$\begin{aligned}
\sum_{j=1}^p \int_0^{u(t_j)} I_j(s)ds &\le \sum_{j=1}^p \int_0^{|u(t_j)|} |I_j(s)|ds \le \sum_{j=1}^p \int_0^{|u(t_j)|} (a_j + b_j|s|)ds \\
&\le (\sum_{j=1}^p a_j)\|u\|_c + \frac{1}{2}(\sum_{j=1}^p b_j)\|u\|_c^2 \\
&\le (\sqrt{T}\sum_{j=1}^p a_j)\|u\| + \frac{T}{2}(\sum_{j=1}^p b_j)\|u\|^2
\end{aligned}$$

and

$$\int_0^T F(t, u)dt \le \int_0^T (a(t)|u|^2 + b(t)|u| + c(t))dt \le \int_0^T a(t)|u|^2dt + C\|u\| + C$$

where and in the following C denotes a universal constant. Then

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_0^T u'^2dt + \frac{\lambda}{2}\int_0^T u^2dt + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t, u)dt \\
&\ge \frac{1}{2}\int_0^T u'^2dt + \frac{\lambda}{2}\int_0^T u^2dt - \sum_{j=1}^p \int_0^{|u(t_j)|} |I_j(s)|ds - \int_0^T a(t)|u|^2dt
\end{aligned}$$

$$
\begin{aligned}
&- C\|u\| - C \\
&\geq \frac{1}{2}\int_0^T u'^2 dt + \int_0^T (\frac{\lambda}{2} - a(t))|u|^2 dt - (\sqrt{T}\sum_{j=1}^p a_j)\|u\| - \frac{T}{2}(\sum_{j=1}^p b_j)\|u\|^2 \\
&- C\|u\| - C \\
&= [\frac{1}{2} - \frac{T}{2}(\sum_{j=1}^p b_j)]\int_0^T u'^2 dt - \int_0^T (a(t) - \frac{\lambda}{2})|u|^2 dt - C\|u\| - C.
\end{aligned}
$$

By Proposition 2.2 and the condition (I2), there exists $\delta > 0$ such that

$$
\Big[\frac{1}{2} - \frac{T}{2}(\sum_{j=1}^p b_j)\Big]\int_0^T u'^2 dt - \int_0^T (a(t) - \frac{\lambda}{2})|u|^2 dt \geq \delta\int_0^T u'^2 dt, \quad \forall u \in H.
$$

Thus

$$
J(u) \geq \delta\|u\|^2 - C\|u\| - C, \quad \forall u \in H.
$$

So

$$
\lim_{\|u\|\to\infty, u\in H} J(u) = +\infty.
$$

Hence every minimizing sequence is bounded. It is easily verified by the condition (A) and a compact imbedding result that $J(u)$ is weakly lower semi-continuous. Hence by a standard result, $J(u)$ has a minimizing point u, which is a classical solution of the problem (3.1). □

Since $\lambda > -\lambda_0$, we can choose the equivalent norm $\|u\|_1^2 = \int_0^T (u'^2 + \lambda u^2)dt$ in H. Hence there exist positive constants m_1 and m_2 such that $m_1\|u\| \leq \|u\|_1 \leq m_2\|u\|$.

Theorem 3.5. *Suppose that $f(t,u)$ satisfies the condition* (A). *Moreover, we assume that* (I1) *and the following conditions hold:*

- (F2) $\lim_{u\to 0}\frac{f(t,u)}{u} < k_1\lambda_0$ *uniformly for a.e.* $t \in [0,T]$, *where* $k_1 < m_1^2$*;*
- (F3) *There exist* $\mu > 2$ *and* $R > 0$ *such that for a.e.* $t \in [0,T]$ *and* $|u| \geq R$, $0 < \mu F(t,u) \leq uf(t,u)$ *(Ambrosetti-Rabinowitz type condition);*
- (I3) $\lim_{u\to 0}\frac{I_j(u)}{u} \to 0$ *for all* $1 \leq j \leq p$*;*
- (I*) $[(\frac{1}{2} - \frac{1}{\mu}) - \frac{1}{m_1^2}(\frac{1}{2} + \frac{1}{\mu})T(\sum_{j=1}^p b_j)] > 0$.

Then problem (3.1) *has at least one nontrivial solution.*

Proof. Obviously, $J(0) = 0$. By (F3) and condition (A), there exist nonnegative functions $d_1(t)$, $d_2(t) \in L^1(0,T)$ such that $d_1(t) > 0$ a.e. on $[0,T]$ and $F(t,u) \geq d_1(t)|u|^\mu - d_2(t)$ for a.e. $t \in [0,T]$ and all $u \in \mathbb{R}$.

Choosing $u \in H\backslash\{0\}$, then for $q > 0$, we have

$$
\begin{aligned}
J(qu) &\leq \frac{q^2}{2}\|u\|_1^2 + \sum_{j=1}^p \int_0^{|qu(t_j)|} |I_j(s)|ds - \int_0^T F(t,qu)dt \\
&\leq \frac{q^2}{2}\|u\|_1^2 + Cq\|u\| + Cq^2\|u\|^2 - \int_0^T (d_1(t)|qu|^\mu - d_2(t))dt \\
&\leq \frac{q^2}{2}\|u\|_1^2 + Cq\|u\|_1 + Cq^2\|u\|_1^2 - |q|^\mu \int_0^T d_1(t)|u|^\mu dt + C \\
&\to -\infty
\end{aligned}
$$

as $q \to \infty$. Setting $e = qu$, then for q large, we obtain $\|e\|_1 > R$ and $J(e) \leq 0$.

By (F2) and (I3), for some proper $\varepsilon > 0$, there exists $0 < r < R$ such that for $|u| \le r$, there holds

$$|F(t,u)| \le \frac{k_1\lambda_0}{2}|u|^2, \quad |I_j(u)| \le \varepsilon |u|.$$

For $u \in H$ with $\|u\| < \frac{r}{\sqrt{T}}$, we have $\|u\|_c < r$. Then $\|u\|_1 \le m_2\|u\| < \frac{m_2 r}{\sqrt{T}}$. So

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|_1^2 + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds - \int_0^T F(t,u)dt \\
&\ge \frac{1}{2}\|u\|_1^2 - \frac{Tp\varepsilon}{2m_1^2}\|u\|_1^2 - \frac{k_1\lambda_0}{2}\frac{1}{\lambda_0}\frac{1}{m_1^2}\|u\|_1^2 \\
&= (\frac{1}{2} - \frac{k_1}{2m_1^2})\|u\|_1^2 - \frac{Tp\varepsilon}{2m_1^2}\|u\|_1^2.
\end{aligned}$$

There exists $0 < \rho < \min\{R, \frac{m_2 r}{\sqrt{T}}\}$ and $\sigma > 0$ such that for $\|u\|_1 = \rho$, we have $J(u) > \sigma$.

Let $\{u_k\}$ be a PS sequence in H; that is,

$$|J(u_k)| \le C,\ \forall k \in \mathbb{N}, \quad |(J'(u_k), h)| \le o(1)\|h\|_1 \quad \text{for all } k \in \mathbb{N} \text{ and } h \in H.$$

We only need to prove that $\{u_k\}$ is bounded. For k large,

$$\begin{aligned}
C + \frac{1}{\mu}\|u_k\|_1 \ge J(u_k) - \frac{1}{\mu}J'(u_k)u_k &= (\frac{1}{2} - \frac{1}{\mu})\int_0^T (|u_k'|^2 + \lambda u_k^2)dt \\
&\quad + \sum_{j=1}^{p}\int_0^{u_k(t_j)} I_j(s)ds - \frac{1}{\mu}\sum_{j=1}^{p} I_j(u_k(t_j))u_k(t_j) - \int_0^T F(t,u_k)dt \\
&\quad + \frac{1}{\mu}\int_0^T f(t,u_k)u_k dt,
\end{aligned}$$

where

$$\begin{aligned}
-\int_0^T F(t,u_k)dt + \frac{1}{\mu}\int_0^T f(t,u_k)u_k\,dt &= -\frac{1}{\mu}\int_0^T (\mu F(t,u_k)dt - f(t,u_k)u_k)dt \\
&= -\frac{1}{\mu}\int_{|u_k|\ge R} (\mu F(t,u_k)dt - f(t,u_k)u_k)dt \\
&\quad - \frac{1}{\mu}\int_{|u_k|\le R} (\mu F(t,u_k)dt - f(t,u_k)u_k)dt.
\end{aligned}$$

The above first term is nonnegative by (F3), the second is bounded by the condition (A). Moreover,

$$\begin{aligned}
&|\sum_{j=1}^{p}\int_0^{u_k(t_j)} I_j(s)ds - \frac{1}{\mu}\sum_{j=1}^{p} I_j(u_k(t_j))u_k(t_j)| \\
&\le |\sum_{j=1}^{p}\int_0^{u_k(t_j)} I_j(s)ds| + \frac{1}{\mu}|\sum_{j=1}^{p} I_j(u_k(t_j))u_k(t_j)| \\
&\le (\sum_{j=1}^{p} a_j)\|u_k\|_c + \frac{1}{2}(\sum_{j=1}^{p} b_j)\|u_k\|_c^2 + \frac{1}{\mu}(\sum_{j=1}^{p} a_j)\|u_k\|_c + \frac{1}{\mu}(\sum_{j=1}^{p} b_j)\|u_k\|_c^2
\end{aligned}$$

$$
\begin{aligned}
&\le (1+\frac{1}{\mu})\sqrt{T}(\sum_{j=1}^{p} a_j)\|u_k\| + (\frac{1}{2}+\frac{1}{\mu})T(\sum_{j=1}^{p} b_j)\|u_k\|^2\\
&\le \frac{1}{m_1}(1+\frac{1}{\mu})\sqrt{T}(\sum_{j=1}^{p} a_j)\|u_k\|_1 + \frac{1}{m_1^2}(\frac{1}{2}+\frac{1}{\mu})T(\sum_{j=1}^{p} b_j)\|u_k\|_1^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
C+\frac{1}{\mu}\|u_k\|_1 &\ge (\frac{1}{2}-\frac{1}{\mu})\|u_k\|_1^2 - \frac{1}{m_1}(1+\frac{1}{\mu})\sqrt{T}(\sum_{j=1}^{p} a_j)\|u_k\|_1\\
&\quad - \frac{1}{m_1^2}(\frac{1}{2}+\frac{1}{\mu})T(\sum_{j=1}^{p} b_j)\|u_k\|_1^2\\
&= [(\frac{1}{2}-\frac{1}{\mu}) - \frac{1}{m_1^2}(\frac{1}{2}+\frac{1}{\mu})T(\sum_{j=1}^{p} b_j)]\|u_k\|_1^2\\
&\quad - \frac{1}{m_1}(1+\frac{1}{\mu})\sqrt{T}(\sum_{j=1}^{p} a_j)\|u_k\|_1.
\end{aligned}
$$

Therefore, by the condition (I*), $\{u_k\}$ is bounded in H.

By the Mountain Pass Lemma [10, 14], $J(u)$ possesses a critical point $u \in H$ such that $J(u) \ge \sigma > 0$; hence u is a nontrivial weak solution of (3.1). □

4. Resonance case

In this section, we study the existence of solutions for the resonant impulsive problem

$$
\begin{gathered}
u'' + \lambda_0 u = f(t,u), \quad \text{a.e. } t \in (0,T),\\
u(0) = 0, \quad u'(T) = 0,\\
\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\dots,p
\end{gathered} \tag{4.1}
$$

where $f(t,u)$ satisfies condition (A), $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\dots,p$, are continuous functions.

The corresponding functional $J : H \to \mathbb{R}$ is defined by

$$
J(u) = \frac{1}{2}\int_0^T u'^2 dt - \frac{\lambda_0}{2}\int_0^T u^2 dt + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)ds + \int_0^T F(t,u)dt.
$$

We will consider the case where the nonlinearity satisfies some kind of sublinear and generalized Ahmad, Lazer, Paul type coercive conditions; e.g. see [6] and [18] and the references therein for some applications to the periodic boundary value problems.

Decompose H as $H = \bar{H} \oplus \widetilde{H}$, where $\bar{H} = \operatorname{span}\{\phi_0\}$ and $\widetilde{H} = \overline{\operatorname{span}\{\phi_1,\phi_2,\dots\}}$.

For all $u \in H$, we write as $u = \bar{u} \oplus \widetilde{u}$, $\bar{u} \in \bar{H}$, $\widetilde{u} \in \widetilde{H}$. We recall the inequality,

$$
\int_0^T \widetilde{u}'^2 dt \ge \lambda_1 \int_0^T \widetilde{u}^2 dt
$$

for all $\widetilde{u} \in \widetilde{H}$.

Theorem 4.1. *Suppose that $f(t,u)$ satisfies condition (A). Moreover, the following conditions hold.*

(F4) *There exists* α *with* $\frac{1}{2} \le \alpha < 1$ *such that* $|f(t,u)| \le g(t)|u|^\alpha + h(t)$, *where* $g(t), h(t) \in L^1(0,T)$;

(F5) $\lim_{|\bar{u}|\to\infty} \int_0^T F(t,\bar{u})dt/|\bar{u}|^{2\alpha} = +\infty$, $\bar{u} \in \bar{H}$;

(I4) $|I_j(u)| \le p_j + q_j|u|^{\gamma_j}$ *where* $p_j, q_j \ge 0$, $j = 1,2,\dots,p$, *and* $0 \le \gamma_j \le 2\alpha - 1$.

Then problem (4.1) *has at least one solution.*

Proof. For all $u \in H$, $u = \bar{u} \oplus \widetilde{u}$, $\bar{u} \in \bar{H}$, $\widetilde{u} \in \widetilde{H}$, we have

$$\begin{aligned} |\sum_{j=1}^p \int_0^{u(t_j)} I_j(s)ds| &\le \sum_{j=1}^p \int_0^{|u(t_j)|} |I_j(s)|ds \le \sum_{j=1}^p \int_0^{|u(t_j)|} (p_j + q_j|s|^{\gamma_j})ds \\ &\le (\sum_{j=1}^p p_j)\|u\|_c + \sum_{j=1}^p q_j \frac{1}{\gamma_j+1}\|u\|^{\gamma_j+1} \\ &\le C|\bar{u}| + C\|\widetilde{u}\| + C|\bar{u}|^{\gamma+1} + C\|\widetilde{u}\|^{\gamma+1} + C, \end{aligned} \tag{4.2}$$

where $\gamma = \max\{\gamma_j, j = 1,2,\dots,p\}$. By (F4), we have

$$\begin{aligned} \int_0^T (F(t,u) - F(t,\bar{u}))dt &\le \int_0^T [\int_0^1 f(t,\bar{u}+s\widetilde{u})\widetilde{u}ds]dt \\ &\le \int_0^T [\int_0^1 (g(t)|\bar{u}+s\widetilde{u}|^\alpha + h(t))\widetilde{u}ds]dt \\ &\le C|\bar{u}|^\alpha\|\widetilde{u}\|_c + C\|\widetilde{u}\|_c^{\alpha+1} + C\|\widetilde{u}\|_c \\ &\le C|\bar{u}|^\alpha\|\widetilde{u}\| + C\|\widetilde{u}\|^{\alpha+1} + C\|\widetilde{u}\| \\ &\le \varepsilon\|\widetilde{u}\|^2 + C(\varepsilon)|\bar{u}|^{2\alpha} + C\|\widetilde{u}\|^{\alpha+1} + C\|\widetilde{u}\|. \end{aligned} \tag{4.3}$$

Then

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^T u'^2dt - \frac{\lambda_0}{2}\int_0^T u^2dt + \sum_{j=1}^p \int_0^{u(t_j)} I_j(s)ds \\ &\quad + \int_0^T (F(t,u) - F(t,\bar{u}))dt + \int_0^T F(t,\bar{u})dt \\ &\ge \frac{1}{2}\int_0^T |\widetilde{u}'|^2dt - \frac{\lambda_0}{2}\cdot\frac{1}{\lambda_1}\int_0^T |\widetilde{u}'|^2dt - C|\bar{u}| - C\|\widetilde{u}\| - C|\bar{u}|^{\gamma+1} - C\|\widetilde{u}\|^{\gamma+1} \\ &\quad - \varepsilon\|\widetilde{u}\|^2 - C(\varepsilon)|\bar{u}|^{2\alpha} - C\|\widetilde{u}\|^{\alpha+1} - C\|\widetilde{u}\| + \int_0^T F(t,\bar{u})dt - C \\ &\ge \big[\frac{1}{2}(1 - \frac{\lambda_0}{\lambda_1}) - \varepsilon\big]\|\widetilde{u}\|^2 - C\|\widetilde{u}\|^{\alpha+1} - C\|\widetilde{u}\|^{\gamma+1} - C\|\widetilde{u}\| - C(\varepsilon)|\bar{u}|^{2\alpha} - C \\ &\quad + \int_0^T F(t,\bar{u})dt \\ &\ge \big[\frac{1}{2}(1 - \frac{\lambda_0}{\lambda_1}) - \varepsilon\big]\|\widetilde{u}\|^2 - C\|\widetilde{u}\|^{\alpha+1} - C\|\widetilde{u}\|^{\gamma+1} - C\|\widetilde{u}\| + |\bar{u}|^{2\alpha}[-C(\varepsilon) \\ &\quad + \frac{1}{|\bar{u}|^{2\alpha}}\int_0^T F(t,\bar{u})dt] - C, \end{aligned}$$

where we have used $0 \le \gamma \le 2\alpha - 1$. Fixing some $0 < \epsilon < \frac{1}{2}(1 - \frac{\lambda_0}{\lambda_1})$, by (F5), we have $J(u) \to +\infty$ as $\|u\| \to \infty$. Noticing also the weak lower semi-continuity of $J(u)$, we complete the proof. □

Theorem 4.2. *Suppose that* (F4), (I4) *and the following condition are satisfied:*

(F6) $\lim_{|\bar{u}|\to\infty}\int_0^T F(t,\bar{u})dt/|\bar{u}|^{2\alpha} = -\infty$, $\bar{u}\in\bar{H}$.

Then problem (4.1) *has at least one solution.*

Proof. We apply the saddle point theorem [10, 14] to prove the theorem.

Step 1. For $u=\bar{u}\in\bar{H}$,

$$
\begin{aligned}
J(\bar{u}) &= \frac{1}{2}\int_0^T |\bar{u}'|^2dt - \frac{\lambda_0}{2}\int_0^T |\bar{u}|^2dt - \sum_{j=1}^{p}\int_0^{\bar{u}(t_j)} I_j(s)ds + \int_0^T F(t,\bar{u})dt\\
&\le C|\bar{u}| + C|\bar{u}|^{\gamma+1} + \int_0^T F(t,\bar{u})dt + C\\
&\le C|\bar{u}|^{2\alpha} + \int_0^T F(t,\bar{u})dt + C\\
&= |\bar{u}|^{2\alpha}(C + \frac{1}{|\bar{u}|^{2\alpha}}\int_0^T F(t,\bar{u})dt) + C.
\end{aligned}
$$

Hence, we have $J(\bar{u})\to-\infty$ as $|\bar{u}|\to\infty$.

Step 2. For $u=\widetilde{u}\in\widetilde{H}$, we have

$$
\begin{aligned}
\int_0^T (F(t,\widetilde{u}) - F(t,0))dt &= \int_0^T [\int_0^1 f(t,s\widetilde{u})\widetilde{u}ds]dt\\
&\le \int_0^T [\int_0^1 (g(t)|s\widetilde{u}|^{\alpha} + h(t))\widetilde{u}ds]dt\\
&\le C\|\widetilde{u}\|^{\alpha+1} + C\|\widetilde{u}\|.
\end{aligned}
$$

By some arguments in the proof of Theorem 4.1, we obtain

$$
\begin{aligned}
J(\widetilde{u}) &\ge \frac{1}{2}(1-\frac{\lambda_0}{\lambda_1})\int_0^T |\widetilde{u}'|^2dt + \sum_{j=1}^{p}\int_0^{\widetilde{u}(t_j)} I_j(s)ds + \int_0^T F(t,\widetilde{u})dt\\
&\ge \frac{1}{2}(1-\frac{\lambda_0}{\lambda_1})\int_0^T |\widetilde{u}'|^2dt - C\|\widetilde{u}\| - C\|\widetilde{u}\|^{\gamma+1} + \int_0^T (F(t,\widetilde{u}) - F(t,0))dt\\
&\quad + \int_0^T F(t,0)dt\\
&\ge \frac{1}{2}(1-\frac{\lambda_0}{\lambda_1})\|\widetilde{u}\|^2 - C\|\widetilde{u}\| - C\|\widetilde{u}\|^{\gamma+1} - C\|\widetilde{u}\|^{\alpha+1} - C\|\widetilde{u}\| - C.
\end{aligned}
$$

Since $\frac{1}{2}\le\alpha<1$ and $0\le\gamma\le2\alpha-1$, we obtain that $J(u)$ is bounded below on $\widetilde{H}$. So there exists $R>0$ such that

$$
\sup_{u\in S_R} J(u) < \inf_{u\in\widetilde{H}} J(u)
$$

where $S_R=\{u|\ \|u\|=R,\ u\in\bar{H}\}$.

Step 3. We show that J satisfies the PS condition. Let $\{u_k\}$ be a PS sequence in H, then there exists constant C such that

$$
|J(u_k)|\le C,\ \forall k\in\mathbb{N},\quad |(J'(u_k),h)|\le o(1)\|h\| \text{ for all } k\in\mathbb{N},\ h\in H.
$$

Since

$$
\begin{aligned}
&|\sum_{j=1}^{p} I_j(u_k(t_j))\widetilde{u}_k(t_j)| \\
&\le \sum_{j=1}^{p} p_j|\widetilde{u}_k(t_j)| + \sum_{j=1}^{p} q_j|\bar{u}_k(t_j)|^{\gamma_j}|\widetilde{u}_k(t_j)| + \sum_{j=1}^{p} q_j|\widetilde{u}_k(t_j)|^{\gamma_j+1} \\
&\le C\|\widetilde{u}_k\| + C|\bar{u}_k|^{\gamma}\|\widetilde{u}_k\| + C\|\widetilde{u}_k\|^{\gamma+1}
\end{aligned}
$$

and

$$
\begin{aligned}
|\int_0^T f(t,u_k)\widetilde{u}_k dt| &\le \int_0^T (g(t)|u_k|^{\alpha} + h(t))|\widetilde{u}_k| dt \\
&\le C|\bar{u}_k|^{\alpha}\|\widetilde{u}_k\| + C\|\widetilde{u}_k\|^{\alpha+1} + C\|\widetilde{u}_k\|,
\end{aligned}
$$

we have, for k large,

$$
\begin{aligned}
\|\widetilde{u}_k\| \ge |(J'(u_k),\widetilde{u}_k)| &= |\int_0^T u_k'\widetilde{u}_k' dt - \lambda_0 \int_0^T u_k\widetilde{u}_k dt - \sum_{j=1}^{p} I_j(u_k(t_j))\widetilde{u}_k(t_j) \\
&\quad + \int_0^T f(t,u_k)\widetilde{u}_k dt| \\
&\ge (1-\frac{\lambda_0}{\lambda_1})\|\widetilde{u}_k\|^2 - C\|\widetilde{u}_k\| - C|\bar{u}_k|^{\gamma}\|\widetilde{u}_k\| - C\|\widetilde{u}_k\|^{\gamma+1} \\
&\quad - C|\bar{u}_k|^{\alpha}\|\widetilde{u}_k\| - C\|\widetilde{u}_k\|^{\alpha+1} - C\|\widetilde{u}_k\|.
\end{aligned}
$$

Therefore, we obtain

$$
\|\widetilde{u}_k\| \le C|\bar{u}_k|^{\gamma} + C|\bar{u}_k|^{\alpha} + C \le C|\bar{u}_k|^{\alpha} + C. \tag{4.4}
$$

Hence, noticing the arguments in (4.2) and (4.3), we have

$$
\begin{aligned}
J(u_k) &= \frac{1}{2}\int_0^T |\tilde{u}_k'|^2 dt - \frac{\lambda_0}{2}\int_0^T \tilde{u}_k^2 dt + \sum_{j=1}^{p}\int_0^{u_k(t_j)} I_j(s) ds \\
&\quad + \int_0^T (F(t,u_k) - F(t,\bar{u}_k)) dt + \int_0^T F(t,\bar{u}_k) dt \\
&\le \frac{1}{2}\|\widetilde{u}_k\|^2 + C|\bar{u}_k| + C\|\widetilde{u}_k\| + C|\bar{u}_k|^{\gamma+1} + C\|\widetilde{u}_k\|^{\gamma+1} \\
&\quad + \varepsilon\|\widetilde{u}_k\|^2 + C(\varepsilon)|\bar{u}_k|^{2\alpha} + C\|\widetilde{u}_k\|^{\alpha+1} + C\|\widetilde{u}_k\| + \int_0^T F(t,\bar{u}_k) dt \\
&\le (\frac{1}{2}+\varepsilon)\|\widetilde{u}_k\|^2 + C\|\widetilde{u}_k\|^{\gamma+1} + C\|\widetilde{u}_k\|^{\alpha+1} + C\|\widetilde{u}_k\| + C|\bar{u}_k| + C|\bar{u}_k|^{\gamma+1} \\
&\quad + C(\varepsilon)|\bar{u}_k|^{2\alpha} + \int_0^T F(t,\bar{u}_k) dt + C.
\end{aligned}
$$

From (4.4), we obtain

$$
\begin{gathered}
\|\widetilde{u}_k\|^2 \le (C|\bar{u}_k|^{\alpha} + C)^2 \le C|\bar{u}_k|^{2\alpha} + C, \\
\|\widetilde{u}_k\|^{\gamma+1} \le (C|\bar{u}_k|^{\alpha} + C)^{\gamma+1} \le C|\bar{u}_k|^{2\alpha} + C, \\
\|\widetilde{u}_k\|^{\alpha+1} \le (C|\bar{u}_k|^{\alpha} + C)^{\alpha+1} \le C|\bar{u}_k|^{2\alpha} + C.
\end{gathered}
$$

So

$$\begin{aligned} J(u_k) &\le C(\varepsilon)|\bar{u}_k|^{2\alpha} + \int_0^T F(t,\bar{u}_k)dt + C \\ &\le |\bar{u}_k|^{2\alpha}[C(\varepsilon) + \frac{1}{|\bar{u}_k|^{2\alpha}}\int_0^T F(t,\bar{u}_k)dt] + C. \end{aligned}$$

Hence, if $\{|\bar{u}_k|\}$ has a unbounded subsequence, we will get a contradiction by (F_6). Therefore, $\{|\bar{u}_k|\}$ is bounded and moreover $\{u_k\}$ is bounded in H. By a standard argument, $\{u_k\}$ has a convergent subsequence. We obtain that $J(u)$ satisfies the PS condition.

Then existence of a critical point for J then follows from the saddle point theorem. The proof is complete. □

Examining carefully the proofs of the Theorems 4.1 and 4.2, we can also get the following two theorems.

Theorem 4.3. *If the conditions of Theorem 4.1 are replaced by the following conditions:*

- (F4') *There exists* α *with* $0 \le \alpha < 1$ *such that* $|f(t,u)| \le g(t)|u|^\alpha + h(t)$, *where* $g(t), h(t) \in L^1(0,T)$;
- (I4') $|I_j(u)| \le p_j + q_j|u|^{\gamma_j}$, *where* $p_j, q_j \ge 0$, $j = 1,2,\dots,p$, $0 \le \gamma_j < 1$;
- (F5') $\lim_{|\bar{u}|\to\infty, \bar{u}\in\bar{H}}(\int_0^T F(t,\bar{u})dt)/(|\bar{u}|^\beta) = +\infty$, *where* $\beta = \max\{\gamma+1, 2\alpha\}$, $\gamma = \max\{\gamma_j,\ j = 1,2,\dots,p\}$.

Then problem (4.1) *has at least one solution.*

Theorem 4.4. *Suppose that the conditions* (F4'), (I4') *and the following condition are satisfied:*

- (F5") $\lim_{|\bar{u}|\to\infty}(\int_0^T F(t,\bar{u})dt)/(|\bar{u}|^\beta) = -\infty$, $\bar{u} \in \bar{H}$ *where* $\beta = \max\{\gamma+1, 2\alpha\}$.

Then problem (4.1) *has at least one solution.*

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