Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 36, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OBLIQUE DERIVATIVE PROBLEMS FOR SECOND-ORDER HYPERBOLIC EQUATIONS WITH DEGENERATE CURVE

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Abstract. The present article concerns the oblique derivative problem for second order hyperbolic equations with degenerate circle arc. Firstly the formulation of the oblique derivative problem for the equations is given, next the representation and estimates of solutions for the above problem are obtained, moreover the existence of solutions for the problem is proved by the successive iteration of solutions of the equations. In this article, we use the complex analytic method, namely the new partial derivative notations, hyperbolic complex functions are introduced, such that the second order hyperbolic equations with degenerate curve are reduced to the first order hyperbolic complex equations with singular coefficients, then the advantage of complex analytic method can be applied.

1. Formulation of the oblique derivative problem

In [1, 2, 3, 4, 5, 8, 9, 10], the authors posed and discussed the Cauchy problem, Dirichlet problem and oblique derivative boundary value problem of second order hyperbolic equations and mixed equations with parabolic degenerate straight lines by using the methods of integral equations, functional analysis, energy integrals, complex analysis and so on, the obtained results possess the important applications. Here we generalize the above results to the oblique derivative problem of hyperbolic equations with degenerate circle arc. In this article, the used notations are the same as in [6, 7, 8, 9, 10].

Let D be a simply connected bounded domain D in the hyperbolic complex plane $\mathbb{C}$ with the boundary $\partial D = L \cup L_0$, where $L = L_1 \cup L_2$. Herein and later on, denote $\hat{y} = y - \sqrt{R^2 - x^2}$, and

$$L_1 = \{x + G(\hat{y}) = R_*, x \in [R_*, 0]\}, \quad L_2 = \{x - G(\hat{y}) = R^*, x \in [0, R^*]\},$$
$$L_0 = \{R_* \le x \le R^*, \hat{y} = 0\},$$

in which $K(\hat{y}) = -|\hat{y}|^m$, m, R are positive numbers, $R_* = -R$, $R^* = R$, $z_0 = z_1 = jy_0 = jy_1$ the intersection of L_1, L_2, $G(\hat{y}) = \int_0^{\hat{y}} \sqrt{|K(t)|}dt$, $H(\hat{y}) = |K(\hat{y})|^{1/2}$. In this article we use the hyperbolic unit j with the condition $j^2 = 1$ in $\overline{D}$, and $x + jy, w(z) = U(z) + jV(z) = [H(\hat{y})u_x - ju_y]/2$ are called the hyperbolic number

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2000 *Mathematics Subject Classification.* 35L20, 35L80.
Key words and phrases. Oblique derivative problem; hyperbolic equations; degenerate curve.

Submitted December 22, 2010. Published March 3, 2011.

and hyperbolic complex function in D. Consider the second-order linear equation of hyperbolic type with degenerate circle arc

$$Lu = K(\hat{y})u_{xx} + u_{yy} + au_x + bu_y + cu = -d \quad \text{in } D, \tag{1.1}$$

where a, b, c, d are real functions of z $(z \in \overline{D})$, and suppose that the equation (1.1) satisfies the following conditions:

Condition C. The coefficients a, b, c, d in $\overline{D}$ satisfy

$$\begin{gathered} \tilde{C}[d, \overline{D}] = C[d, \overline{D}] + C[d_x, \overline{D}] \le k_1, \quad \tilde{C}[\eta, \overline{D}] \le k_0, \quad \eta = a, b, c, \\ |a(x,y)||\hat{y}|^{1-m/2} = \varepsilon_1(\hat{y}) \quad \text{as } \hat{y} \to 0,\ m \ge 2,\ z \in \overline{D}, \end{gathered} \tag{1.2}$$

in which $\varepsilon_1(\hat{y})$ is a non-negative function satisfying the condition: $\varepsilon_1(\hat{y}) \to 0$ as $\hat{y} \to 0$.

To write the complex form of the above equation, denote $Y = G(\hat{y})$, $\hat{y} = y - \sqrt{R^2 - x^2}$, $\hat{x} = x$, and

$$\begin{gathered} W(z) = U + jV = \frac{1}{2}[H(\hat{y})u_x - ju_y] = \frac{H(\hat{y})}{2}[u_x - ju_Y] = H(\hat{y})u_Z, \\ H(\hat{y})W_{\overline{Z}} = \frac{H(\hat{y})}{2}[W_x + jW_Y] = \frac{1}{2}[H(\hat{y})u_x + jW_y] = W_{\overline{z}} \quad \text{in } \overline{D}, \end{gathered}$$

where $Z = Z(z) = x + jY = x + jG(\hat{y})$ in $\overline{D}$, $G(\hat{y}) = \int_0^{\hat{y}} H(t)dt$, $H(\hat{y}) = \sqrt{|K(\hat{y})|}$. Moreover,

$$\begin{aligned} -K(\hat{y})u_{xx} - u_{yy} &= H(\hat{y})[H(\hat{y})u_x - ju_y]_x + j[H(\hat{y})u_x - ju_y]_y - [jH_y + HH_x]u_x \\ &= 4H(\hat{y})W_{\overline{Z}} - [jH_y/H + H_x]Hu_x \\ &= au_x + bu_y + cu + d, \end{aligned}$$

$$\begin{aligned} &H(\hat{y})W_{\overline{Z}} \\ &= H[W_x + jW_Y]/2 \\ &= H[(U + jV)_x + j(U + jV)_Y]/2 \\ &= \frac{1}{4}(e_1 - e_2)(H_{\hat{y}}/H)Hu_x + (e_1 + e_2)[(H_x + a/H)Hu_x + bu_y + cu + d], \\ &(U + V)_\mu = \frac{1}{4H}\{2[H_{\hat{y}}/H + H_x + a/H]U - 2bV + cu + d\}, \quad \text{in } D, \\ &(U - V)_\nu = \frac{1}{4H}\{-2[H_{\hat{y}}/H - H_x - a/H]U - 2bV + cu + d\}, \quad \text{in } D, \end{aligned} \tag{1.3}$$

where $e_1 = (1 + j)/2$, $e_2 = (1 - j)/2$, $x = \mu + \nu$, $Y = \mu - \nu$, $\partial x/\partial\mu = 1/2 = \partial Y/\partial\mu$, $\partial x/\partial\nu = 1/2 = -\partial Y/\partial\nu$. Hence the complex form of (1.1) can be written as

$$\begin{gathered} W_{\overline{z}} = A_1W + A_2\overline{W} + A_3u + A_4 \quad \text{in } \overline{D}, \\ u(z) = 2\operatorname{Re}\int_{z_0}^{z}[\frac{U(z)}{H(\hat{y})} - jV(z)]dz + b_0 \quad \text{in } \overline{D}, \end{gathered} \tag{1.4}$$

where $b_0 = u(z_0)$, $z_0 = jy_0$, and the coefficients $A_l = A_l(z)$ $(l = 1, 2, 3, 4)$ are as follows

$$\begin{gathered} A_1 = \frac{1}{4}[\frac{a}{H} + \frac{jH_{\hat{y}}}{H} + H_x - jb], \quad A_2 = \frac{1}{4}[\frac{a}{H} + \frac{jH_{\hat{y}}}{H} + H_x + jb], \\ A_3 = \frac{c}{4}, \quad A_4 = \frac{d}{4} \quad \text{in } \overline{D}. \end{gathered}$$

For convenience, sometimes the hyperbolic complex number $\hat{z} = \hat{x} + j\hat{y} = x + j(y - \sqrt{R^2 - x^2})$ and the function $F[z(Z)]$ are simply written as $z = x + j\hat{y}$ and $F(Z)$ respectively. We mention that in this article, three domains; i.e., the original domain D, the characteristic domain $D_{\hat{z}}$ and the image domain D_Z are used, and the corresponding characteristic domain $D_{\hat{z}}$ almost is written as the original domain D.

The oblique derivative problem for (1.1) may be formulated as follows.

Problem O. Find a continuous solution $u(z)$ of (1.1) in $\overline{D}\backslash L_0$, which satisfies the boundary conditions

$$\begin{gathered}
\frac{1}{2}\frac{\partial u}{\partial l} = \frac{1}{H(y)}\operatorname{Re}[\overline{\lambda(z)}u_{\hat{z}}] = \operatorname{Re}[\overline{\Lambda(z)}u_z] = r(z), \quad z \in L = L_1 \cup L_2, \\
u(z_0) = b_0, \quad \frac{1}{H(\hat{y})}\operatorname{Im}[\overline{\lambda(z)}u_{\hat{z}}]|_{z=z_0} = \operatorname{Im}[\overline{\Lambda(z)}u_z]|_{z=z_0} = b_1,
\end{gathered} \tag{1.5}$$

in which l is a given vector at every point $z \in L$, $u_{\hat{z}} = [H(\hat{y})u_x - ju_y]/2$, $u_{\bar{\hat{z}}} = [H(\hat{y})u_x + ju_y]/2, b_0, b_1$ are real constants, $\lambda(z) = \lambda_1(x) + j\lambda_2(x)$, $\Lambda(z) = \cos(l, x) + j\cos(l, y)$, $R(z) = H(\hat{y})r(z)$, $z \in L$, $b_1' = H(\hat{y}_1)b_1$, $\lambda_1(z)$ and $\lambda_2(x)$ are real functions, $\lambda(z), r(z), b_0, b_1$ satisfy the conditions

$$\begin{gathered}
C^1[\lambda(z), L] \le k_0, \quad C^1[r(z), L] \le k_2, \quad |b_0|, |b_1| \le k_2, \\
\max_{z\in L_1}\frac{1}{|\lambda_1(x) - \lambda_2(x)|}, \quad \max_{z\in L_2}\frac{1}{|\lambda_1(x) + \lambda_2(x)|} \le k_0,
\end{gathered} \tag{1.6}$$

in which k_0, k_2 are positive constants.

For the Dirichlet problem (Problem D) with the boundary condition:

$$u(z) = \phi(x) \quad \text{on } L = L_1 \cup L_2, \tag{1.7}$$

where L_1, L_2 are as stated before, we find the derivative for (1.7) according to the parameter $s = x$ on L_1, L_2, and obtain

$$\begin{gathered}
u_s = u_x + u_y y_x = u_x - \frac{u_y}{H(\hat{y})} = \phi'(x) \quad \text{on } L_1, \\
u_s = u_x + u_y y_x = u_x + \frac{u_y}{H(\hat{y})} = \phi'(x) \quad \text{on } L_2;
\end{gathered}$$

i. e.,

$$\begin{gathered}
U(z) + V(z) = \frac{1}{2}H(\hat{y})\phi'(x) = R(z) \quad \text{on } L_1, \\
U(z) - V(z) = \frac{1}{2}H(\hat{y})\phi'(x) = R(z) \quad \text{on } L_2;
\end{gathered}$$

i. e.,

$$\begin{gathered}
\operatorname{Re}[(1+j)(U+jV)] = U(z) + V(z) = R(z) \quad \text{on } L_1, \\
\operatorname{Im}[(1+j)(U+jV)]|_{z=z_0-0} = [U(z) + V(z)]|_{z=z_0-0} = R(z_0 - 0), \\
\operatorname{Re}[(1-j)(U+jV)] = U(z) - V(z) = R(z) \quad \text{on } L_2, \\
\operatorname{Im}[(1-j)(U+jV)]|_{z=z_0+0} = [-U(z) + V(z)]|_{z=z_0+0} = -R(z_0 + 0),
\end{gathered}$$

where

$$\begin{gathered}
U(z) = \frac{1}{2}H(\hat{y})u_x, \quad V(z) = -\frac{u_y}{2}, \\
\lambda_1 + j\lambda_2 = 1 - j, \quad \lambda_1 = 1 \ne \lambda_2 = -1 \quad \text{on } L_1,
\end{gathered}$$

$$\lambda_1+j\lambda_2=1+j,\quad \lambda_1=1\neq-\lambda_2=-1\quad\text{on } L_2.$$

From the above formulas, we can write the complex forms of boundary conditions of $U+jV$:

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}(U+jV)]=R(z)\quad\text{on } L,\\ \operatorname{Im}[\overline{\lambda(z)}(U+jV)]|_{z=z_0-0}=R(z_0-0)=b_1',\\ \lambda(z)=\begin{cases}1-j=\lambda_1+j\lambda_2,\\ 1+j=\lambda_1+j\lambda_2,\end{cases}\qquad R(z)=\begin{cases}H(\hat{y})\phi'(x)/2 & \text{on } L_1,\\ H(\hat{y})\phi'(x)/2 & \text{on } L_2,\end{cases}\\ u(z)=2\operatorname{Re}\int_{z_0}^{z}[\frac{U(z)}{H(\hat{y})}-jV(z)]dz+\phi(z_0)\quad\text{in } D.\end{gathered}\tag{1.8}$$

Hence Problem D is a special case of Problem O.

Noting that the condition (1.6), we can find a twice continuously differentiable functions $u_0(z)$ in $\overline{D}$, for instance, which is a solution of the oblique derivative problem with the boundary condition in (1.5) for harmonic equations in D (see [6, 7]), thus the functions $v(z)=u(z)-u_0(z)$ in D is the solution of the following boundary value problem in the form

$$K(\hat{y})v_{xx}+v_{yy}+av_x+bv_y+cv=-\hat{d}\quad\text{in } D,\tag{1.9}$$

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}v_{\tilde{z}}(z)]=r(z)\quad\text{on } L,\\ v(z_0)=b_0,\quad \operatorname{Im}[\overline{\lambda(z_0)}v_{\tilde{z}}(z_0)]=b_1',\end{gathered}\tag{1.10}$$

where $W(z)=U+jV=v_{\tilde{z}}$ in $\overline{D}$, $r(z)=0$ on L, $b_0=b_1'=0$. Hence later on we only discuss the case of the homogeneous boundary condition. From $v(z)=u(z)-u_0(z)$ in $\overline{D}$, we have $u(z)=v(z)+u_0(z)$ in $\overline{D}$, and $v_y=2\tilde{R}_0(x)$ on $L_0=D_{\hat{z}}\cap\{\hat{y}=0\}$, in which $\tilde{R}_0(x)$ is an undermined real function. The boundary vale problem (1.9), (1.10) is called Problem $\tilde{O}$.

2. Properties of solutions to the oblique derivative problem

In this section, we consider the special mixed equation

$$\begin{gathered}u_{\tilde{z}\overline{\tilde{z}}}=W_{\overline{\tilde{z}}}=0,\quad\text{i.e.,}\\ (U+V)_\mu=0,\quad (U-V)_\nu=0\quad\text{in } \overline{D},\end{gathered}\tag{2.1}$$

where $U(z)=\operatorname{Re}W(z)$, $V(z)=\operatorname{Im}W(z)$.

Theorem 2.1. *Any solution $u(z)$ of Problem O for the hyperbolic equation (2.1) can be expressed as*

$$u(z)=u(x)-2\int_0^{\hat{y}}V(\hat{y})d\hat{y}=2\operatorname{Re}\int_{z_0}^{z}[\frac{\operatorname{Re}W(z)}{H(\hat{y})}-j\operatorname{Im}W(z)]dz+b_0\quad\textit{in } \overline{D},\tag{2.2}$$

where

$$\begin{aligned}W(z)=U+jV&=f(x-Y)e_1+g(x+Y)e_2\\ &=f(\nu)e_1+g(\mu)e_2\\ &=\frac{1}{2}\{f(x-Y)+g(x+Y)+j[f(x-Y)-g(x+Y)]\},\end{aligned}\tag{2.3}$$

in which $Y = G(\hat{y})$*. For convenience denote by the functions* $\lambda_1(x), \lambda_2(x), r(x)$ *of* x *the functions* $\lambda_1(z), \lambda_2(z), r(z)$ *of* z *in* (1.10)*, and* $f(x-Y) = f(\nu)$*,* $g(x+Y) = g(\mu)$ *possess the forms*

$$\begin{gathered}
f(\nu) = f(x-Y) = \frac{2r((x-Y+R_*)/2)}{\lambda_1((x-Y+R_*)/2) - \lambda_2((x-Y+R_*)/2)} \\
- \frac{[\lambda_1((x-Y+R_*)/2) + \lambda_2((x-Y+R_*)/2)]g(R^*)}{\lambda_1((x-Y+R_*)/2) - \lambda_2((x-Y+R_*)/2)}, \\
R_* \le x - Y \le R^*, \\
(\lambda_1(0) + \lambda_2(0))g(R_*) = (\lambda_1(0) + \lambda_2(0))(U(z_1) - V(z_1)) = r(0) - b_1 \quad \textit{or } 0, \\
g(\mu) = g(x+Y) = \\
= \frac{2r((x+Y+R^*)/2) - [\lambda_1((x+Y+R^*)/2) - \lambda_2((x+Y+R^*)/2)]f(R^*)}{\lambda_1((x+Y+R^*)/2) + \lambda_2((x+Y+R^*)/2)}, \\
R_* \le x + Y \le R^*, \\
(\lambda_1(0) - \lambda_2(0))f(R^*) = (\lambda_1(0) - \lambda_2(0))(U(z_1) + V(z_1)) = r(0) + b_1 \quad \textit{or } 0.
\end{gathered} \tag{2.4}$$

Moreover $u(z)$ *satisfies the estimate*

$$C^1_\delta[u(z), \overline{D}] \le M_1, \quad C^1_\delta[u(z), \overline{D}] \le M_2 k_1, \tag{2.5}$$

where $\delta = \delta(\alpha, k_0, k_1, D) < 1$*,* $M_1 = M_1(\alpha, k_0, k_1, D)$*,* $M_2 = M_2(\alpha, k_0, D)$ *are positive constants.*

Proof. Let the general solution

$$W(z) = u_{\tilde{z}} = \frac{1}{2}\{f(x-Y) + g(x+Y) + j[f(x-Y) - g(x+Y)]\}$$

of (2.1) be substituted in the boundary condition (1.10), thus (1.10) can be rewritten as

$$\begin{gathered}
\lambda_1(x)U(z) - \lambda_2(x)V(z) = r(z) \quad \text{on } L, \\
\overline{\lambda(z_1)}W(z_1) = r(z_1) + jb_1;
\end{gathered}$$

i.e.,

$$\begin{gathered}
[\lambda_1(x) - \lambda_2(x)]f(2x - R_*) + [\lambda_1(x) + \lambda_2(x)]g(R_*) = 2r(x) \quad \text{on } L_1, \\
[\lambda_1(x) - \lambda_2(x)]f(R^*) + [\lambda_1(x) + \lambda_2(x)]g(2x - R^*) = 2r(x) \quad \text{on } L_2,
\end{gathered}$$

the above formulas can be rewritten as

$$\begin{aligned}
&\Big[\lambda_1\Big(\frac{t+R_*}{2}\Big) - \lambda_2\Big(\frac{t+R_*}{2}\Big)\Big]f(t) + \Big[\lambda_1\Big(\frac{t+R_*}{2}\Big) + \lambda_2\Big(\frac{t+R_*}{2}\Big)\Big]g(R_*) \\
&= 2r\Big(\frac{t+R_*}{2}\Big), \quad t \in [R_*, R^*], \\
&(\lambda_1(0) + \lambda_2(0))g(R_*) = (\lambda_1(0) + \lambda_2(0))(U(z_1) - V(z_1)) = r(0) - b_1 \quad \text{or} \quad 0, \\
&\Big[\lambda_1\Big(\frac{t+R^*}{2}\Big) - \lambda_2\Big(\frac{t+R^*}{2}\Big)\Big]f(R^*) + \Big[\lambda_1\Big(\frac{t+R^*}{2}) + \lambda_2\Big(\frac{t+R^*}{2}\Big)\Big]g(t) \\
&= 2r(\frac{t+R^*}{2}), \quad t \in [R_*, R^*], \\
&(\lambda_1(0) - \lambda_2(0))f(R^*) = (\lambda_1(0) - \lambda_2(0))(U(z_1) + V(z_1)) = r(0) + b_1 \quad \text{or} \quad 0,
\end{aligned}$$

thus the solution $W(z)$ can be expressed as (2.3). Here we mention that for the oblique derivative boundary condition, by (1.10), we have $(\lambda_1(0)+\lambda_2(0))g(R_*)=0$, $(\lambda_1(0)-\lambda_2(0))f(R^*)=0$. If $(\lambda_1(x)+\lambda_2(x))g(R_*)$ on L_1 and $(\lambda_1(x)-\lambda_2(x))f(R^*)$ on L_2 are known. From the condition (1.6) and the relation (2.2), we see that the estimate (2.5) of the solution $u(z)$ for (2.1), (2.2) is obviously true. □

3. UNIQUENESS OF SOLUTIONS TO THE OBLIQUE DERIVATIVE PROBLEM

The representation of solutions of Problem O for equation (1.1) is as follows.

Theorem 3.1. *Under Condition C, any solution $u(z)$ of Problem O for equation* (1.1) *in D can be expressed as*

$$\begin{gathered}
u(z)=2\operatorname{Re}\int_{z_0}^{z}[\frac{\operatorname{Re}W}{H(\hat{y})}-j\operatorname{Im}W]\mathrm{d}z+b_0,\\
W(z)=w(z)+\Phi(z)+\Psi(z)\quad \text{in } D,\\
w(z)=f(\nu)e_1+g(\mu)e_2, \Phi(z)=\tilde{f}(\nu)e_1+\tilde{g}(\mu)e_2,\\
\Psi(z)=\int_{R_*}^{\mu}g_1(z)e_1\mathrm{d}\mu+\int_{R^*}^{\nu}g_2(z)e_2d\nu,\\
g_l(z)=A_l\xi+B_l\eta+Cu+D,\quad l=1,2.
\end{gathered}\tag{3.1}$$

Here

$$\begin{gathered}
A_1=\frac{1}{4H}[\frac{a}{H}+H_x+\frac{H_y}{H}-b],\quad B_1=\frac{1}{4H}[\frac{a}{H}+H_x+\frac{H_y}{H}+b],\quad C=\frac{c}{4H},\\
A_2=\frac{1}{4H}[\frac{a}{H}+H_x-\frac{H_y}{H}-b],\quad B_2=\frac{1}{4H}[\frac{a}{H}+H_x-\frac{H_y}{H}+b],\quad D=\frac{d}{4H},
\end{gathered}\tag{3.2}$$

where $f(\nu),g(\mu)$ are as stated in (2.4), *and $\tilde{f}(\nu),\tilde{g}(\mu)$ are similar to $f(\nu),g(\mu)$, and $\Phi(z)$ satisfy the boundary condition*

$$\begin{gathered}
\operatorname{Re}[\overline{\lambda(z)}(\Phi(z)+\Psi(z))]=0,\quad z\in L,\\
\operatorname{Im}[\overline{\lambda(z_0)}(\Phi(z_0)+\Psi(z_0))]=0.
\end{gathered}\tag{3.3}$$

Proof. Since Problem O is equivalent to the Problem A for (1.4), from Theorem 2.1 and (1.3), it is not difficult to see that the function $\Psi(z)$ satisfies the complex equation

$$[\Psi]_{\bar{\bar{z}}}=H\{[A_1\xi+B_1\eta+Cu+D]e_1+[A_2\xi+B_2\eta+Cu+D]e_2\}\quad \text{in } D,\tag{3.4}$$

and $\Phi(z)=W(z)-w(z)-\Psi(z)$ satisfies (2.1) and the boundary conditions

$$\begin{gathered}
\operatorname{Re}[\overline{\lambda(z)}\Phi(z)]=-\operatorname{Re}[\overline{\lambda(z)}\Psi(z)]\quad \text{on } L,\\
\operatorname{Im}[\overline{\lambda(z_0)}\Phi(z_0)]=-\operatorname{Im}[\overline{\lambda(z_0)}\Psi(z_0)].
\end{gathered}\tag{3.5}$$

By the representation of solutions of Problem A for (1.4) as stated in the final four formulas of (3.1), we can obtain the representation of solutions of Problem O for (1.1) as stated in the first formula of (3.1). □

Next, we prove the uniqueness of solutions of Problem O for equation (1.1).

Theorem 3.2. *Suppose that* (1.1) *satisfies the Condition C. Then Problem O for* (1.1) *in D has a unique solution.*

Proof. Let $u_1(z), u_2(z)$ be two solutions of Problem O for (1.1). Then $u(z) = u_1(z) - u_2(z)$ is a solution of the homogeneous equation

$$K(\hat{y})u_{xx} + u_{yy} + au_x + bu_y + cu = 0 \quad \text{in } D, \tag{3.6}$$

satisfying the boundary conditions

$$\begin{gathered} u(z) = 0; \quad \text{i.e., } \operatorname{Re}[\overline{\lambda(z)}u_{\tilde{z}}(z)] = 0 \quad \text{on } L, \\ u(z_0) = 0, \quad \operatorname{Im}[\overline{\lambda(z_0)}u_{\tilde{z}}(z_0)] = 0, \end{gathered} \tag{3.7}$$

where the function $W(z) = [H(\hat{y})u_x - ju_y]/2$ is a solution of the homogeneous problem of Problem A; namely $W(z)$ satisfies the homogeneous equation and boundary conditions

$$\begin{gathered} W_{\tilde{z}} = A_1 W + A_2 \overline{W} + A_3 u \quad \text{in } D, \\ u(z) = 2\operatorname{Re}\int_{z_0}^{z} [\frac{\operatorname{Re} W}{H(\hat{y})} - j\operatorname{Im} W]dz, \\ \operatorname{Re}[\overline{\lambda(z)}W(z)] = 0 \quad \text{on } L, \quad \operatorname{Im}[\overline{\lambda(z_0)}W(z_0)] = 0. \end{gathered} \tag{3.8}$$

On the basis of Theorem 3.1, the function $W(z)$ can be expressed in the form

$$\begin{aligned} W(z) &= \Phi(z) + \Psi(z), \\ \Psi(z) &= \int_{R_*}^{\mu} [A_1\xi + B_1\eta + Cu]e_1 d\mu + \int_{R^*}^{\nu} [A_2\xi + B_2\eta + Cu]e_2 d\nu \\ &= \int_{y_1'}^{\hat{y}} 2H(\hat{y})[A_1\xi + B_1\eta + Cu]e_1 dy - \int_{y_1''}^{\hat{y}} 2H(\hat{y})[A_2\xi + B_2\eta + Cu]e_2 dy \end{aligned} \tag{3.9}$$

in D, where $z_1' = x_1' + j\hat{y}_1', z_1'' = x_1'' + j\hat{y}_1''$ are two intersection points of L_1, L_2 and two families of characteristics lines

$$s_1 : \frac{\mathrm{d}x}{\mathrm{d}y} = \sqrt{|K(\hat{y})|} = H(\hat{y}), \quad s_2 : \frac{\mathrm{d}x}{\mathrm{d}y} = -\sqrt{|K(\hat{y})|} = -H(\hat{y}) \tag{3.10}$$

passing through $z = x + \hat{y} \in \overline{D}$ respectively. Suppose $w(z) \not\equiv 0$ in the neighborhood of the point z_1. We may choose a sufficiently small positive number R_0, such that $8M_2MR_0 < 1$, where $M_2 = \max\{C[A_1, Q_0], C[B_1, Q_0], C[A_2, Q_0], C[B_2, Q_0], C[C, Q_0]\}$, $M = 1 + 4k_0^2(1 + 2k_0^2)$ is a positive constant, and $M_0 = C[W(z), \overline{Q_0}] + C[u(z), \overline{Q_0}] > 0$. Herein

$$\|W(z)\| = \hat{C}[W(z), \overline{Q_0}] = C[\operatorname{Re} W(z)/H(\hat{y}) + j\operatorname{Im} W(z), \overline{Q_0}],$$

$Q_0 = \{R_* \le \mu \le R_* + R_0\} \cap \{R^* - R_0 \le \nu \le R^*\}$. From (2.4)–(3.3), (3.9) and Condition C, we have

$$\|\Psi(z)\| \le 8M_2M_0R_0, \quad \|\Phi(z)\| \le 32M_2k_0^2(1 + 2k_0^2)M_0R_0,$$

thus an absurd inequality $M_0 \le 8M_2MM_0R_0 < M_0$ is derived. It shows $W(z) = 0$, $(x, \hat{y}) \in Q_0$. Moreover, we extend along the positive direction of $\mu = x + Y$ and the negative direction of $\nu = x - Y$ successively, and finally obtain $W(z) = 0$ in D. This proves the uniqueness of solutions of Problem A for (3.8), and then $u(z) = u_1(z) - u_2(z) = 0$ in D, this shows that Problem O for (1.1) has a unique solution. □

4. Solvability of the oblique derivative problem

In this section, we prove the existence of solutions of Problem O for (1.1) by the method of the successive approximations.

Theorem 4.1. *If* (1.1) *satisfies Condition C, then Problem O for* (1.1) *has a solution.*

Proof. To find a solution $u(z)$ of Problem O in D, we first find a solution $[W(z), u(z)]$ of Problem A for (1.4) in the closed domain $D_\delta = \overline{D} \cap \{\hat{y} \le -\delta\}$, where δ is a small positive constant. In the following, a solution of Problem A for the equation (1.1) in D_δ can be found by using successive approximations. First of all, substituting the solution $[W_0(z), u_0(z)] = [\xi_0 e_1 + \eta_0 e_2, u_0(z)]$ of Problem A for (1.4) into the position of $W = \xi e_1 + \eta e_2$ on the right-hand side of (3.1), the functions

$$\begin{gathered}
W_1(z) = W_0(z) + \Phi_1(z) + \Psi_1(z),\\
\begin{aligned}
\Psi_1(z) = {}& \int_{R_*}^{\mu} [A_1\xi_0 + B_1\eta_0 + Cu_0 + D]e_1 d\mu\\
&+ \int_{R^*}^{\nu} [A_2\xi_0 + B_2\eta_0 + Cu_0 + D]e_2 d\nu,
\end{aligned}\\
u_1(z) = 2\operatorname{Re}\int_{z_0}^{z} [\frac{\operatorname{Re} W_1}{H(\hat{y})} - j\operatorname{Im} W_1]dz + b_0 \quad \text{in } D_\delta,
\end{gathered} \tag{4.1}$$

are determined, where $\mu = x + Y, \nu = x - Y, \Phi_1(z)$ is a solution of (2.1) in D_δ satisfying the boundary conditions

$$\begin{gathered}
\operatorname{Re}[\overline{\lambda(z)}\Phi_1(z)] = -\operatorname{Re}[\overline{\lambda(z)}\Psi_1(z)] \quad \text{on } L,\\
\operatorname{Im}[\overline{\lambda(z_0)}\Phi_1(z_0)] = -\operatorname{Im}[\overline{\lambda(z_0)}\Psi_1(z_0)].
\end{gathered} \tag{4.2}$$

Thus from (4.1), we have

$$\begin{aligned}
\|W_1(z) - W_0(z)\| &= C[W_1(z) - W_0(z), D_\delta] + C[u_1(z) - u_0(z), D_\delta]\\
&\le 2M_3 M(4M_0 + 1)R',
\end{aligned} \tag{4.3}$$

where $M_3 = \max_{z\in D_\delta}(|A_1|, |B_1|, |A_2|, |B_2|, |C|)$, $M_0 = C[w_0(z), D_\delta] + C[u_0(z), D_\delta]$, $R' = \max(R^*, |R_*|), M = 1 + 4k_0^2(1 + 2k_0^2)$ is a positive constant similar to the one in the proof of Theorem 3.2. Moreover, we substitute $W_1(z) = W_0(z) + \Phi_1(z) + \Psi_1(z)$ and the corresponding functions $\xi_1(z) = \operatorname{Re} W_1(z) + \operatorname{Im} W_1(z)$, $\eta_1(z) = \operatorname{Re} W_1(z) - \operatorname{Im} W_1(z), u_1(z)$ into the positions of $W(z), \xi(z), \eta(z), u(z)$ in (3.1), and similarly to (4.1)–(4.3), we can find the corresponding functions $\Psi_2(z), \Phi_2(z), u_2(z)$ in $\overline{D}$ and the function

$$\begin{gathered}
W_2(z) = W_0(z) + \Phi_2(z) + \Psi_2(z) \quad \text{in } D_\delta,\\
u_2(z) = 2\operatorname{Re}\int_{z_0}^{z} [\frac{\operatorname{Re} W_2}{H(\hat{y})} - j\operatorname{Im} W_2]dz + b_0.
\end{gathered}$$

It is clear that the function $W_2(z) - W_1(z)$ satisfies the equality

$$\begin{aligned}
W_2(z) - W_1(z) = \Phi_2(z) - \Phi_1(z) + \Psi_2(z) - \Psi_1(z) &= \Phi_2(z) - \Phi_1(z)\\
&+ \int_{R_*}^{\mu} [A_1(\xi_1 - \xi_0) + B_1(\eta_1 - \eta_0) + C(u_1 - u_0)]e_1 d\mu\\
&+ \int_{R^*}^{\nu} [A_2(\xi_1 - \xi_0) + B_2(\eta_1 - \eta_0) + C(u_1 - u_0)]e_2 d\nu,
\end{aligned}$$

$$u_2(z)-u_1(z)=2\operatorname{Re}\int_{z_0}^{z}[\frac{\operatorname{Re}W_1}{H(\hat{y})}-j\operatorname{Im}W_1]dz \quad \text{in } D_\delta,$$

and then

$$\|W_2-W_1\|\le[2M_3M(4M_0+1)]^2\int_0^{R'}R'\mathrm{d}R'\le\frac{[2M_3M(4M_0+1)R']^2}{2\,!},$$

where M_3 is a constant as stated in (4.3). Thus we can find a sequence of functions $\{W_n(z)\}$ satisfying

$$\begin{gathered}W_n(z)=W_0(z)+\Phi_n(z)+\Psi_n(z),\\ \Psi_n(z)=\int_{R_*}^{\mu}[A_1\xi_n+B_1\eta_n+Cu_n]e_1d\mu+\int_{R^*}^{\nu}[A_2\xi_n+B_2\eta_n+Cu_n]e_2d\nu,\\ u_n(z)=2\operatorname{Re}\int_{z_0}^{z}[\frac{\operatorname{Re}W_n}{H(\hat{y})}-j\operatorname{Im}W_n]dz+b_0,\end{gathered}\tag{4.4}$$

and $W_n(z)-W_{n-1}(z)$ satisfies

$$\begin{gathered}W_n(z)-W_{n-1}(z)=\Phi_n(z)-\Phi_{n-1}(z)+\Psi_n(z)-\Psi_{n-1}(z),\\ \begin{aligned}&\Phi_n(z)-\Phi_{n-1}(z)\\ &=\int_{R_*}^{\mu}[A_1(\xi_{n-1}-\xi_{n-2})+B_1(\eta_{n-1}-\eta_{n-2})+C[u_{n-1}-u_{n-2})]e_1d\mu\\ &\quad+\int_{R^*}^{\nu}[A_2(\xi_{n-1}-\xi_{n-2})+B_2(\eta_{n-1}-\eta_{n-2})]e_2d\nu,\end{aligned}\\ u_n(z)-u_{n-1}(z)=2\operatorname{Re}\int_{z_0}^{z}[\frac{\operatorname{Re}(W_n-W_{n-1})}{H(\hat{y})}-j\operatorname{Im}(W_n-W_{n-1})]dz \quad \text{in } D_\delta,\end{gathered}\tag{4.5}$$

and then

$$\begin{aligned}\|W_n-W_{n-1}\|&\le[2M_3M(4M_0+1)]^n\int_0^{R'}\frac{R'^{n-1}}{(n-1)!}\mathrm{d}R'\\ &\le\frac{[2M_3M(4M_0+1)R']^n}{n\,!}\quad\text{in } D_\delta.\end{aligned}$$

From the above inequality, we see that the sequences of the functions $\{W_n(z)\}$, $\{u_n(z)\}$; i.e.,

$$\begin{gathered}W_n(z)=W_0(z)+[W_1(z)-W_0(z)]+\dots+[W_n(z)-W_{n-1}(z)],\\ u_n(z)=u_0(z)+[u_1(z)-u_0(z)]+\dots+[u_n(z)-u_{n-1}(z)],\quad n=1,2,\dots\end{gathered}$$

converges uniformly to a function $[W_*(z),u^*(z)]$ and $[W_*(z),u_*(z)]$ satisfies

$$\begin{gathered}W_*(z)=W_0(z)+\Phi_*(z)+\Psi_*(z),\\ \begin{aligned}\Psi_n(z)=&\int_{R_*}^{\mu}[A_1\xi_*+B_1\eta_*+Cu_*+D]e_1d\mu\\ &+\int_{R^*}^{\nu}[A_2\xi_*+B_2\eta_*+Cu_*+D]e_2d\nu,\end{aligned}\\ u_*(z)=u_0(z)+2\operatorname{Re}\int_{z_0}^{z}[\frac{\operatorname{Re}W_*}{H(\hat{y})}-j\operatorname{Im}W_*]dz+b_0\quad\text{in } D_\delta.\end{gathered}\tag{4.6}$$

It is easy to see that $[W_*(z),u_*(z)]$ satisfies (1.4) in D_δ and the boundary condition (1.10), hence $u_*(z)$ is just a solution of Problem O for (1.1) in the domain D_δ.

Finally letting $\delta \to 0$, we can choose a limit function $u(z)$, which is a solution of Problem O for (1.1) in D. $\square$

5. OBLIQUE DERIVATIVE PROBLEM IN GENERAL DOMAINS

Now we consider some general domains with non-characteristic boundary and prove the unique solvability of Problem O for (1.1). Denote by D a simply connected bounded domain D in the hyperbolic complex plane $\mathbb{C}$ with the boundary $\partial D = L_0 \cup L$, where $L_0, L = L_1 \cup L_2$ are as stated in Section 1.

(1) We consider the domain D' with the boundary $L_0 \cup L', L' = L_1' \cup L_2'$, where the parameter equations of the curves L_1', L_2' are as follows:

$$L_1' = \{\hat{y} = -\gamma_1(s), 0 \le s \le s_0\}, \quad L_2' = \{x - G(\hat{y}) = R^*, 0 \le x \le R^*\}. \tag{5.1}$$

Herein $Y = G(\hat{y}) = \int_0^{\hat{y}} \sqrt{K(t)}dt$, s is the parameter of arc length of L_1', $\gamma_1(s)$ on $\{0 \le s \le s_0\}$ is continuously differentiable, $\gamma_1(0) = 0, \gamma_1(s) > 0$ on $\{0 < s \le s_0\}$, and the slope of curve L_1' at a point z^* is not equal to $dy/dx = -1/H(\hat{y})$ of the characteristic curve $s_2 : dy/dx = -1/H(\hat{y})$ at the point, where z^* is an intersection point of L_1' and the characteristic curve of s_2, and $z_0' = x_0' - j\gamma_1(s_0)$ is the intersection point of L_1' and L_2'.

The boundary conditions of the oblique derivative problem (Problem O') for (1.1) in D' are as follows:

$$\begin{gathered} \frac{1}{2}\frac{\partial u}{\partial \nu} = \frac{1}{H(\hat{y})}\operatorname{Re}[\overline{\lambda(z)}u_{\tilde{z}}] = r(z), \quad z \in L' = L_1' \cup L_2', \\ u(z_0') = b_0, \quad \frac{1}{H(\hat{y})}\operatorname{Im}[\overline{\lambda(z)}u_{\tilde{z}}]|_{z=z_0'} = b_1, \end{gathered} \tag{5.2}$$

where $\lambda(z) = \lambda_1(x) + j\lambda_2(x)$, $R(z) = H(\hat{y})r(z)$ on $L', b_1' = H(\hat{y}_0')b_1 = H(\operatorname{Im} z_0')b_1$, and $\lambda(z)$, $r(z)$, b_1' satisfy the conditions

$$\begin{gathered} C^1[\lambda(z), L'] \le k_0, \quad C^1[r(z), L'] \le k_2, \quad |b_0|, |b_1| \le k_2, \\ \max_{z \in L_1'} \frac{1}{|\lambda_1(x) - \lambda_2(x)|} \le k_0, \quad \max_{z \in L_2'} \frac{1}{|\lambda_1(x) + \lambda_2(x)|} \le k_0, \end{gathered} \tag{5.3}$$

in which k_0, k_2 are positive constants.

Set $Y = G(\hat{y}) = \int_0^{\hat{y}} \sqrt{K(t)}dt$. By the conditions in (5.1), the inverse function $x = \sigma(\nu) = (\mu + \nu)/2$ of $\nu = x - G(\hat{y})$ can be found, and then $\mu = 2\sigma(\nu) - \nu$, $R_* \le \nu \le R^*$. We make a transformation

$$\begin{gathered} \tilde{\mu} = R_*[\mu - 2\sigma(\nu) + \nu]/[2\sigma(\nu) - \nu] + R_*, \quad \tilde{\nu} = \nu, \\ 2\sigma(\nu) - \nu \le \mu \le 0, \quad R_* \le \nu \le R^*, \end{gathered} \tag{5.4}$$

where μ, ν are real variables, its inverse transformation is

$$\begin{gathered} \mu = [2\sigma(\nu) - \nu](\tilde{\mu} - R_*)/R_* + 2\sigma(\nu) - \nu, \quad \nu = \tilde{\nu}, \\ R_* \le \tilde{\mu} \le R^*, \quad R_* \le \tilde{\nu} \le R^*. \end{gathered} \tag{5.5}$$

It is not difficult to see that the transformation in (5.5) maps the domain D' onto D, $\tilde{x} = (\tilde{\mu} + \tilde{\nu})/2, \tilde{Y} = (\tilde{\mu} - \tilde{\nu})/2$, and $x = (\mu + \nu)/2$, $Y = (\mu - \nu)/2$. Denote by $\tilde{Z} = \tilde{x} + j\tilde{Y} = f(Z)$, $Z = x + jY = f^{-1}(\tilde{Z})$ the transformation (5.4) and the inverse transformation (5.5) respectively. In this case, the system (1.3) can be rewritten as

$$\begin{gathered} \xi_\mu = A_1\xi + B_1\eta + C_1(\xi + \eta) + Du + E, \quad z \in D', \\ \eta_\nu = A_2\xi + B_2\eta + C_2(\xi + \eta) + Du + E, \quad z \in D'. \end{gathered} \tag{5.6}$$

Suppose that (1.1) in D' satisfies Condition C, through the transformation (5.5), we obtain $\xi_{\tilde{\mu}} = [2\sigma(\nu) - \nu]\xi_\mu / R_*$, $\eta_{\tilde{\nu}} = \eta_\nu$, in D', where $\xi = U + V$, $\eta = U - V$, and then

$$\begin{gathered}\xi_{\tilde{\mu}} = [2\sigma(\nu) - \nu][A_1\xi + B_1\eta + C_1(\xi+\eta) + Du + E]/R_*,\\ \eta_{\tilde{\nu}} = A_2\xi + B_2\eta + C_2(\xi+\eta) + Du + E \quad \text{in } D,\end{gathered} \tag{5.7}$$

and through the transformation (5.5), the boundary condition (5.2) is reduced to

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(f^{-1}(\tilde{Z}))}W(f^{-1}(\tilde{Z}))] = H[\hat{y}(Y)]r(f^{-1}(\tilde{Z})), \quad \tilde{Z} = \tilde{x} + j\tilde{Y} \in L = L_1 \cup L_2,\\ \operatorname{Im}[\overline{\lambda(f^{-1}(\tilde{Z}_0'))}W(f^{-1}(\tilde{Z}_0')] = b_1, \quad u(z_0) = b_0,\end{gathered} \tag{5.8}$$

in which $Z = f^{-1}(\tilde{Z})$, $\tilde{Z}_0' = f(Z_0')$, $Z_0' = x_0' + jG[-\gamma_1(s_0)]$. Therefore, the boundary value problem (5.6), (5.2) (Problem A') is transformed into the boundary value problem (5.7), (5.8); i.e., the corresponding Problem A in D. On the basis of Theorem 4.1, we see that the boundary value problem (5.7)-(5.8) has a unique solution $w(\tilde{Z})$, and

$$u(z) = 2\operatorname{Re}\int_{z_0'}^{z} [\frac{\operatorname{Re} W}{H(\hat{y})} - j \operatorname{Im} W]dz + b_0 \quad \text{in } \begin{pmatrix} D^+ \\ D^- \end{pmatrix} \tag{5.9}$$

is just a solution of Problem O' for (1.1) in D' with the boundary conditions (5.2), where $W = W(\tilde{Z}(z))$.

Theorem 5.1. *If* (1.1) *in D' satisfies Condition C in the domain D' with the boundary $L_0 \cup L_1' \cup L_2'$, where L_1', L_2' are as stated in* (5.1)*, then Problem O' for* (1.1) *with the boundary conditions* (5.2) *has a unique solution $u(z)$.*

(2) Next let the domain D'' be a simply connected domain with the boundary $L_0 \cup$ $L_1'' \cup L_2''$, where L_0 is as stated before and

$$L_1'' = \{\hat{y} = \gamma_1(s), 0 \le s \le s_0\}, \quad L_2'' = \{\hat{y} = \gamma_2(s), 0 \le x \le s_0'\}, \tag{5.10}$$

in which s is the parameter of arc length of L_1'' or L_2'', $\gamma_1(0) = 0$, $\gamma_2(0) = 0$, $\gamma_1(s) > 0$, $0 < s \le s_0$, $\gamma_2(s) > 0$, $0 < x \le s_0'$, and $\gamma_1(s)$ on $0 \le x \le s_0$ and $\gamma_2(s)$ on $0 \le s \le s_0'$ are continuously differentiable, $z_0'' = x_0'' - j\gamma_1(s_0) = x_0'' - j\gamma_2(s_0')$. Denote by two points z_1^*, z_2^* the intersection points of L_1'', L_2'' and the characteristic curves $s_2 : dy/dx = -1/H(\hat{y})$, $s_1 : dy/dx = 1/H(\hat{y})$ respectively, we require that the slopes of curves L_1'', L_2'' at z_1^*, z_2^* are not equal to those at the characteristic curves s_2, s_1 at the corresponding points, hence $\gamma_1(s), \gamma_2(s)$ can be expressed by $\gamma_1[s(\mu)]$ $(R_* \le \mu \le R^*)$, $\gamma_2[s(\nu)]$ $(R_* \le \nu \le R^*)$. We consider the oblique derivative problem (Problem O") for (1.1) in D'' with the boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}u_{\tilde{z}}] = R(z), \quad z \in L'' = L_1'' \cup L_2'',\\ u(z_0'') = b_0, \quad \operatorname{Im}[\overline{\lambda(z)}u_{\tilde{z}}]\,|_{z=z_0''} = b_1,\end{gathered} \tag{5.11}$$

where $\lambda(z) = \lambda_1(x) + j\lambda_2(x)$, $r(z)$ satisfy the corresponding conditions

$$\begin{gathered}C^1[\lambda(z), L''] \le k_0, \quad C^1[r(z), L''] \le k_2, \quad |b_0|, |b_1| \le k_2,\\ \max_{z\in L_1''} \frac{1}{|\lambda_1(x) - \lambda_2(x)|}, \quad \max_{z\in L_2''} \frac{1}{|\lambda_1(x) + \lambda_2(x)|} \le k_0,\end{gathered} \tag{5.12}$$

in which k_0, k_2 are positive constants. By the conditions in (5.10), the inverse function $x = (\mu+\nu)/2 = \tau(\mu)$, $x = (\mu+\nu)/2 = \sigma(\nu)$ of $\mu = x + G(y), \nu = x - G(y)$

can be found, namely

$$\mu = 2\sigma(\nu) - \nu, \quad \nu = 2\tau(\mu) - \mu, \quad R_* \le \mu \le R^*, \quad R_* \le \nu \le R^*. \tag{5.13}$$

We make the transformation

$$\begin{gathered} \tilde{\mu} = \mu, \quad \tilde{\nu} = R^*[\nu - 2\tau(\mu) + \mu]/[2\tau(\mu) - \mu] + R^*, \\ R_* \le \mu \le R^*, \quad 0 \le \nu \le 2\tau(\mu) - \mu. \end{gathered} \tag{5.14}$$

It is clear that its inverse transformation is

$$\begin{gathered} \mu = \tilde{\mu}, \quad \nu = \frac{[\tilde{\nu} - R^*][2\tau(\mu) - \mu]}{R^*} + 2\tau(\mu) - \mu, \\ R_* \le \tilde{\mu} \le R^*, \quad R_* \le \tilde{\nu} \le R^*. \end{gathered} \tag{5.15}$$

Hence $\tilde{x} = (\tilde{\mu} + \tilde{\nu})/2$, $\tilde{Y} = (\tilde{\mu} - \tilde{\nu})/2$, $x = (\mu + \nu)/2$, $Y = (\mu - \nu)/2$. Denote by $\tilde{Z} = \tilde{x} + j\tilde{Y} = g(z)$, $Z = x + jY = g^{-1}(\tilde{Z})$ the transformation (5.14) and its inverse transformation in (5.15) respectively. Through the transformation (5.15), we obtain $(u+v)_{\tilde{\mu}} = (u+v)_{\mu}$, $(u-v)_{\tilde{\nu}} = [2\tau(\mu) - \mu](u-v)_{\nu}/R^*$ in D''. Thus the system (5.6) in D'' is reduced to

$$\begin{gathered} \xi_{\tilde{\mu}} = A_1\xi + B_1\eta + C_1(\xi + \eta) + Du + E \quad \text{in } D', \\ \eta_{\tilde{\nu}} = [2\tau(\mu) - \mu][A_2\xi + B_2\eta + C_2(\xi + \eta) + Du + E]/R^* \quad \text{in } D'. \end{gathered} \tag{5.16}$$

Moreover, through the transformation (5.15), the boundary condition (5.11) on L_1'', L_2'' is reduced to

$$\begin{gathered} \operatorname{Re}[\overline{\lambda(g^{-1}(\tilde{Z}))}W(g^{-1}(\tilde{Z}))] = H_1[\hat{y}(Y)]r[g^{-1}(\tilde{Z})], \quad z = x + jy \in L_1' \cup L_2', \\ \operatorname{Im}[\overline{\lambda(g^{-1}(Z_0'))}W(g^{-1}(Z_0')] = b_1', \quad u(z_0') = b_0, \end{gathered} \tag{5.17}$$

in which $Z = g^{-1}(\tilde{Z})$, $\tilde{Z}_0' = g(Z_0'')$, $Z_0'' = l_0' + jG[-\gamma_2(s_0')]$. Therefore the boundary-value problem (5.6), (5.11) in D'' is transformed into the boundary-value problem (5.16), (5.17), where we require that the boundaries L_1', L_2' satisfy the similar conditions in (5.1). According to the method in the proof of Theorem 5.1, we can see that the boundary-value problem (5.6), (5.11) has a unique solution $u(\tilde{Z})$, and then the corresponding $u = u(z)$ is a solution of the oblique derivative problem (Problem O") of (1.1).

Theorem 5.2. *If* (1.1) *satisfies Condition C in the domain* D'' *bounded by the boundary* $L_0 \cup L_1'' \cup L_2''$, *where* L_1'', L_2'' *are as stated in* (5.10), *then Problem O" for* (1.1) *in* D'' *with the boundary condition* (5.11) *on* L'' *has a unique solution* $u(z)$.

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