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EXISTENCE OF ENTIRE SOLUTIONS FOR SEMILINEAR ELLIPTIC SYSTEMS UNDER THE KELLER-OSSERMAN CONDITION

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Abstract. Under the Keller-Osserman condition on $f+g$, we show the existence and nonexistence of entire solutions for the semilinear elliptic system $\Delta u = p(x)f(v), \quad \Delta v = q(x)g(u), \quad x \in \mathbb{R}^N$, where $p, q : \mathbb{R}^N \to [0,\infty)$ are continuous functions.

1. Introduction

The purpose of this paper is to investigate the existence and nonexistence of entire solutions to the semilinear elliptic system

$$\begin{gathered} \Delta u = p(x)f(v), \quad x \in \mathbb{R}^N \ (N \geq 3), \\ \Delta v = q(x)g(u), \quad x \in \mathbb{R}^N. \end{gathered} \tag{1.1}$$

By an entire large solution (u,v), we mean a pair of functions $u, v \in C^2(\mathbb{R}^N)$ that satisfies (1.1) and

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = +\infty. \tag{1.2}$$

In this article, we assume that p, q, f and g satisfy the following hypotheses:

- (H1) $p, q : \mathbb{R}^N \to [0,\infty)$ and $f, g : [0,\infty) \to [0,\infty)$ are continuous and nontrivial;
- (H2) f and g are nondecreasing on $[0,\infty)$ and $f(t) > 0$, $g(t) > 0$ for all $t > 0$;
- (H3) $H(\infty) := \lim_{r\to\infty} H(r) = \infty$,

where

$$H(r) := \int_a^r \frac{dt}{\sqrt{2(F(t)+G(t))}}, \quad r \geq a > 0, \tag{1.3}$$

$$F(t) := \int_0^t f(s)ds, \quad G(t) := \int_0^t g(s)ds. \tag{1.4}$$

We see that

$$H'(r) = \frac{1}{\sqrt{2(F(r)+G(r))}} > 0, \quad \forall r > a$$

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and H has the inverse function H^{-1} on $[a,\infty)$. Denote

$$\begin{aligned}\phi_1(r) &:= \max_{|x|=r} p(x), \quad \phi_2(r) := \min_{|x|=r} p(x),\\ \psi_1(r) &:= \max_{|x|=r} q(x), \quad \psi_2(r) := \min_{|x|=r} q(x).\end{aligned} \tag{1.5}$$

First we review the single elliptic equation

$$\Delta u = p(x)f(u), \quad x \in \mathbb{R}^N. \tag{1.6}$$

For $p \equiv 1$ on $\mathbb{R}^N$ and f satisfying (H1) and (H2), Keller-Osserman [8, 15] first supplied the necessary and sufficient condition

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} = \infty \tag{1.7}$$

for the existence of entire radial large solutions to (1.6). For the weight $p(x) = p(|x|)$ and $f(u) = u^\alpha$ with $\alpha \in (0,1]$, Lair and Wood [10] proved that (1.6) has a non-negation entire radial large solution if and only if

$$\int_0^\infty rp(r)dr = \infty. \tag{1.8}$$

Recently, Lair [11] obtained the following results.

Lemma 1.1. *Let f and b satisfy* (H1) *and* (H2) *with $f(0) = 0$. Suppose*

(i) (1.7) *holds;*
(ii) *there exists a positive constant ε such that $\int_0^\infty r^{1+\varepsilon}\phi_1(r)dr < \infty$,*
(iii) *$r^{2N-2}\phi_1(r)$ is nondecreasing near ∞.*

Then (1.6) *has one nonnegative nontrivial entire bounded solution. If, on the other hand, p satisfies*

$$\int_0^\infty r\phi_2(r)dr = \infty$$

and (iii) holds, then (1.6) *has no nonnegative nontrivial entire bounded solution.*

Lemma 1.2. *Let f and b satisfy* (H1) *and* (H2) *with $f(0) = 0$ and $p(x) = p(|x|)$. Suppose* (1.7) *holds. Then* (1.6) *has one nonnegative nontrivial entire solution. Suppose further that (iii) and* (1.8) *hold, then any nonnegative nontrivial entire solution of* (1.6) *is large. Conversely, if* (1.6) *has a nonnegative nontrivial entire large solution, then p satisfies*

$$\int_0^\infty r^{1+\varepsilon}\phi_1(r)dr = \infty, \quad \forall \varepsilon > 0.$$

For more works, see for example [1, 2, 4, 9, 10, 11, 18, 20, 21, 22] and the references therein.

Now let us return to (1.1).

When $p(x) = p(|x|)$, $q(x) = q(|x|)$, $f(v) = v^\alpha$, $g(u) = u^\gamma$, and $0 < \alpha \le \gamma$, Lair and Wood [12] considered the existence and nonexistence of entire positive radial solutions to system (1.1). Moreover, when $0 < \alpha \le 1$ and $0 \le \gamma \le 1$, Lair [13] showed that (1.1) has a nonnegative entire radial large solution if and only if p and

q satisfy both of the following conditions

$$\int_0^\infty tp(t)\Big(t^{2-N}\int_0^t s^{N-3}Q_1(s)ds\Big)^\alpha dt=\infty, \tag{1.9}$$

$$\int_0^\infty tq(t)\Big(t^{2-N}\int_0^t s^{N-3}P_1(s)ds\Big)^\gamma dt=\infty, \tag{1.10}$$

where

$$P_1(r)=\int_0^r \tau p(\tau)d\tau,\quad Q_1(r)=\int_0^r \tau q(\tau)d\tau.$$

Ghanmi, Mâagli, Rădulescu and Zeddini [5] generalized the results in [12] to the case when f and g are satisfy the condition that: For all $c>0$, there exists $L_c>0$ such that for all $s_1,s_2\in[c,\infty)$,

$$|f(s_2)-f(s_1)|+|g(s_2)-g(s_1)|\le L_c|s_2-s_1|. \tag{1.11}$$

Recently, the authors in [14] showed the existence of entire positive radial large solutions for (1.1) under the condition

$$\int_1^\infty \frac{ds}{f(s)+g(s)}=\infty. \tag{1.12}$$

For related works, see [3, 4, 5, 16, 19, 21, 22, 23] and the references therein.

In this paper, we extend some of the existence results for entire positive solutions in Keller [8], Osserman [15] and Lair [11] to (1.1). Our main results are as the following.

Theorem 1.3. *Under the hypotheses* (H1)–(H3)*. Suppose that*

(H4) $r^{2N-2}\big(\phi_1(r)+\psi_1(r)\big)$ *is nondecreasing for large* r*;*
(H5) *there exists a positive constant* ε *such that*

$$\int_0^\infty r^{1+\varepsilon}\big(\phi_1(r)+\psi_1(r)\big)dr<\infty,$$

then (1.1) *has a positive entire bounded solution* (u,v)*.*

From Theorem 1.3, we have the following corollaries for the spherically symmetric case $p(x)=p(|x|)$ and $q(x)=q(|x|)$.

Corollary 1.4. *Under hypotheses* (H1)–(H3), (1.1) *has one positive solution* (u,v)*. Suppose furthermore that*

(H6) $P(\infty)=Q(\infty)=\infty$*, where*

$$P(\infty):=\lim_{r\to\infty}P(r),\quad P(r):=\int_0^r t^{1-N}\Big(\int_0^t s^{N-1}p(s)ds\Big)dt,\quad r\ge 0,$$

$$Q(\infty):=\lim_{r\to\infty}Q(r),\quad Q(r):=\int_0^r t^{1-N}\Big(\int_0^t s^{N-1}q(s)ds\Big)dt,\quad r\ge 0.$$

Then every positive radial entire solution (u,v) *of* (1.1) *is large and satisfies*

$$u(r)\ge u(0)+f(v(0))P(r),\quad v(r)\ge v(0)+g(u(0))Q(r),\quad \forall r\ge 0.$$

Corollary 1.5. *Assume* (H1)–(H4)*. If* (1.1) *has a non-negative radial entire large solution, then*

$$\int_0^\infty r^{1+\varepsilon}\big(p(r)+q(r)\big)dr=\infty,\quad \forall\varepsilon>0. \tag{1.13}$$

Corollary 1.6. *Under hypotheses* (H1)–(H3)*,* (1.1) *has no radial entire large solutions if $p+q$ satisfies one of the following conditions:*

(i) $p(r)+q(r) \le Cr^{2-2N}$ *for large r;*

(ii) $r^{2N-2}\big(p(r)+q(r)\big)$ *is nondecreasing near ∞ and*
$$\int_0^\infty \sqrt{p(r)+q(r)}dr < \infty;$$

(iii) $\int_0^\infty \sqrt{\Lambda(r)}dr < \infty$*, where*
$$\Lambda(r) = \max_{t\in[0,r]} \big(p(t)+q(t)\big), \quad r \ge 0. \tag{1.14}$$

Theorem 1.7. *Under hypotheses* (H1)–(H3)*,* (1.1) *has no radial entire large solutions if $p+q$ satisifes*
$$0 < \liminf_{r\to\infty} \frac{p(r)+q(r)}{r^\beta} \le \limsup_{r\to\infty} \frac{p(r)+q(r)}{r^\beta} < \infty, \quad \beta < -2. \tag{1.15}$$

Remark 1.8. By (H1) and (H2), we see that (H3) implies
$$\int_a^\infty \frac{ds}{\sqrt{F(s)}} = \int_a^\infty \frac{ds}{\sqrt{G(s)}} = \infty.$$

Remark 1.9. By [10], we see that $P(\infty)=\infty$ if and only if $\int_0^\infty rp(r)dr = \infty$.

Remark 1.10. By [9], we see that if $\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty$, then $\int_1^\infty \frac{dt}{f(t)} < \infty$. In other words, if $\int_1^\infty \frac{dt}{f(t)} = \infty$, then $\int_1^\infty \frac{dt}{\sqrt{F(t)}} = \infty$. Conversely, if $\int_1^\infty \frac{dt}{\sqrt{F(t)}} = \infty$, then $\int_1^\infty \frac{dt}{f(t)} = \infty$ does not hold. For example,
$$f(t) = 2(1+t)(\ln(t+1))^{2\sigma-1}\big(\ln(t+1)+\sigma\big), \quad F(t) = (t+1)^2\big(\ln(t+1)\big)^{2\sigma},$$
where $\sigma > 0$. We can see that $\int_1^\infty \frac{dt}{f(t)} = \infty$ if and only if $\sigma \in (0,1/2]$ and $\int_1^\infty \frac{dt}{\sqrt{F(t)}} = \infty$ if and only if $\sigma \in (0,1]$.

2. Proof of main theorems

Proof of Theorem 1.3. Suppose (H4) holds. We will show that (1.1) has a solution by finding a supersolution, $(\bar u, \bar v)$ and a subsolution, $(\underline{u}, \underline{v})$, for which $\underline{u} \le \bar u$ and $\underline{v} \le \bar v$. To do this, we first prove the existence of $(\underline{u}, \underline{v})$ to (1.1) by considering the system of the integral equations
$$\begin{gathered} \underline{u}(r) = \beta + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1}\phi_1(s)f(\underline{v}(s))ds\Big)dt, \quad r \ge 0, \\ \underline{v}(r) = \beta + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1}\psi_1(s)g(\underline{u}(s))ds\Big)dt, \quad r \ge 0, \end{gathered} \tag{2.1}$$
where $\beta \ge a > 0$, a is in (1.3). Let $\{\underline{v}_m\}_{m\ge0}$ and $\{\underline{u}_m\}_{m\ge1}$ be the sequences of positive continuous functions defined on $[0,\infty)$ by
$$\begin{gathered} \underline{v}_0(r) = \beta, \\ \underline{u}_m(r) = \beta + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1}\phi_1(s)f(\underline{v}_{m-1}(s))ds\Big)dt, \quad r \ge 0, \\ \underline{v}_m(t) = \beta + \int_0^r t^{1-N}\Big(\int_0^t s^{N-1}\psi_1(s)g(\underline{u}_m(s))ds\Big)dt, \quad r \ge 0. \end{gathered} \tag{2.2}$$

Obviously, for all $r \geq 0$ and $m \in \mathbb{N}$, $\underline{u}_m(r) \geq \beta$, $\underline{v}_m(r) \geq \beta$ and $\underline{v}_0 \leq \underline{v}_1$. $(\mathbf{H_2})$ yields $u_1(r) \leq u_2(r)$ for all $r \geq 0$, then $\underline{v}_1(r) \leq \underline{v}_2(r)$ for all $r \geq 0$. By the same argument, we obtain that the sequences $\{\underline{u}_m(r)\}$ and $\{\underline{v}_m(r)\}$ are increasing with respect to m for $r \in [0, \infty)$. Moreover, for each $r > 0$,

$$\underline{u}_m'(r) = r^{1-N}\Big(\int_0^r s^{N-1}\phi_1(s) f(\underline{v}_{m-1}(s))ds\Big) \geq 0,$$
$$\underline{v}_m'(r) = r^{1-N}\Big(\int_0^r s^{N-1}\psi_1(s) g(\underline{u}_m(s))ds\Big) \geq 0$$

and

$$\begin{aligned}
&\Big(r^{N-1}\big(\underline{u}_m(r) + \underline{v}_m(r)\big)'\Big)' \\
&= r^{N-1}\big(\phi_1(r) f(\underline{v}_{m-1}(r)) + \psi_1(r) g(\underline{u}_m(r))\big) \\
&\leq r^{N-1}\big(\phi_1(r) + \psi_1(r)\big)\Big(f(\underline{v}_m(r) + \underline{u}_m(r)) + g(\underline{v}_m(r) + \underline{u}_m(r))\Big).
\end{aligned}$$

Let

$$\Lambda(r) = \max_{t \in [0,r]} \big(\phi_1(t) + \psi_1(t)\big), \quad r \geq 0.$$

Multiplying this by $2r^{N-1}\big(\underline{u}_m(r) + \underline{v}_m(r)\big)'$ and integrate on $[0, r]$, we obtain

$$\begin{aligned}
&\Big(r^{N-1}\big(\underline{u}_m(r) + \underline{v}_m(r)\big)'\Big)^2 \\
&\leq 2\int_0^r t^{2(N-1)}\big(\phi_1(t) + \psi_1(t)\big) \\
&\Big(f(\underline{v}_m(t) + \underline{u}_m(t)) + g(\underline{v}_m(t) + \underline{u}_m(t))\Big)\big(\underline{u}_m(t) + \underline{v}_m(t)\big)' dt \\
&\leq 2r^{2(N-1)}\Lambda(r)\int_{2\beta}^{\underline{u}_m(r) + \underline{v}_m(r)} \big(f(\sigma) + g(\sigma)\big)d\sigma \\
&\leq 2r^{2(N-1)}\Lambda(r)\big(F(\underline{u}_m(r) + \underline{v}_m(r)) + G(\underline{u}_m(r) + \underline{v}_m(r))\big),
\end{aligned}$$

and

$$\big(\underline{u}_m(r) + \underline{v}_m(r)\big)' \leq \sqrt{2\Lambda(r)}\Big(\big(F(\underline{v}_m(r) + \underline{u}_m(r)) + G(\underline{v}_m(r) + \underline{u}_m(r))\big)\Big)^{1/2}. \tag{2.3}$$

Thus

$$\begin{aligned}
&\int_0^r \frac{\underline{u}_m'(t) + \underline{v}_m'(t)}{\sqrt{2}\big(F(\underline{u}_m(t) + \underline{v}_m(t)) + G(\underline{u}_m(t) + \underline{v}_m(t))\big)^{1/2}} dt \\
&= \int_{2\beta}^{\underline{u}_m(r) + \underline{v}_m(r)} \frac{d\tau}{\sqrt{2(F(\tau) + G(\tau))}} \\
&= H(\underline{u}_m(r) + \underline{v}_m(r)) - H(2\beta) \leq \int_0^r \sqrt{M(t)}dt.
\end{aligned}$$

Since H^{-1} is increasing on $[0, \infty)$, we have

$$\underline{u}_m(r) + \underline{v}_m(r) \leq H^{-1}\Big(H(2\beta) + \int_0^r \sqrt{M(t)}dt\Big), \quad \forall r \geq 0. \tag{2.4}$$

It follows by (H3) and (2.2) that the sequences $\{\underline{u}_m\}$ and $\{\underline{v}_m\}$ are bounded and equi-continuous on $[0, c_0]$ for arbitrary $c_0 > 0$. By Arzela-Ascoli theorem, $\{\underline{u}_m\}$

and $\{\underline{v}_m\}$ have subsequences converging uniformly to $\underline{u}$ and $\underline{v}$ on $[0, c_0]$. By the arbitrariness of $c_0 > 0$, we see that $(\underline{u}, \underline{v})$ is a positive entire solution of

$$\begin{aligned} \Delta \underline{u} &= \phi_1(r) f(\underline{v}) \geq p(x) f(\underline{v}), \quad x \in \mathbb{R}^N, \\ \Delta \underline{v} &= \psi_1(r) g(\underline{u}) \geq q(x) g(\underline{u}), \quad x \in \mathbb{R}^N; \end{aligned} \tag{2.5}$$

i.e., $(\underline{u}, \underline{v})$ is a positive entire subsolution of (1.1).

Next we prove that $(\underline{u}, \underline{v})$ is bounded. Since $(\underline{u}, \underline{v})$ satisfies

$$\big(r^{N-1} \underline{u}'(r)\big)' = r^{N-1} \phi_1(r) f(\underline{v}), \tag{2.6}$$

$$\big(r^{N-1} \underline{v}'(r)\big)' = r^{N-1} \psi_1(r) g(\underline{u}). \tag{2.7}$$

Choose $R > 0$ so that $r^{2N-2}\big(\phi_1(r) + \psi_1(r)\big)$ is nondecreasing on $[R, \infty)$ and

$$\underline{u}(r) > 0, \quad \underline{v}(r) > 0, \quad \forall r \geq R.$$

Now, since $\underline{u}'(r) \geq 0$ and $\underline{v}'(r) \geq 0$ for $r \geq 0$, and (H2) holds, multiplying (2.6) and (2.7) by $r^{N-1}\underline{u}'(r)$ and $r^{N-1}\underline{v}'(r)$, respectively, and integrating from 0 to r, we have

$$\begin{aligned} \big(r^{N-1} \underline{u}'(r)\big)^2 &\leq \big(R^{N-1} \underline{u}'(R)\big)^2 + 2\Big(\int_R^r t^{2(N-1)} p(t) f(\underline{v}(t)) \underline{u}'(t) dt\Big) \\ &\leq C + 2r^{2(N-1)}\big(\phi_1(r) + \psi_1(r)\big)\Big(\int_R^r \frac{d}{dt} F(\underline{v}(t) + \underline{u}(t)) dt\Big) \\ &\leq C + 2r^{2(N-1)}\big(\phi_1(r) + \psi_1(r)\big) F(\underline{v}(r) + \underline{u}(r)), \end{aligned}$$

and

$$\big(r^{N-1} \underline{v}'(r)\big)^2 \leq C + 2r^{2(N-1)}\big(\phi_1(r) + \psi_1(r)\big) G(\underline{v}(r) + \underline{u}(r)),$$

for $r > R$, where $C = \big(R^{N-1}\big(\underline{u}'(R) + \underline{v}'(R)\big)\big)^2$, which yields

$$\begin{aligned} &\underline{u}'(r) + \underline{v}'(r) \\ &\leq \sqrt{2C} r^{-(N-1)} + \sqrt{2(\phi_1(r) + \psi_1(r))}\big(G(\underline{u}(r) + \underline{v}(r)) + F(\underline{v}(r) + \underline{u}(r))\big)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dr} \int_{\underline{u}(R)+\underline{v}(R)}^{\underline{u}(r)+\underline{v}(r)} \frac{d\tau}{\sqrt{2\big(F(\tau) + G(\tau)\big)}} \\ &\leq \sqrt{C} r^{1-N}\big(G(\underline{u}(r) + \underline{v}(r)) + F(\underline{v}(r) + \underline{u}(r))\big)^{-1/2} + \sqrt{\phi_1(r) + \psi_1(r)}. \end{aligned}$$

Integrating the above inequality and using the facts that

$$G(\underline{u}(r) + \underline{v}(r)) + F(\underline{v}(r) + \underline{u}(r)) \geq G(\underline{u}(R) + \underline{v}(R)) + F(\underline{v}(R) + \underline{u}(R)) = C_1,$$

for all $r \geq R$, and

$$\sqrt{\phi_1(r) + \psi_1(r)} \leq \sqrt{2r^{1+\varepsilon}\big(\phi_1(r) + \psi_1(r)\big) r^{-1-\varepsilon}} \leq r^{1+\varepsilon}\big(\phi_1(r) + \psi_1(r)\big) + r^{-(1+\varepsilon)}$$

for $\varepsilon > 0$, we have

$$\begin{aligned} H(\underline{u}(r) + \underline{v}(r)) \leq H(\underline{u}(R) + \underline{v}(R)) &+ \int_R^r s^{1+\varepsilon}\big(\phi_1(s) + \psi_1(s)\big) ds + (\varepsilon R^{\varepsilon})^{-1} \\ &+ \sqrt{CC_1^{-1}} (NR^N)^{-1}. \end{aligned}$$

Letting $r \to \infty$, we find that $(\underline{u}, \underline{v})$ is bounded since $\phi_1 + \psi_1$ satisfies (H5) and $f + g$ satisfies (H3). Thus, Since $(\underline{u}, \underline{v})$ is nondecreasing, we have

$$\lim_{r\to\infty} \underline{u}(r) = M_1 > 0, \qquad \lim_{r\to\infty} \underline{v}(r) = M_2 > 0.$$

In the same way, we can see that the system

$$\begin{gathered}
\bar{u}(0) = \bar{v}(0) = \max\{M_1, M_2\}, \quad \bar{u}'(r) = \bar{v}'(r) = 0, \\
\Delta \bar{u}(x) = \bar{u}''(r) + \frac{N-1}{r}\bar{u}'(r) = \phi_2(r) f(\bar{v}(r)), \quad r > 0, \\
\Delta \bar{v}(x) = \bar{v}''(r) + \frac{N-1}{r}\bar{v}'(r) = \psi_2(r) g(\bar{u}(r)), \quad r > 0
\end{gathered} \tag{2.8}$$

has a bounded solution $(\bar{u}, \bar{v})$ which is a supersolution for (1.1). It is also clear that

$$\bar{u}(r) \ge M_1 \ge \underline{u}(r), \quad \bar{v}(r) \ge M_2 \ge \underline{v}(r), \quad \forall r \ge 0.$$

Hence the standard super-sub solution principle (see [17, 7]) implies that (1.1) has a bounded solution (u, v) such that $\underline{u}(x) \le u(x) \le \bar{u}(x)$ and $\underline{v}(x) \le v(x) \le \bar{v}(x)$ on $\mathbb{R}^N$. This completes the proof. $\square$

Proof of Theorem 1.7. We follow the arguments in ([6, Theorem 4.3] and [22, Theorem 3.4]) for studying the nonexistence of entire radial large solutions to (1.6). Let

$$a(r) = r^\theta \int_r^\infty t\big(p(t) + q(t)\big)dt, \quad r \ge 0. \tag{2.9}$$

By (1.15), there exist $R_0 > 0, C_2 > C_1 > 0$ such that

$$C_1 r^\beta \le p(r) + q(r) \le C_2 r^\beta, \quad r \ge R_0,$$

so

$$\begin{aligned}
a'(r) &= \theta r^{\theta-1} \int_r^\infty t\big(p(t) + q(t)\big)dt - r^{\theta+1}\big(p(r) + q(r)\big) \\
&= -r^{\beta+\theta+1}\Big(C_1 - \frac{C_2\theta}{-\beta-2}\Big) < 0
\end{aligned}$$

provided $\theta \in \big(0, C_1 C_2^{-1}(-\beta-2)\big)$; i.e., a is decreasing in $[R_0, \infty)$. Define

$$b(r) = \int_r^\infty t\big(p(t) + q(t)\big)dt, \quad r \ge 0. \tag{2.10}$$

Now suppose that (1.1) has a radial entire large solution (u, v) with $u(r) > 0$ and $v(r) > 0$ for all $r \ge R$, then for $r \ge R_0$

$$\begin{aligned}
u(r) + v(r) &= u(0) + v(0) + \frac{1}{N-2}\int_0^r \Big(1 - \big(\frac{\tau}{r}\big)^{N-2}\Big)\tau\big(p(\tau) f(v(\tau)) \\
&\quad + q(\tau) g(u(\tau))\big)d\tau, \\
&\le u(0) + v(0) + \frac{1}{N-2}\int_0^r \Big(1 - \big(\frac{\tau}{r}\big)^{N-2}\Big)\tau\big(p(\tau) + q(\tau)\big) \\
&\quad \times \big(f(v(\tau) + u(\tau)) + g(u(\tau) + v(\tau))\big)d\tau \\
&= C + \frac{C}{N-2}\int_{R_0}^r \Big(1 - \big(\frac{\tau}{r}\big)^{N-2}\Big)\tau\big(p(\tau) + q(\tau)\big) \\
&\quad \times \big(f(v(\tau) + u(\tau)) + g(u(\tau) + v(\tau))\big)d\tau.
\end{aligned}$$

Let $\tau = b^{-1}(s)$, $w = (u+v)\circ b^{-1}$. By the monotonicity of b and $a = r^\theta b$ in $[R_0, \infty)$, $t\big(b^{-1}(t)\big)^\theta$ is increasing in $(0, t_0]$, where $t_0 = b(R_0)$, and

$$1 - r^\alpha \le C_\alpha(1-r), \quad \forall r \in [0,1] \text{ and and fixed } \alpha > 0, \tag{2.11}$$

we obtain, for $t \in (0, t_0]$,

$$\begin{aligned}
w(t) &= C + \frac{1}{N-2}\int_t^{t_0} \Big(1 - \big(\frac{b^{-1}(s)}{b^{-1}(t)}\big)^{N-2}\Big)\big(f(w(s)) + g(w(s))\big)ds \\
&\le C + \frac{1}{N-2}\int_t^{t_0} \Big(1 - \big(\frac{t}{s}\big)^{(N-2)/\theta}\Big)\big(f(w(s)) + g(w(s))\big)ds \\
&\le C + \frac{1}{N-2}\int_t^{t_0} \Big(1 - \frac{t}{s}\Big)\big(f(w(s)) + g(w(s))\big)ds = z(t).
\end{aligned}$$

It is easy to see that $z'(t) \le 0$ for $t \in (0, t_0]$ and

$$z''(t) = \frac{C\big(f(w(t)) + g(w(t))\big)}{t} \le \frac{C\big(f(z(t)) + g(z(t))\big)}{t},$$

which yields

$$\begin{aligned}
z'^2(t_0) - z'^2(t) = 2\int_t^{t_0} z''(s)z'(s)ds \\
&\ge 2C\int_t^{t_0} \frac{\big(f(z(s)) + g(z(s))\big)z'(s)}{s}ds \\
&\ge \frac{2C}{t}\int_t^{t_0} \big(f(z(s)) + g(z(s))\big)z'(s)ds \\
&= \frac{2C}{t}\big(F(z(t_0)) + G(z(t_0)) - F(z(t)) - G(z(t))\big).
\end{aligned}$$

Since $\lim_{t\to 0} w(t) = \infty$, so is $F(z(t)) + G(z(t))$. We obtain, for $0 < t < t_1$ small enough,

$$z'^2(t) \le \frac{C\big(F(z(t)) + G(z(t))\big)}{t},$$

and

$$-\frac{C}{\sqrt{t}} \le \frac{z'(t)}{\sqrt{F(z(t)) + G(z(t))}} \le 0.$$

Integrating from t to t_1 and letting $t \to 0$, we obtain

$$\int_{z(t_1)}^\infty \frac{d\sigma}{\sqrt{F(\sigma) + G(\sigma)}} \le C\int_0^{t_1} \frac{dt}{\sqrt{t}} = 2C\sqrt{t_1} < \infty.$$

This is a contradiction. The proof is completed. $\square$

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