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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR DIVERGENCE TYPE ELLIPTIC EQUATIONS

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Abstract. We establish the existence and multiplicity of weak solutions of a problem involving a uniformly convex elliptic operator in divergence form. We find one nontrivial solution by the mountain pass lemma, when the nonlinearity has a $(p-1)$-superlinear growth at infinity, and two nontrivial solutions by minimization and mountain pass when the nonlinear term has a $(p-1)$-sublinear growth at infinity.

1. Introduction

In this article we study the boundary-value problem

$$-\operatorname{div}(a(x,\nabla u)) + |u|^{p-2}u = \lambda f(x,u), \quad x \in \Omega, \tag{1.1}$$

$$u(x) = \text{constant}, \quad x \in \partial\Omega, \tag{1.2}$$

$$\int_{\partial\Omega} a(x,\nabla u)\cdot n\,ds = 0, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary. We obtain the existence and multiplicity for the equation

$$-\operatorname{div}(a(x,\nabla u)) = f(x,u). \tag{1.4}$$

Such operators arise, for example, from the expression of the p-Laplacian in curvilinear coordinates. We refer to the books [5, 11, 13] for the foundation of the variational methods and refer to the overview papers [1, 2, 4, 6, 7, 8, 9, 10, 12] for the advances and references of this area. Recently, the Dirichlet problem (1.4) was studied and obtained one weak solution by the mountain pass lemma in [8], when the potential satisfies a set of assumptions and f is $(p-1)$-superlinear at infinity. Duc and Vu [2] extended the result of [8], considering the Dirichlet problem (1.4) in the nonuniform case. Kristály, Lisei and Varga [4] study the Dirichlet problem (1.4), and obtain three solutions when f is $(p-1)$-sublinear at infinity. Yang, Geng and Yan [12] deal with the singular p-Laplacian type equation and get three solutions with f having $(p-1)$-sublinear growth at infinity. Papageorgiou, Rocha and Staicu [9] consider the nonsmooth p-Laplacian problem, and obtain at least two solutions. In [7], the sub-supersolution method has been applied to find one solution to the

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problem (1.4) with the boundary condition (1.2) and (1.3) where the nonlinearity f satisfies the condition: $|f(x,u)| \le a_3(x)$, with $a_3 \in L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$.

The first result of this paper is about the existence of solution of (1.1)-(1.3). We assume that the nonlinear term $f : \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Ambrosetti-Rabinowitz type condition and obtain one weak solution by the mountain pass lemma in Theorem 3.1.

The second result of this paper is about the existence and multiplicity of solutions for the problem (1.1)-(1.3). Under the growth on f, saying, f is $(p-1)$-sublinear at infinity, we obtain two nontrivial solutions by minimization and mountain pass lemma in [1, 4, 9], where they do the same thing under different assumptions on f.

We remark that in [2, 4, 8], the function A, with $\nabla_\xi A = a(x,\xi)$, satisfies the p-uniformly convex condition: there exists a constant $k>0$ such that

$$A(x,\frac{\xi+\eta}{2}) \le \frac{1}{2}A(x,\xi)+\frac{1}{2}A(x,\eta)-k|\xi-\eta|^p, \quad x\in\Omega,\ \xi,\eta\in\mathbb{R}^N.$$

However, for the case $A(\xi)=|\xi|^p$, the p-uniform convexity condition is satisfied only for $p\in[2,+\infty)$. We assume the function A satisfies the condition (UC) in this paper, while the condition (UC) is satisfied for $A(\xi)=|\xi|^p$ for all $p\in(1,+\infty)$ (see [3]).

2. Preliminaries

Let X be a Banach space and X^* is its topological dual. We denote the duality brackets for the pair (X^*,X) by $\langle\cdot,\cdot\rangle$ and $W^{1,p}(\Omega)$ $(p>1)$ is the usual Sobolev space, equipped with the norm

$$\|u\| = \|u\|_{W^{1,p}(\Omega)} = \Big(\int_\Omega |\nabla u|^p + |u|^p dx\Big)^{1/p}. \tag{2.1}$$

Let

$$V=\{u\in W^{1,p}(\Omega) : u|_{\partial\Omega} = \text{constant}\}.$$

We next claim that V is a closed subspace of $W^{1,p}(\Omega)$ and thus a reflexive Banach space with the restricted norm of (2.1).

Lemma 2.1 ([7]). *V is a Banach space equipped with the norm of* (2.1).

Proof. From the definition of V, we set $V=\{u+c : u\in W_0^{1,p}(\Omega), c\in\mathbb{R}\}$. We assume that $v_n\in V$, then $v_n = u_n + c_n$, with $u_n\in W_0^{1,p}(\Omega)$. If $\{v_n\}$ is Cauchy sequence in $W^{1,p}(\Omega)$, then for all $\varepsilon>0$, we have

$$\begin{aligned}\varepsilon > \|v_n - v_m\|_{W^{1,p}} &= \|u_n + c_n - (u_m+c_m)\|_{W^{1,p}}\\ &= \|\nabla(u_n-u_m)\|_{L^p} + \|u_n - u_m + c_n - c_m\|_{L^p}\\ &\ge \|\nabla(u_n-u_m)\|_{L^p}.\end{aligned}$$

We obtain that $\{u_n\}$ is Cauchy sequence in $W_0^{1,p}(\Omega)$, so there exists $\tilde u\in W_0^{1,p}(\Omega)$ such that

$$u_n\to\tilde u \quad \text{in} W_0^{1,p}(\Omega).$$

As

$$\|u_n-u_m\|_{L^p} \le c_p\|\nabla(u_n-u_m)\|_{L^p} \le c_p\varepsilon,$$

we have

$$\|c_n - c_m\|_{L^p} = \|u_n + c_n - (u_m+c_m) - u_n + u_m\|_{L^p}$$

$$
\begin{aligned}
&\le \|u_n + c_n - (u_m + c_m)\|_{L^p} + \|u_n - u_m\|_{L^p} \\
&\le \|v_n - v_m\|_{L^p} + c_p\|u_n - u_m\|_{L^p} \\
&\le \varepsilon + c_p\varepsilon.
\end{aligned}
$$

We conclude that $\{c_n\}$ is a Cauchy sequence in $L^p(\Omega)$, and so is in $\mathbb{R}$. We conclude that there exists $\tilde{c} \in \mathbb{R}$, such that

$$u_n + c_n \to \tilde{u} + \tilde{c} \quad \text{in } V \text{ as } c_n \to \tilde{c} \text{ in } \mathbb{R}.$$

□

Definition 2.2. We say that $u \in V$ is a weak solution of the boundary-value problem (1.1)-(1.3)if

$$\int_\Omega a(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} uv \, dx - \lambda \int_\Omega f(x, u)v \, dx = 0, \quad \forall v \in V. \tag{2.2}$$

Definition 2.3 ([3])**.** Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}, A = A(x, \xi)$ be a continuous function in $\Omega \times \mathbb{R}^N$ with continuous derivative with respect to ξ, $a(x, \xi) = \nabla_\xi A(x, \xi) = A'$. Define $A^{|\vee|} : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows,

$$A^{|\vee|}(x, t) = \sup_{|\xi|=t} A(x, \xi), \quad \forall x \in \Omega.$$

For every $\varepsilon, b \in (0, 1)$ and $x \in \Omega$, define

$$
\begin{aligned}
E_{\varepsilon,b}(x) = \Big\{ &(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N : A(x, \frac{\xi - \eta}{2}) \ge \frac{1}{2} \max\{A(x, \varepsilon\xi), A(x, \varepsilon\eta)\}, \\
&A(x, \frac{\xi + \eta}{2}) > (1 - b)\frac{A(x, \xi) + A(x, \eta)}{2} \Big\},
\end{aligned}
$$

and

$$q_{\varepsilon,b}(x) = \sup\{\frac{|\xi - \eta|}{2} : (\xi, \eta) \in E_{\varepsilon,b}(x)\}.$$

We say that A satisfies condition (UC) if

$$\lim_{b\to 0} \int_\Omega A^{|\vee|}(x, q_{\varepsilon,b}(x))dx = 0 \quad \text{for every } \varepsilon \in (0, 1).$$

So a function A is said to be uniformly convex if A satisfies condition (UC).

As in [3], we remark that for $A(\xi) = |\xi|^p$, the p-uniform convexity condition

$$A(x, \frac{\xi + \eta}{2}) \le \frac{1}{2} A(x, \xi) + \frac{1}{2} A(x, \eta) - k|\xi - \eta|^p, \quad \forall x \in \Omega,\ \xi, \eta \in \mathbb{R}^N,$$

where k is a positive constant, is satisfied only if $p \in [2, +\infty)$, but (UC) is satisfied for all $p \in (1, +\infty)$.

Lemma 2.4 ([5, 11, 13])**.** *Let X be a Banach space and $I \in C^1(X; \mathbb{R})$ satisfy the Palais-Smale condition. Suppose*

(i) *$I(0) = 0$;*
(ii) *there exists constants $r > 0, a > 0$ such that $I(u) \ge a$ if $\|u\| = r$;*
(iii) *there exists $u_1 \in X$ such that $\|u_1\| \ge r$ and $I(u_1) < a$.*

Define

$$\Gamma = \{\gamma \in C([0, 1]; X) : \gamma(0) = 0, \gamma(1) = u_1\}.$$

Then

$$\beta = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} I(u) \ge a$$

is a critical value.

3. Main result

Let $p > 1$, $A : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $a(x, \xi)$ be derivative of $A(x, \xi)$ with respect to ξ, and we assume that the following conditions hold

- (A1) $A(x, 0) = 0$ for all $x \in \Omega$;
- (A2) a satisfies the growth condition $|a(x, \xi)| \le c_2(1 + |\xi|^{p-1})$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$, for some constant $c_2 > 0$;
- (A3) A is uniformly convex;
- (A4) A is p-subhomogeneous, $0 \le a(x, \xi)\xi \le pA(x, \xi)$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$.
- (A5) A satisfies $A(x, \xi) \ge \Lambda|\xi|^p$ for all $x \in \Omega$, $\xi \in \mathbb{R}^N$, where $\Lambda > 0$ is a constant.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following conditions:

- (F1) The subcritical growth condition
$$|f(x, s)| \le c_3(1 + |s|^{q-1}), \quad \forall x \in \Omega, s \in \mathbb{R},$$
where $p < q < p^* = \frac{Np}{N-p}$ if $p < N$ or $p < q < +\infty$ if $p > N$;
- (F2) (The Ambrosetti-Rabinowitz condition) $F(x, s) = \int_0^s f(x, t)dt$ is θ-superhomogeneous at infinity; i.e., there exists $s_0 > 0$ such that
$$0 < \theta F(x, s) \le f(x, s)s, \quad \text{for } |s| \ge s_0,\ x \in \Omega,$$
where $\theta > p$;
- (F3) $\lim_{|s|\to 0} \frac{f(x,s)}{|s|^{p-1}} = 0$;
- (F4) $\lim_{|s|\to\infty} \frac{f(x,s)}{|s|^{p-1}} = 0$;
- (F5) There exists $s^* > 0, s^* \in \mathbb{R}$ such that $F(x, s^*) > 0$, $\forall x \in \Omega$.

Our main result is as follows.

Theorem 3.1. *Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a potential which satisfies* (A1)–(A5)*, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. If f satisfies* (F1)–(F3)*, then* (1.1)-(1.3) *has at least one nontrivial weak solution in V, for every $\lambda \in \mathbb{R}$.*

Theorem 3.2. *Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a potential which satisfies* (A1)–(A5)*, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. If f satisfies* (F3)–(F5)*, then there exists a constant $\mu > 0$, such that for $\lambda \in (\mu, +\infty)$, problem* (1.1)-(1.3) *has at least two nontrivial weak solutions in V.*

3.1. **Proof of Theorem 3.1.** Under the assumptions of Theorem 3.1 we define the functional
$$J(u) = \int_\Omega A(x, \nabla u)\, dx + \frac{1}{p} \int_\Omega |u|^p dx - \lambda \int_\Omega F(x, u)\, dx.$$
It is easy to see that $J : V \to \mathbb{R}$ is well defined and $J \in C^1(V; \mathbb{R})$. Its derivative is given by
$$\langle J'(u), \varphi\rangle = \int_\Omega a(x, \nabla u) \cdot \nabla\varphi\, dx + \int_\Omega |u|^{p-2}u\varphi\, dx - \lambda \int_\Omega f(x, u)\varphi\, dx,$$
for all $u, \varphi \in V$. Thus the weak solution of (1.1)–(1.3) corresponds to the critical point of the functional J on V.

To prove Theorem 3.1, we apply the mountain pass lemma to this functional. We will show J satisfies the Palais-Smale condition in the first. Let $\{u_n\} \subset V$ be a

Palais-Smale sequence; i.e., $J'(u_n) \to 0$ in X^* and $J(u_n) \to l$, where l is a constant. We first show that $\{u_n\}$ is bounded in V,

$$\begin{aligned}J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle &= \int_\Omega [A(x, \nabla u_n) - \frac{1}{\theta} a(x, \nabla u_n) \cdot \nabla u_n] dx \\ &\quad + (\frac{1}{p} - \frac{1}{\theta}) \int_\Omega |u_n|^p dx + \lambda \int_\Omega [\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)] dx,\end{aligned}$$

where $\theta > p$. From condition (A4), we have

$$\begin{aligned}J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle &\ge (1 - \frac{p}{\theta}) \int_\Omega A(x, \nabla u_n) dx + (\frac{1}{p} - \frac{1}{\theta}) \int_\Omega |u_n|^p dx \\ &\quad + \lambda \int_\Omega [\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)] dx,\end{aligned}$$

then

$$\begin{aligned}&(1 - \frac{p}{\theta}) \int_\Omega A(x, \nabla u_n) dx + (\frac{1}{p} - \frac{1}{\theta}) \int_\Omega |u_n|^p dx \\ &\le J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle - \lambda \int_{\{x:|u_n(x)|>s_0\}} [\frac{1}{\theta} f(x, u_n) u_n - F(x, u_n)] dx + Mm(\Omega),\end{aligned}$$

where $M = \sup\{|\frac{1}{\theta} f(x, s)s - F(x, s)| : x \in \Omega, |s| \le s_0\}$, and $m(\Omega)$ denotes the Lebesgue measure of Ω.

By (F2) (the Ambrosetti-Rabinowitz condition), we have

$$(1 - \frac{p}{\theta}) \int_\Omega A(x, \nabla u_n) dx + (\frac{1}{p} - \frac{1}{\theta}) \int_\Omega |u_n|^p dx \le J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle + Mm(\Omega).$$

By (A5),

$$(1 - \frac{p}{\theta}) \min\{\Lambda, \frac{1}{p}\}(\int_\Omega |\nabla u_n|^p + |u_n|^p) dx \le J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle + Mm(\Omega),$$

where $\min\{\Lambda, \frac{1}{p}\}$ denotes the minimum of Λ and $\frac{1}{p}$. As

$$\|u_n\| = \Big(\int_\Omega |\nabla u_n|^p + |u_n|^p dx\Big)^{1/p},$$

we conclude that $\{u_n\}$ is bounded in V. Since V is a closed subspace of $W^{1,p}(\Omega)$ and the reflexivity of $W^{1,p}(\Omega)$, we may extract a weakly convergent subsequence that we call $\{u_n\}$ for simplicity. So we may assume that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.

Next, we will prove that u_n converges strongly to $u \in V$. From the derivative of J we obtain

$$\begin{aligned}&\int_\Omega a(x, \nabla u_n) \cdot \nabla(u_n - u) dx + \int_\Omega |u_n|^{p-2} u_n (u_n - u) dx \\ &= \langle J'(u_n), u_n - u\rangle - \lambda \int_\Omega f(x, u_n)(u_n - u)\, dx.\end{aligned} \tag{3.1}$$

Since $\|J'(u_n)\|_{W^{-1,p'}} \to 0$ and $\{u_n - u\}$ is bounded in $V \subset W^{1,p}(\Omega)$, by the $|\langle J'(u_n), u_n - u\rangle| \le \|J'(u_n)\|_{W^{-1,p'}} \|u_n - u\|$ it follows that

$$\langle J'(u_n), u_n - u\rangle \to 0.$$

From (F1), we have

$$\int_\Omega |f(x, u_n(x))||u_n(x) - u(x)| dx$$

$$
\begin{aligned}
&\le c_3 \int_\Omega |u_n(x)-u(x)|dx + c_3 \int_\Omega |u_n(x)|^{q-1}|u_n(x)-u(x)|dx \\
&\le c_3((m(\Omega))^{1/q'} + \|u_n\|_{L^q}^{q-1})\|u_n-u\|_{L^q},
\end{aligned}
$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Since the embedding $W^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ is compact, with $q<\frac{Np}{N-p}$, we obtain $u_n\to u$ strongly in $L^q(\Omega)$. So we obtain

$$
\int_\Omega |f(x,u_n(x))||u_n(x)-u(x)|dx \to 0.
$$

Considering the inequality

$$
\begin{aligned}
\int_\Omega ||u_n(x)|^{p-2}u_n(x)(u_n(x)-u(x))|dx &= \int_\Omega |u_n(x)|^{p-1}|u_n(x)-u(x)|dx \\
&\le \|u_n\|_{L^p}^{p-1}\|u_n-u\|_{L^p},
\end{aligned}
$$

and $u_n\to u$ strongly in $L^p(\Omega)$, we have

$$
\int_\Omega ||u_n(x)|^{p-2}u_n(x)(u_n(x)-u(x))|dx.
$$

From (3.1), we may conclude

$$
\limsup_{n\to\infty}\langle a(x,u_n),u_n-u\rangle = \limsup_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n-u)dx \le 0,
$$

where $\langle a(x,u_n),u_n-u\rangle$ denotes $\int_\Omega a(x,\nabla u_n)\cdot\nabla(u_n-u)dx$.

Therefore, from condition (A3), A is uniformly convex, and the operator $a(x,\xi)=D_\xi A(x,\xi)$ satisfies the (S_+) property. From the (S_+) condition in [8, Proposition 2.1], so we have $u_n\to u$ strongly in $W^{1,p}(\Omega)$. Since $\{u_n\}\subset V$, V is a closed subspace of $W^{1,p}(\Omega)$, and we have $u\in V$. So $u_n\to u$ strongly in V.

Next, we show that J satisfies the geometry condition of the mountain pass lemma; i.e.,

1. There exists $r>0$, such that $\inf_{\|u\|=r} J(u)=b>0$.
2. There exists $u_0\in V$ such that $J(tu_0)\to-\infty$, as $t\to+\infty$.

Step 1. Fix $\lambda\in\mathbb{R}$, we choose $\varepsilon>0$ small enough satisfying $\Lambda>\frac{\lambda\varepsilon}{pc_p}$. Then by (F3), there exists $\delta>0$ such that $|f(x,s)|\le\varepsilon|s|^{p-1}$ for $|s|\le\delta$, for all $x\in\Omega$. Integrating the above inequality, we deduce that

$$
F(x,s)\le\frac{\varepsilon}{p}|s|^p,\quad \text{for } |s|\le\delta.
$$

Consequently, using (F1) and the Sobolev embedding, we have

$$
\begin{aligned}
J(u) &\ge \int_\Omega A(x,\nabla u)dx + \frac{1}{p}\int_\Omega |u|^p dx - \lambda\int_{\{x\in\Omega:|u(x)|\le\delta\}}\frac{\varepsilon}{p}|u|^p dx \\
&\quad - \lambda\int_{\{x\in\Omega:|u|>\delta\}} c_4|u|^q dx \\
&\ge \min\{\Lambda,\frac{1}{p}\}\|u\|^p - \frac{\lambda\varepsilon}{p}c_p\|u\|^p - \lambda c_4\|u\|^q \\
&\ge \Big(\min\{\Lambda,\frac{1}{p}\} - \frac{\lambda\varepsilon}{p}c_p\Big)\|u\|^p - \lambda c_4\|u\|^q = \Phi(r),
\end{aligned}
$$

where $r=\|u\|^p$, $\min\{\Lambda,\frac{1}{p}\}>\frac{\lambda\varepsilon}{p}c_p$, as ε is small enough. Moreover, $\Phi(r)>0$ for $r>0$ small enough, since $q>p$.

Step 2. Since A is p-subhomogeneous, can be restated as a differential inequality for the function F in the form

$$s|s|^{\theta}\frac{d}{ds}(|s|^{-\theta}F(x,s)) \geq 0, \quad \text{for } |s| \geq s_0.$$

We infer that for $|s| \geq s_0$, we have $F(x,s) \geq \gamma_0(x)|s|^{\theta}$, where

$$\gamma_0 = s_0^{-\theta}\min\{F(x,s_0), F(x,-s_0)\} > 0.$$

Considering condition (A4), we obtain that for some constant $k(u) > 0$ there holds

$$\begin{aligned} J(tu_0) &= \int_\Omega A(x,t\nabla u_0)dx + \frac{1}{p}\int_\Omega |tu_0|^p dx - \lambda\int_\Omega F(x,tu_0)dx \\ &\leq t^p\int_\Omega A(x,\nabla u_0)dx + \frac{1}{p}t^p\int_\Omega |u_0|^p dx - k(u)|\lambda|t^{\theta} + |\lambda|M_1 m(\Omega), \end{aligned}$$

where $M_1 = \sup\{|F(x,s)| : x \in \Omega, |s| \leq s_0\}$. Since $\theta > p$, we choose u_0 such that $m\{x \in \Omega : u_0(x) \geq s_0\} > 0$. We deduce that $J(tu_0) \to -\infty$, as $t \to +\infty$. For fixed $u_0 \neq 0$ and sufficiently large $t > 0$, we let $u_1 = tu_0$. By Lemma 2.4 (mountain pass lemma), we obtain the existence of a non-trivial solution u to (1.1)-(1.3). The proof is completed.

3.2. **Proof of Theorem 3.2.**

We denote

$$\mathcal{A}(u) = \int_\Omega A(x,\nabla u)dx + \frac{1}{p}\int_\Omega |u|^p dx$$

and $\mathcal{F}(u) = \int_\Omega F(x,u)dx$, then the functional J is given by $J(u) = \mathcal{A}(u) - \lambda\mathcal{F}(u)$.

Lemma 3.3 ([4])**.** *For every $\lambda \in \mathbb{R}$, the functional $J : V \to \mathbb{R}$ is sequentially weakly lower semicontinuous.*

Proof. The functional $\mathcal{A}$ being locally uniformly convex is weakly lower semicontionous. From the condition (F_4), we have $|f(x,s)| \leq c_5(1+|s|^{p-1})$ for every $s \in \mathbb{R}$. Since the embedding $V \subset W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we obtain that $\mathcal{F}$ is sequentially weakly lower semicontinuous in the standard method. $\square$

Lemma 3.4. *For every $\lambda \in \mathbb{R}$, the functional J is coercive and satisfies the Palais-Smale condition.*

Proof. By (F4), for $\varepsilon > 0$ small enough, there exists δ such that $|f(x,s)| \leq \varepsilon|s|^{p-1}$ for every $|s| \geq \delta$. Integrating this inequality, we have

$$|F(x,s)| \leq \frac{\varepsilon}{p}|s|^p + \max_{|t|\leq\delta}|f(x,t)||s|, \quad \forall s \in \mathbb{R}.$$

Thus, for every $u \in V$, we obtain

$$\begin{aligned} J(u) &\geq \mathcal{A}(u) - |\lambda||\mathcal{F}(u)| \\ &\geq \min\{\Lambda, \frac{1}{p}\}\|u\|^p - |\lambda|\frac{\varepsilon}{p}\int_\Omega |u|^p dx - |\lambda|\max_{|t|\leq\delta}|f(x,t)|\int_\Omega |u|dx \\ &\geq \min\{\Lambda, \frac{1}{p}\}\|u\|^p - \frac{\varepsilon|\lambda|}{p}\int_\Omega |u|^p dx - |\lambda|m(\Omega)^{1/p'}\max_{|t|\leq\delta}|f(x,t)|\Big(\int_\Omega |u|^p dx\Big)^{1/p} \\ &\geq \Big(\min\{\Lambda, \frac{1}{p}\} - \frac{\varepsilon|\lambda|c_p}{p}\Big)\|u\|^p - c_p^{1/p}|\lambda|m(\Omega)^{1/p'}\max_{|t|\leq\delta}|f(x,t)|\|u\|. \end{aligned}$$

Since ε is small enough, $\min\{\Lambda, \frac{1}{p}\} > \frac{\varepsilon|\lambda|c_p}{p}$, so we have $J(u) \to +\infty$, whenever $\|u\| \to +\infty$. Hence J is coercive.

The proof of the functional J satisfying the Palais-Smale condition is similar to Theorem 3.1 The proof is complete. □

Proof of Theorem 3.2. From condition (F5), we have

$$\rho := \sup_{u \in V, u \neq 0} \frac{\mathcal{F}(u)}{\mathcal{A}(u)} \geq \frac{\mathcal{F}(s^*)}{\mathcal{A}(s^*)} > 0.$$

Let $\mu = 1/\rho$. Fix $\lambda \in (\mu, +\infty)$. From the definition of ρ, there exists some $u^* \in V$, with $\min\{\mathcal{A}(u^*), \mathcal{F}(u^*)\} > 0$, such that

$$\frac{1}{\lambda} < \frac{\mathcal{F}(u^*)}{\mathcal{A}(u^*)}.$$

This implies $J(u^*) = \mathcal{A}(u^*) - \lambda\mathcal{F}(u^*) < 0$. By Lemma 3.4, the functional J is bounded from below, coercive and satisfies the (P-S) condition on V for every $\lambda > 0$. This implies the functional J has a global minimizer u_1; i.e.,

$$J(u_1) \leq J(u) \quad \forall u \in V.$$

Let $u = u^*$. We have

$$J(u_1) \leq J(u^*) < 0.$$

By (F3), there exists $\delta > 0$ such that $|f(x,s)| \leq \varepsilon|s|^{p-1}$ for $|s| < \delta$, for all $x \in \Omega$. We have

$$|F(x,s)| \leq \frac{\varepsilon}{p}|u|^p \text{for} |s| \leq \delta. \tag{3.2}$$

Using (F4), there exists $k(\delta) > 0$ such that $|F(x,s)| \leq k(\delta)|s|^p \leq k(\delta)|s|^q$, $p < q < \frac{Np}{N-p}$, for $|s| > \delta$. Considering this fact and (3.2), for $\lambda \in (\mu, +\infty)$ we have

$$\begin{aligned}
J(u) &\geq \int_\Omega A(x, \nabla u)dx + \frac{1}{p}\int_\Omega |u|^p dx - \lambda \int_{\{x\in\Omega:|u(x)|\leq\delta\}} \frac{\varepsilon}{p}|u|^p dx \\
&\quad - \lambda \int_{\{x\in\Omega:|u|>\delta\}} k(\delta)|u|^q dx \\
&\geq \min\{\Lambda, \frac{1}{p}\}\|u\|^p - \frac{\lambda\varepsilon}{p}c_p\|u\|^p - \lambda k(\delta)\|u\|^q \\
&\geq \Big(\min\{\Lambda, \frac{1}{p}\} - \frac{\lambda\varepsilon}{p}c_p\Big)\|u\|^p - \lambda k(\delta)\|u\|^q = \Phi(r),
\end{aligned}$$

where $r = \|u\|^p$ and $q > p$. We can take ε small enough, such that $\min\{\Lambda, \frac{1}{p}\} > \frac{\lambda\varepsilon}{p}c_p$. Moreover, $\exists r > 0$ small enough and $a > 0$, such that $\Phi(r) \geq a > 0$.
Obviously, $J(0) = 0$. If we denote by Γ the set of all continuous functions $\gamma : [0,1] \to V$, such that $\gamma(0) = 0$ and $\gamma(1) = u_1$. From the mountain pass lemma, there exists u_2 such that $J'(u_2) = 0$ and

$$J(u_2) = \beta = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma} J(u) \geq a > 0.$$

This completes the proof. □

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