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OPTIMIZATION PROBLEMS INVOLVING POISSON'S EQUATION IN $\mathbb{R}^3$

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ABSTRACT. In this article, we prove the existence of minimizers for integrals associated with a second-order elliptic problem. For this three-dimensional optimization problem, the admissible set is a rearrangement class of a given function.

1. Introduction

We consider the Poisson's equation

$$\begin{gathered} -\Delta u = f - 2h \quad \text{in } \mathbb{R}^3 \\ \lim_{|x|\to+\infty} u(x) = 0, \quad f \in L^p_b(\mathbb{R}^3), \end{gathered} \tag{1.1}$$

where $L^p_b(\mathbb{R}^3) = \{f \in L^p(\mathbb{R}^3) : f \text{ has bounded support}\}$ and $p > 3$. Here h is a given non-negative function in $h \in L^\infty(\mathbb{R}^3)$ with bounded support. For the sake of convenience in the discussions, we have $2h$ instead of h, but it can be replaced by h. By standard results on elliptic equations, problem (1.1) has a unique solution $u \in W^{2,p}_{\rm loc}(\mathbb{R}^3)$; see [8]. Let u_f be the solution of (1.1), we define energy functional corresponding to (1.1) as

$$\Psi_\lambda(f) = \frac{1}{2}\int_{\mathbb{R}^3} f u_f + \lambda \int_{\mathbb{R}^3} g f, \tag{1.2}$$

for $f \in L^p_b(\mathbb{R}^3)$ where $g \in C^2(\mathbb{R}^3)$, $\lim_{|x|\to+\infty} g = +\infty$ and $\Delta g > c$ for some $c > 0$ and $\lambda \geq 0$. In this paper we minimize the functional Ψ_λ on rearrangement class of a fixed function. We separate the investigation of the particular case $\lambda = 0$, since the discussion in the case $\lambda > 0$ does not carry over the case $\lambda = 0$. The same optimization problems have been investigated in bounded domains for the Laplacian operator in [1, 4, 6], for the p-Laplacian operator in [3, 10], for semilinear operators in [7]. For the current problem we face two mathematical difficulties: firstly the awkward nature of rearrangements class, and secondly a loss of compactness which is caused by the unboundness of the domain $\mathbb{R}^3$. To overcome these difficulties we first investigate the problem in a bounded domain. Then using Burton's theory on

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rearrangements class, we show that a solution valid in a sufficiently large bounded domain is in fact valid in the whole space.

2. Notation, definitions and statement of the main result

Henceforth we assume $p \in (3, \infty)$ and p' is the conjugate exponent of p; that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Points in $\mathbb{R}^3$ are denoted by $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, and so on. By $B_r(x)$ we denote the ball centered at $x \in \mathbb{R}^3$ with radius r; if the center is the origin, we write B_r. Measure will refer to Lebesgue measure on $\mathbb{R}^3$, and if $A \subseteq \mathbb{R}^3$ is measurable then $|A|$ will denote the measure of A. If $A \subset \mathbb{R}^3$ is a measurable set, then we say $x \in A$ is a density point of A whenever

$$|B_\varepsilon(x) \cap A| > 0,$$

for all positive ε.

For a measurable function $f : \mathbb{R}^3 \to \mathbb{R}^+$, the strong support or simply the support of f is denoted $\operatorname{supp}(f)$ and is defined by

$$\operatorname{supp}(f) = \{x : f(x) > 0\}.$$

For a measurable function $f : \mathbb{R}^3 \to \mathbb{R}^+$ we define

$$\|f\|_{-\infty} = \operatorname{ess\,inf}(f) = \sup\{M \geq 0 : f(x) \geq M, \text{ for almost all } x\}.$$

When f and g are non-negative measurable functions that vanish outside sets of finite measure in $\mathbb{R}^3$, we say f is a rearrangement of g whenever

$$|\{x \in \mathbb{R}^3 : f(x) \geq \alpha\}| = |\{x \in \mathbb{R}^3 : g(x) \geq \alpha\}|,$$

for every positive α.

For any real integrable and non-negative function f vanishing outside a bounded set $\Omega \subset \mathbb{R}^3$ of measure m, we can define a decreasing rearrangement f^Δ which is a decreasing function on the interval $(0, m)$ satisfying

$$|\{s \in (0, m) : f^\Delta(s) \geq \alpha\}| = |\{x \in \Omega : f(x) \geq \alpha\}|,$$

for every positive α. Also there exists a Schwarz rearrangement f^* for f, that is a rearrangemet of f as a radial decreasing function on a ball.

Let us fix $f_0 \in L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ to be a measurable and non-negative function vanishing outside a set of measure $4\pi a^3/3$, for some positive $a \in \mathbb{R}$. The set of all rearrangements on $\mathbb{R}^3$ of f_0 with bounded support is denoted by $\mathcal{R}$. The subset of $\mathcal{R}$ containing functions vanishing outside the ball B_r, where $r \geq a$, is denoted by $\mathcal{R}(r)$; henceforth we assume $r \geq a$ in order that $\mathcal{R}(r)$ is non-empty. The weak closure in $L^p(B_r)$ of $\mathcal{R}(r)$ is denoted by $\overline{\mathcal{R}(r)}^w$.

Now we are ready to introduce our minimizing problems P_λ as follows:

$$\min_{f \in \mathcal{R}} \Psi_\lambda(f). \tag{2.1}$$

The set of solutions of P_λ is denoted by S_λ. Similarly, for $r \geq a$ we define $P_\lambda(r)$ as follows:

$$\min_{f \in \mathcal{R}(r)} \Psi_\lambda(f), \tag{2.2}$$

and the set of solutions is denoted by $S_\lambda(r)$. Our main results are the following:

Theorem 2.1. *There exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, the optimization problem P_λ has a solution. Moreover, if $f_\lambda \in S_\lambda$ and u_{f_λ} be the solution of (1.1) corresponding with energy minimizer, then there exists a decreasing function φ_λ such that*

$$f_\lambda = \varphi_\lambda \circ (u_{f_\lambda} + \eta + \lambda g), \tag{2.3}$$

almost everywhere in $\mathbb{R}^3$ where η will be presented later.

Theorem 2.2. *Let f_0 and h be as introduced above. Let $|\operatorname{supp}(f_0)| = 4\pi a^3/3$ and $|\operatorname{supp}(h)| = 4\pi b^3/3$ for some a, b positive real numbers. We assume*

$$b > \sqrt{3}a, \quad \|f_0\|_\infty < \|h\|_{-\infty}. \tag{2.4}$$

Then the optimization problem P_0 has a solution.

3. Preliminary results

In this section we state and/or prove some lemmas which are essential in our analysis. We begin with a result proved by Burton in [2].

Lemma 3.1. *For $r \geq a$ and $q \geq 1$, we have*

(i) $\|f\|_q = \|f_0\|_q$, *for $f \in \mathcal{R}(r)$;*
(ii) $\overline{\mathcal{R}(r)^w}$ *is weakly sequentially compact in $L^q(B_r)$;*
(iii) $\overline{\mathcal{R}(r)^w} = \{f \in L^1(B_r) : \int_0^s f^\Delta(t)dt \leq \int_0^s f_0^\Delta(t)dt,\ 0 < s \leq 4\pi r^3/3,\ \int_{B_r} f = \int_{B_r} f_0\}$.

Lemma 3.2. *Let $\lambda \geq 0$ and $f \in L_b^p(\mathbb{R}^3)$. Then*

(i) *for the energy functional Ψ_λ we have*

$$\Psi_\lambda(f) = \frac{1}{2}\int_{\mathbb{R}^3} fKf - \int_{\mathbb{R}^3} \eta f + \lambda \int_{\mathbb{R}^3} gf, \tag{3.1}$$

where

$$Kf(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)dy,$$

and $\eta = Kh$.

(ii) *for $f \in L_b^p(\mathbb{R}^3)$*

$$|Kf(x)| \leq C\|f\|_p, \quad \forall x \in \mathbb{R}^3, \tag{3.2}$$

where C depends only on p and $|\operatorname{supp}(f)|$.

Proof. Using the fundamental solution of $-\Delta$ on $\mathbb{R}^3$ and asymptotic behavior of the solutions in (1.1), we derive the unique solution of the problem (1.1), $u_f = Kf - 2Kh$, this yields (3.1). We note that $u_f(x) = O(\frac{1}{|x|})$ as $|x| \to +\infty$. Indeed, for large $|x|$, we have $|x-y| > |x|/2$ for $y \in \operatorname{supp}(f) \cup \operatorname{supp}(h)$. Thus, $K(f-2h)$ is dominated by $2\|f-2h\|_1/|x|$.

To prove (ii), let f be as in the lemma, we have

$$|Kf(x)| \leq \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|}dy \leq \frac{1}{4\pi}\int_{B_{r^*}(x)} \frac{f^*(y)}{|x-y|}dy,$$

where f^* is the Schwarz rearrangement of f with respect to x and

$$r^* = \Big(\frac{3|\operatorname{supp}(f)|}{4\pi}\Big)^{1/3}.$$

The inequality is a consequence of Hardy-Littlewood inequality [9]. Now by Hölder's inequality, we obtain the assertion where

$$C = \frac{1}{4\pi}\Big(\int_{B_{r^*}(x)} \frac{1}{|x-y|^{p'}}dy\Big)^{1/p'} = \frac{(3|\operatorname{supp}(f)|)^{\frac{1}{p'}-\frac{1}{3}}}{(4\pi)^{2/3}(3-p')^{1/p'}}, \tag{3.3}$$

and p' is the conjugate exponent of p. □

Lemma 3.3. *Let K be as the above lemma.*

- (i) *If U is a bounded open subset in $\mathbb{R}^3$, $K : L^p(U) \to L^{p'}(U)$ is a linear compact operator.*
- (ii) *For $f \in L^p(U)$, $Kf \in W^{2,p}(U)$ and $-\Delta Kf = f$, almost everywhere in U.*

Proof. Since $W^{1,2}(U)$ is compactly embedded into $L^{p'}(U)$ for $p > 3$, in order to show the compactness of K it is sufficient to prove the bondedness of K as a map from $L^p(U)$ into $W^{1,2}(U)$. To do this, let $f \in L^p(U)$ we have

$$|\nabla Kf(x)| \le \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|^2}dy, \quad x \in U. \tag{3.4}$$

Similar to the proof of the lemma above and the fact that $p' < \frac{3}{2}$, we deduce

$$\|\nabla Kf\|_2 \le C\|f\|_p,$$

where C depends on $|U|$ and p. This completes the proof. For a proof of part (ii) see [5]. □

The following lemma is a simple variation of [2, Lemma 2.15].

Lemma 3.4. *Let $r \ge a$ and $v \in L^{p'}(B_r)$. Denote by $L_\alpha(v)$ the level set of v at height α; that is,*

$$L_\alpha(v) = \{x \in B_r : v(x) = \alpha\}.$$

Let $T : L^p(B_r) \to \mathbb{R}$ be the linear functional defined by

$$T(f) = \int_{B_r} fv.$$

If $\hat{f}$ is a minimizer of T relative to $\overline{\mathcal{R}(r)^w}$ and if

$$\Big|L_\alpha(v) \cap \operatorname{supp}(\hat{f})\Big| = 0,$$

for every $\alpha \in \mathbb{R}$, then $\hat{f} \in \mathcal{R}(r)$ and

$$\hat{f} = \varphi o v,$$

almost everywhere in B_r, for some decreasing function φ.

4. Investigation in the case: $\lambda > 0$

In this section we consider the case in which $\lambda > 0$. First we are concerned with the existence of minimizers for the energy functional in a bounded domain, then we will demonstrate the problem in the unbounded domain.

4.1. **Bounded domains.** We begin with the following lemma.

Lemma 4.1. (i) *The energy functional Ψ_λ attains its minimum relative to $\overline{\mathcal{R}(r)^w}$ for $r \geq a$.*

(ii) *If $f_{r,\lambda}$ is any minimizer for Ψ_λ relative to $\overline{\mathcal{R}(r)^w}$, then $f_{r,\lambda}$ is a solution of the following variational problem*
$$\inf_{f \in \overline{\mathcal{R}(r)^w}} \int_{\mathbb{R}^3} f(u_{f_{r,\lambda}} + \eta + \lambda g), \tag{4.1}$$
where $u_{f_{r,\lambda}}$ is the solution of (1.1) corresponding to $f_{r,\lambda}$ and $\eta = Kh$.

Proof. From Lemma 3.2, the optimization problem (2.2) is equivalent to
$$\inf_{f \in \mathcal{R}(r)} \Big\{ \frac{1}{2} \int_{\mathbb{R}^3} fKf - \int_{\mathbb{R}^3} \eta f + \lambda \int_{\mathbb{R}^3} gf \Big\}.$$
By Lemma 3.3, K is compact and symmetric, then Ψ_λ is a weakly sequentially continuous and Gâteaux differentiable functional. From Lemma 3.1, $\overline{\mathcal{R}(r)^w}$ is weakly sequentially compact, hence Ψ_λ attains its minimum on it. If $f_{r,\lambda}$ is a minimizer of Ψ_λ on $\overline{\mathcal{R}(r)^w}$, since the Gâteaux differential of Ψ_λ at $f_{r,\lambda}$ is $Kf_{r,\lambda} - \eta + \lambda g$, then by [2, Theorem 3.3], $f_{r,\lambda}$ is a solution of the variational problem (4.1). □

Lemma 4.2. *Let $r \geq a$ and $f_{r,\lambda}$ be a minimizer of Ψ_λ relative to $\overline{\mathcal{R}(r)^w}$. Let $\psi_{r,\lambda} = Kf_{r,\lambda} - \eta + \lambda g$ and denote by $L_\alpha(\psi_{r,\lambda})$ the level set of $\psi_{r,\lambda}$ at height α. Then there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$,*
$$|L_\alpha(\psi_{r,\lambda}) \cap \operatorname{supp}(f_{r,\lambda})| = 0, \quad \forall \alpha \in \mathbb{R}.$$

Proof. Let $r \geq a$. From Lemma 4.1, for every $\lambda > 0$, the minimizer $f_{r,\lambda}$ of Ψ_λ on $\overline{\mathcal{R}(r)^w}$ exists. Suppose there exists $\hat{\alpha} \in \mathbb{R}$ such that $|L_{\hat{\alpha}}(\psi_{r,\lambda}) \cap \operatorname{supp}(f_{r,\lambda})| > 0$. Let $S_{\hat{\alpha}} = L_{\hat{\alpha}}(\psi_{r,\lambda}) \cap \operatorname{supp}(f_{r,\lambda})$. Since $\psi_{r,\lambda} = u_{f_{r,\lambda}} + \eta + \lambda g$, using [8, Theorem 7.7], lemma 3.3 and equation (1.1), we have
$$-\Delta\psi_{r,\lambda} = f_{r,\lambda} - h - \lambda\Delta g = 0, \quad \text{a.e. in } S_{\hat{\alpha}}. \tag{4.2}$$
On the other hand, by Lemma 3.1,
$$\int_0^s f_{r,\lambda}^{\Delta}(t)dt \leq \int_0^s f_0^{\Delta}(t)dt, \quad s > 0. \tag{4.3}$$
Then we deduce
$$\|f_{r,\lambda}^{\Delta}\|_\infty \leq \|f_0^{\Delta}\|_\infty. \tag{4.4}$$
Since $f_{r,\lambda}^{\Delta}$ is a rearrangement of $f_{r,\lambda} \in \overline{\mathcal{R}(r)^w}$ and f_0^{Δ} is a rearrangement of f_0, from equation (4.4), we conclude
$$\|f_{r,\lambda}\|_\infty \leq \|f_0\|_\infty\,. \tag{4.5}$$
If we assume that $\lambda_0 = \|f_0\|_\infty / \|\Delta g\|_{-\infty}$, then for every $\lambda > \lambda_0$, we have
$$\|f_0\|_\infty < \|h + \lambda\Delta g\|_{-\infty} \tag{4.6}$$
Finally, from (4.5) and (4.6) for every $\lambda > \lambda_0$, we deduce
$$\|f_{r,\lambda}\|_\infty < \|h + \lambda\Delta g\|_{-\infty} \tag{4.7}$$
which is a contradiction to (4.2). This completes the proof. □

Lemma 4.3. *Let λ_0 be as in the lemma above. Then for every $\lambda > \lambda_0$, the variational problem $P_\lambda(r)$ has a solution for $r \geq a$. If $f_{r,\lambda}$ is any solution of $P_\lambda(r)$, then*

$$f_{r,\lambda} = \varphi_\lambda \circ (u_{f_{r,\lambda}} + \eta + \lambda g), \tag{4.8}$$

almost everywhere in B_r, for a decreasing unknown function φ_λ.

Proof. Let $r \geq a$. From Lemma 4.1, there exists $f_{r,\lambda} \in \overline{\mathcal{R}(r)^w}$ such that $f_{r,\lambda}$ is a minimizer of Ψ_λ relative to $\overline{\mathcal{R}(r)^w}$ and a solution of (4.1). By Lemma 4.2, for every $\lambda > \lambda_0$, the level sets of $\psi_{r,\lambda} = Kf_{r,\lambda} - \eta + \lambda g$ on $\operatorname{supp}(f_{r,\lambda})$ have zero measure. We can use Lemma 3.4 to deduce equation (4.8). □

4.2. **Unbounded domain.** We proved that the variational problem $P_\lambda(r)$ has a solution for $\lambda > \lambda_0$ and $r \geq a$. Now we will show that if r is chosen large enough, it ceases to have any influence whatever on the variational problem, $P_\lambda(r)$. To do this, we now perform some calculations to deduce the following result.

Lemma 4.4. *Let $\lambda > \lambda_0$. Then, there exists $r_0 > a$ such that for $r \geq r_0$ and $f_{r,\lambda} \in S_\lambda(r)$ we have*

$$\operatorname{supp}(f_{r,\lambda}) \subset B_{r_0}.$$

Proof. To prove this lemma, it is sufficient to show that the support of $f_{r,\lambda}$ does not have any dense point on the boundary of B_r when r is chosen large enough. Let $r_h > a$ be the smallest positive number for which $\operatorname{supp}(h) \subset B_{r_h}$. We consider $r > r_h + 1$ and $f_{r,\lambda} \in S_\lambda(r)$. From Lemma 4.3 we have

$$f_{r,\lambda} = \varphi_\lambda \circ (u_{f_{r,\lambda}} + \eta + \lambda g), \tag{4.9}$$

almost everywhere in B_r, for a decreasing unknown function φ_λ where $u_{f_{r,\lambda}}$ is the solution of (1.1) corresponding with $f_{r,\lambda}$. To seek a contradiction suppose the assertion is false. Then there exists $x_0 \in den(\operatorname{supp}(f_{r,\lambda}))$(set of dense points of support) such that $|x_0| = r$. Let $A = \operatorname{supp}(f_{r,\lambda}) \cap B_1(x_0)$, then $|A| > 0$. For $x \in A$

$$Kf_{r,\lambda}(x) = \frac{1}{4\pi}\int_{B_r} \frac{1}{|x-y|} f_{r,\lambda}(y)dy \geq \frac{1}{4\pi}\frac{\|f_0\|_1}{2r} \tag{4.10}$$

and

$$\eta(x) = \frac{1}{4\pi}\int_{B_{r_h}} \frac{1}{|x-y|} h(y)dy \leq \frac{1}{4\pi}\frac{\|h\|_1}{r - r_h - 1} \tag{4.11}$$

From (4.10), (4.11) and relation $u_{f_{r,\lambda}} = Kf_{r,\lambda} - 2\eta$, we obtain

$$u_{f_{r,\lambda}}(x) + \eta(x) + \lambda g(x) \geq \frac{1}{4\pi}\Big(\frac{1}{2r}\|f_0\|_1 - \frac{1}{r - r_h - 1}\|h\|_1\Big) + \lambda g(x). \tag{4.12}$$

Since $|\operatorname{supp}(f_{r,\lambda})| = 4\pi a^3/3$ and $r_h > a$, there exists $D \subset B_{r_h}$ such that $D \cap \operatorname{supp}(f_{r,\lambda})$ is empty and $|D| > 0$. For $z \in D$ from Lemma 3.2 we have

$$Kf_{r,\lambda}(z) = \frac{1}{4\pi}\int_{B_r} \frac{1}{|z-y|} f_{r,\lambda}(y)dy \leq C\|f_0\|_p, \tag{4.13}$$

where C depends on p and $|\operatorname{supp}(f_{r,\lambda})|$. Also

$$\eta(z) = \frac{1}{4\pi}\int_{B_{r_h}} \frac{1}{|z-y|} h(y)dy \geq \frac{1}{4\pi}\frac{1}{2r_h}\|h\|_1. \tag{4.14}$$

Then, from (4.13) and (4.14) we derive

$$u_{f_{r,\lambda}}(z) + \eta(z) + \lambda g(z) \leq \lambda g(z) - C_1. \tag{4.15}$$

Now, since $|z| \leq r_h$ for $z \in D$ and (4.15), we deduce

$$u_{f_{r,\lambda}}(z) + \eta(z) + \lambda g(z) \leq C_2, \quad z \in D, \tag{4.16}$$

where C_2 is a constant independent of r. If we make r large we derive from (4.12) and (4.16)

$$(u_{f_{r,\lambda}}(x) + \eta(x) + \lambda g(x)) - (u_{f_{r,\lambda}}(z) + \eta(z) + \lambda g(z)) > 0,$$

for $x \in A$ and $z \in D$. Since $|A| > 0$, $|D| > 0$, this is a contradiction to (4.9). □

4.3. **Proof of Theorem 2.1.** Let r_0 be as in Lemma 4.4. Assume $f_{r,\lambda}$ to be a solution of $P_\lambda(r)$ for $r \geq r_0$ and $\lambda > \lambda_0$. From Lemma 4.4, $\operatorname{supp}(f_{r,\lambda}) \subset B_{r_0}$ for $r > r_0$, therefore we obtain the inclusion $S_\lambda(r_0) \subset S_\lambda$ that it means P_λ has a solution. Let $f_\lambda \in S_\lambda$ for $\lambda > \lambda_0$. To prove the last part of theorem, if $f_\lambda \in S_\lambda$ we have by applying Lemma 4.3

$$f_\lambda = \varphi_\lambda \circ (u_{f_\lambda} + \eta + \lambda g), \tag{4.17}$$

almost everywhere in B_r for $r > r_0$ and a decreasing unknown function φ_λ. Notice that we can suppose $\varphi_\lambda \geq 0$. Since $u_{f_\lambda} + \eta + \lambda g$ is a continuous function on the compact set B_{r_0}, and $\operatorname{supp}(f_\lambda) \subset B_{r_0}$, there exists $k \in \mathbb{R}$ such that

$$u_{f_\lambda} + \eta + \lambda g < k \quad \text{a.e } \operatorname{supp}(f_\lambda). \tag{4.18}$$

On the other hand, by applying condition (4.12) we have $u_{f_\lambda} + \eta + \lambda g \to +\infty$, as $|x| \to \infty$. Then we can find $r > r_0$ such that

$$u_{f_\lambda} + \eta + \lambda g \geq k \quad \text{a.e outside } B_r. \tag{4.19}$$

Now define

$$\hat{\varphi}_\lambda(t) = \begin{cases} \varphi_\lambda(t) & t < k \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\hat{\varphi}_\lambda$ is a decreasing function and $f_\lambda = \hat{\varphi}_\lambda \circ (u_{f_\lambda} + \eta + \lambda g)$ almost everywhere on $\mathbb{R}^3$.

5. The case $\lambda = 0$

To derive the existence result in this case we assume some conditions. Here we suppose f_0 and h satisfy all conditions mentioned in the Theorem 2.2. Now we deduce the following result in bounded domain.

Lemma 5.1. *Let $r \geq a$ and f_r be a minimizer of Ψ_0 relative to $\overline{\mathcal{R}(r)^w}$. Let $\psi_r = u_{f_r} + \eta$ where u_r is a solution of* (1.1) *corresponding to f_r. Denote by $L_\alpha(\psi_r)$ the level set of ψ_r at height α. Then*

$$|L_\alpha(\psi_r) \cap \operatorname{supp}(f_r)| = 0, \quad \forall \alpha \in \mathbb{R}.$$

Proof. Let $r \geq a$. Suppose there exists $\hat{\alpha} \in \mathbb{R}$ such that $|L_{\hat{\alpha}}(\psi_r) \cap \operatorname{supp}(f_r)| > 0$. Let $A_{\hat{\alpha}} = L_{\hat{\alpha}}(\psi_r) \cap \operatorname{supp}(f_r)$. Then from equation (1.1), we have

$$-\Delta\psi_r = f_r - h = 0, \quad \text{a.e. in } A_{\hat{\alpha}}. \tag{5.1}$$

So $A_{\hat{\alpha}} \subset \operatorname{supp}(h)$. On the other hand, by Lemma 3.1, we have

$$\int_0^s f_r^\Delta(t)dt \leq \int_0^s f_0^\Delta(t)dt, \quad s > 0.$$

Then

$$\|f_r^\Delta\|_\infty \leq \|f_0^\Delta\|_\infty. \tag{5.2}$$

Since f_r^{Δ} is a rearrangement of f_r and f_0^{Δ} is a rearrangement of f_0, from (5.2) we obtain

$$\|f_r\|_\infty \leq \|f_0\|_\infty. \tag{5.3}$$

Finally, from (5.3) and condition (2.4), we deduce

$$\|f_r\|_\infty < \|h\|_{-\infty}. \tag{5.4}$$

which is a contradiction to (5.1). □

5.1. **Proof of Theorem 2.2.** Since the level sets of ψ_r have zero measure, similar to the proof of Lemma 4.3 we can claim that there exists minimizer f_r for P_r such that

$$f_r = \varphi o(u_{f_r} + \eta), \tag{5.5}$$

almost everywhere in B_r, for a decreasing unknown function φ. To prove the existence in unbounded domain, it is enough to show that the support of f_r does not have any dense point at the boundary of B_r when r is chosen large enough. Let $r_h > a$ be the smallest positive number for which $\operatorname{supp}(h) \subset B_{r_h}$. Since $b > \sqrt{3}a$, then similar to presented trend in the proof of Lemma 4.4 there exits $A \subset \operatorname{supp}(f_r)$ with positive measure and $D \subset \operatorname{supp}(h) \cap (\operatorname{supp}(f_r))^c$ such that $|D| > 0$ and $|z - y| < b$ for almost every $z, y \in D$. Then, for $r > r_h + 1$ we have

$$u_{f_r}(x) + \eta(x) \geq \frac{1}{4\pi}\Big(\frac{1}{2r}\|f_0\|_1 - \frac{1}{r - r_h - 1}\|h\|_1\Big), \quad \text{a.e. in } A, \tag{5.6}$$

$$u_{f_r}(z) + \eta(z) \leq \frac{a^2}{2}\|f_0\|_\infty - \frac{1}{8\pi b}\|h\|_{-\infty}|\operatorname{supp}(h)|, \quad \text{a.e. in } D. \tag{5.7}$$

Utilizing conditions mentioned in (2.4), there exists $C < 0$ such that

$$u_{f_r}(z) + \eta(z) \leq C \quad \text{a.e. in } D. \tag{5.8}$$

If we make r large enough we derive from (5.6) and (5.8),

$$\big(u_{f_r}(x) + \eta(x)\big) - \big(u_{f_r}(z) + \eta(z)\big) > 0,$$

for $x \in A$ and $z \in D$. Since $|A| > 0$ and $|D| > 0$, this is a contradiction to (5.5). Let r_0 be such that $r > r_0$, support of f_r does not touch the boundary of B_r where f_r is a solution of $P(r)$ for $r \geq r_0$. Then, $\operatorname{supp}(f_r)$ does not have any density point on the boundary of B_r for $r > r_0$. This means that $\operatorname{supp}(f_r)$ has a positive distance from the boundary of B_r. Hence $\operatorname{supp}(f_r) \subset B(r_0)$. Therefore we obtain the inclusion $S(r_0) \subset S$. It yields that P has a solution.

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References

[1] G. R. Burton; Rearrangements of functions, maximization of convex functionals and vortex rings, *Math. Ann.* 276, 225-253(1987).
[2] G. R. Burton; Variational problems on classes of rearrangements and multiple configurations for steady vorticies. *Ann. Inst. H. Poincaré – Anal. Non Linéaire*, 6, 295-319(1989).
[3] F. Cuccu, B. Emamizadeh and G. Porru; Nonlinear elastic membrance involving P-Lapllacian operator, *Electronic Journal of Differential Equations,* Vol. 2009, no. 49, 110 (2006).
[4] F. Cuccu, G. Porru and A. Vitolo; Optimization of the energy integral in two classes of rearrangements, *Nonlinear Studies*, Vol. 17, No. I, 23-35(2010).
[5] B. Emamizadeh; Steady vortex in a uniform shear flow of an ideal fluid, *Proc. Roy. Soc. Edinburgh*, 130A(2000), 801–812.

[6] B. Emamizadeh and J. V. Prajapat; Symmetry in rearrangemet optimization problems, *Electronic Journal of Differential Equations,* Vol. 2009, no. 149, 110(2009).
[7] B. Emamizadeh and M. Zivari-Rezapor; Rearrangement Optimization for Some Elleptic Equations, *J. Optim. Theory Appl.* 135: 367-379(2007).
[8] D. Gilbarg and N. S. Trudinger; Elliptic partial differential equations of second order. *Springer-Verlag*, second edt, New York, 1998.
[9] G. H. Hardy, J. E. Littlewood, G. Pólya; Inequalities, *Cambridge: Cambridge University Press*, 1934.
[10] M. Marras; Optimization in problems involving the P-Laplacian, *Electronic Journal of Differential Equations,* Vol. 2010, No. 02, 1–10 (2010).

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