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QUASILINEAR ELLIPTIC PROBLEMS WITH NONSTANDARD GROWTH

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Abstract. We prove the existence of solutions to Dirichlet problems associated with the $p(x)$-quasilinear elliptic equation

$$Au = -\operatorname{div} a(x, u, \nabla u) = f(x, u, \nabla u).$$

These solutions are obtained in Sobolev spaces with variable exponents.

1. Introduction

Partial differential equations with non-standard growth in Lebesgue and Sobolev spaces with variable exponent have been a very active field of investigation in recent years. The present line of investigation goes back to an article by Ková(c)ik and Rákosnik [9] in 1991.

The development, mainly by Ružička [15], of a theory modelling the behavior of electro-rhelogical fluid, an important class of non-Newtonian fluids, seems to have boosted a still far from completed effort to study and understand nonlinear PDEs involving variable exponents by several researches. Samko [16, 17, 18] working based on earlier Russian work (Sharapudinov [19] and Zhikov [20]), Fan and collaborators [5, 6, 7, 8] drawing inspiration from the study of differential equations(e.g. Marcellini [14]). More recently, an application to image processing was proposed by Chen, Levine and Rao [4]. To give the reader a feeling for the idea behind this application we mention that the proposed model requires the minimization over u of the energy,

$$E(u) = \int_\Omega |\nabla u(x)|^{p(x)} + |u(x) - I(x)|^2 dx, \tag{1.1}$$

where I is a given input. Recall that in the constant exponent case, the power $p = 2$ corresponds to isotropic smoothing, which corresponds to minimizing the energy,

$$E_2(u) = \int_\Omega |\nabla u(x)|^2 + |u(x) - I(x)|^2 dx. \tag{1.2}$$

Unfortunately, the smoothing will destroy all small details from the image, so this procedure is not very useful. Where as $p = 1$ gives total variations smoothing which

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corresponds to minimizing the energy,

$$E_1(u) = \int_\Omega |\nabla u(x)| + |u(x) - I(x)|^2 dx. \tag{1.3}$$

The benefit of this approach not only preserves edges, it also creates edges where there were none in the original image (the so-called staircase effect).

As the strengths and weaknesses of these two methods for image restoration are opposite, it is a natural to try to combine them. That was the idea of Chen, Levine and Rao [4], looking at E_1 and E_2 suggests that the appropriate energy is $E(u)$ (see 1.1), where $p(x)$, is a function varying between 1 and 2. This function should be close to 1 where there are likely to be edges, and close to 2 where there are likely not to be edges, and depends on the location , x, in the image. In this way the direction and speed of diffusion at each location depends on the local behavior.

We point out that, this model is linked with energy which can be associated to the $p(x)$-Laplacian operators; i.e.,

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u). \tag{1.4}$$

Moreover, the choice of the exponent yields a variational problem which has an Euler-Lagrange equation, and the solution can be found by solving corresponding evolutionary PDE.

In this paper, we consider a problem with potential applications. This problem has already been treated for constant exponent but it seems to be more realistic to assume the exponent to be variable. More precisely, we are interested in this paper to the following Dirichlet problems

$$\begin{gathered} Au = f(x, u, \nabla u) \quad \text{in } D'(\Omega), \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.5}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and $p \in \mathcal{C}(\bar\Omega)$, $p(x) > 1$, and where A is a Leray-Lions operator defined from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$ by the formula

$$Au = -\operatorname{div} a(x, u, \nabla u) \tag{1.6}$$

and where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies the growth condition

$$|f(x, r, \xi)| \leq g(x) + |r|^{\eta(x)} + |\xi|^{\delta(x)}, \tag{1.7}$$

where $0 \leq \eta(x) < p(x) - 1$ and $0 \leq \delta(x) < \big(p(x) - 1\big)/p'(x)$. In the case of non-variables exponents, Boccardo, Murat and Puel have studied in [3] the problem (1.5) with f satisfying the condition

$$|f(x, r, \xi)| \leq h(|r|)(1 + |\xi|^p), \tag{1.8}$$

where h is an increasing function from $\mathbb{R}^+ \to \mathbb{R}^+$.

Kuo and Tsai [10], proved the existence results under the growth condition

$$|f(x, r, \xi)| \leq C(1 + |r|^\delta + |\xi|^p). \tag{1.9}$$

However, in the case of variable exponent, we can list the work of Fan and Zhang [11] who studied the particular case

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u) \quad x \in \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.10}$$

where f satisfies the growth condition

$$|f(x,r)| \le C_1 + C_2|r|^{\beta(x)-1}, \tag{1.11}$$

with $1 \le \beta < p^- := \operatorname{ess\,inf}_{x\in\overline{\Omega}} p(x)$ and we denote $p^+ := \operatorname{ess\,sup}_{x\in\overline{\Omega}} p(x)$.

The aim of this article is to study the existence of a solution to the problem (1.5) in the Sobolev spaces with variable exponents. The model example of our problem is

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = |u|^{\eta(x)} + |\nabla u|^{\delta(x)} + g(x) \quad \text{in } D'(\Omega) \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.12}$$

where $p \in \mathcal{C}_+(\Omega)$, $1 < p^- \le p(x) \le p^+ < N$, $g \in L^{p'(x)}(\Omega)$, η and δ are two continuous functions on Ω such that $0 \le \eta(x) < p(x) - 1$ and $0 \le \delta(x) < \frac{p(x)-1}{p'(x)}$. Let us point that our work can be seen as a generalization of [11], [10] and [3] in the sense that in the first work the authors have considered $Au = -\triangle_{p(x)} u$, $f = f(x,u)$, however in the two last works the exponent is constant $p(x) = p$.

This article is organized as follows: In section 2, we introduce the mathematical preliminaries . In section 3, we introduce basic assumptions and we give and prove some main lemmas. Section 4, is devoted to the proof of our general existence result.

2. Preliminaries

For each open bounded subset Ω of $\mathbb{R}^N$ $(N \ge 2)$, we denote

$$\mathcal{C}_+(\bar{\Omega}) = \{p \in \mathcal{C}(\bar{\Omega}) : p(x) > 1 \text{ for any } x \in \bar{\Omega}\},$$

and we define the variable exponent Lebesque space by:

$$L^{p(x)}(\Omega) = \{u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\big\{\lambda > 0, \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \le 1\big\}.$$

Remark 2.1. Note that the variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces (Ková(c)ik and Rákosnik [9, Theorem 2.5]), the Hölder inequality holds (Ková(c)ik and Rákosnik [9, Theorem 2.1]), they are reflexive if and only if $1 < p^- \le p^+ < \infty$, (Ková(c)ik and Rákosnik [9, Coro. 2.7]) and continuous functions are dense, if $p^+ < \infty$ (Ková(c)ik and Rákosnik [9, Theorem 2.11]).

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [12], [22]).

Proposition 2.2 (Generalized Hölder inequality [12, 22])**.**

(i) *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$|\int_\Omega uv dx| \le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)|u|_{p(x)}|v|_{p'(x)}.$$

(ii) *If $p_1(x), p_2(x) \in \mathcal{C}_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and the imbedding is continuous.*

Proposition 2.3 ([12],[21]). *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies*
$$|f(x,s)| \le a(x) + b|s|^{p_1(x)/p_2(x)} \quad \text{for any } x \in \Omega, s \in \mathbb{R},$$
where $p_1, p_2 \in \mathcal{C}_+(\bar{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $a(x) \ge 0$ and $b \ge 0$ is a constant, then the Nemytskii operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_f(u))(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.4 ([12], [22]). *Let $\rho(u) = \int_\Omega |u|^{p(x)}dx$ for $u \in L^{p(x)}(\Omega)$. Then the following assertions hold:*

- (i) *$|u|_{p(x)} < 1$ (resp. $= 1$, > 1) if and only if $\rho(u) < 1$ (resp. $= 1$, > 1);*
- (ii) *$|u|_{p(x)} > 1$ implies $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$; $|u|_{p(x)} < 1$ implies $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
- (iii) *$|u|_{p(x)} \to 0$ if and only if $\rho(u) \to 0$; $|u|_{p(x)} \to \infty$ if and only if $\rho(u) \to \infty$.*

We define the variable Sobolev space by
$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$
with the norm
$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)} \quad \forall u \in W^{1,p(x)}(\Omega). \tag{2.1}$$
We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $p^*(x) = \frac{Np(x)}{N-p(x)}$, for $p(x) < N$.

Proposition 2.5 ([12]).

- (i) *Assuming $p^- > 1$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
- (ii) *if $q \in \mathcal{C}_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.*
- (iii) *There is a positive constant C, such that*
$$|u|_{p(x)} \le C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Remark 2.6. By (iii) of Proposition 2.5, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(x)}$.

3. Basic assumptions and some Lemmas

Let $p \in \mathcal{C}_+(\bar{\Omega})$ such that $1 < p^- \le p(x) \le p^+ < N$, and denote
$$Au = -\operatorname{div} a(x, u, \nabla u),$$
where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a carathéodory function satisfying the following assumptions:

- (H1) $|a(x,r,\xi)| \le \beta[k(x) + |r|^{p(x)-1} + |\xi|^{p(x)-1}]$;
- (H2) $[a(x,r,\xi) - a(x,r,\eta)](\xi - \eta) > 0$ for all $\xi \ne \eta \in \mathbb{R}^N$;
- (H3) $a(x,r,\xi)\xi \ge \alpha|\xi|^{p(x)}$;

for a.e. $x \in \Omega$, all $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $k(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\beta, \alpha > 0$.

Let f be a Carathéodory function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ such that

- (H4) $|f(x,r,\xi)| \le g(x) + |r|^{\eta(x)} + |\xi|^{\delta(x)}$ for a.e. $x \in \Omega$, all $(r,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $g : \Omega \to \mathbb{R}^+$, $g \in L^{p'(x)}(\Omega)$ and $0 \le \eta(x) < p(x) - 1$, $0 \le \delta(x) < \frac{p(x)-1}{p'(x)}$.

Definition 3.1. Let Y be a separable reflexive Banach space. An operator B defined from Y to its dual Y^* is called an operator of the calculus of variations type, if B is bounded and is of the form

$$B(u) = B(u, u), \tag{3.1}$$

where $(u, v) \to B(u, v)$ is an operator defined from $Y \times Y$ into Y^* which satisfying the following properties:

$$\begin{gathered}\text{For } u \in Y \text{, the mapping } v \to B(u, v) \text{ is bounded hemicontinuous}\\ \text{from } Y \text{ into } Y^* \text{ and } (B(u, u) - B(u, v), u - v) \geq 0;\end{gathered} \tag{3.2}$$

for $v \in Y$, the mapping $u \to B(u, v)$ is bounded hemicontinous from Y into Y^*;

$$\begin{gathered}\text{if } u_n \rightharpoonup u \text{ in } Y \text{ and if } (B(u_n, u_n) - B(u_n, u), u_n - u) \to 0 \text{ , then}\\ B(u_n, v) \rightharpoonup B(u, v) \text{ in } Y^* \text{ for all } v \in Y.\end{gathered} \tag{3.3}$$

and

$$\begin{gathered}\text{if } u_n \rightharpoonup u \text{ in } Y \text{ and if } B(u_n, v) \rightharpoonup \psi \text{ in } Y^*\text{, then } (B(u_n, v), u_n) \to\\ (\psi, u).\end{gathered} \tag{3.4}$$

The symbol $\rightharpoonup$ denote the weak convergence.

Lemma 3.2. *Assume that* (H1)–(H4) *are satisfied and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p(x)}(\Omega)$ *and let* $u \in W_0^{1,p(x)}(\Omega)$. *If* $u_n \rightharpoonup u$ *in* $W_0^{1,p(x)}(\Omega)$*, then for some subsequence denoted again* (u_n)*, we have*

$$a(x, u_n, \nabla v) \to a(x, u, \nabla v) \quad \text{in } \left(L^{p'(x)}(\Omega)\right)^N, \forall v \in W_0^{1,p(x)}(\Omega).$$

Proof. From (H1), it follows that

$$\begin{aligned}
&|a(x, u_n, \nabla v)|^{p'(x)}\\
&\leq \beta^{p'(x)}[k(x) + |u_n|^{p(x)-1} + |\nabla v|^{p(x)-1}]^{p'(x)}\\
&\leq (\beta+1)^{p'^+} 2^{p'^+-1}[k(x)^{p'(x)} + 2^{p'^+-1}(|u_n|^{(p(x)-1)p'(x)} + |\nabla v|^{(p(x)-1)p'(x)})]\\
&\leq (\beta+1)^{p'^+} 2^{2(p'^+-1)}[k(x)^{p'(x)} + |u_n|^{p(x)} + |\nabla v|^{p(x)}].
\end{aligned} \tag{3.5}$$

In the second inequality above we have used [2]. Since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and according to proposition 2.5, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}$ is compact and continuous, there exists a subsequence denoted again (u_n) such that, $u_n \to u$ in $L^{p(x)}(\Omega)$, and therefore a.e. in Ω; hence

$$|a(x, u_n, \nabla v)|^{p'(x)} \to |a(x, u, \nabla v)|^{p'(x)} \quad \text{a.e. in } \Omega, \tag{3.6}$$

and

$$\begin{aligned}
&(\beta+1)^{p'^+} 2^{2(p'^+-1)}[k(x)^{p'(x)} + |u_n|^{p(x)} + |\nabla v|^{p(x)}]\\
&\to (\beta+1)^{p'^+} 2^{2(p'^+-1)}[k(x)^{p'(x)} + |u|^{p(x)} + |\nabla v|^{p(x)}] \text{ a.e. in } \Omega.
\end{aligned} \tag{3.7}$$

For each measurable subset E, we have

$$\begin{aligned}
&\int_E |a(x, u_n, \nabla v)|^{p'(x)} dx\\
&\leq (\beta+1)^{p'^+} 2^{2(p'^+-1)}\Big[\int_E k(x)^{p'(x)} dx + \int_E |u_n|^{p(x)} dx + \int_E |\nabla v|^{p(x)} dx\Big],
\end{aligned} \tag{3.8}$$

in view of (3.7) and (3.8), there exists $\eta(\varepsilon)$ such that

$$\int_E |a(x,u_n,\nabla v)|^{p'(x)}dx < \varepsilon$$

for all E with $\mathrm{meas}(E) < \eta(\varepsilon)$, which implies the equi-integrability of $a(x,u_n,\nabla v)$. Finaly by Vitali's theorem,

$$a(x,u_n,\nabla v) \to a(x,u,\nabla v) \quad \text{in } \big(L^{p'(x)}(\Omega)\big)^N. \tag{3.9}$$

□

Lemma 3.3. *Let $g \in L^{r(x)}(\Omega)$ and $g_n \in L^{r(x)}(\Omega)$ with $|g_n|_{L^{r(x)}(\Omega)} \le C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega)$.*

Proof. Let

$$E(N) = \{x \in \Omega : |g_n(x) - g(x)| \le 1, \, \forall n \ge N\}.$$

Since $\mathrm{meas}(E(N)) \to \mathrm{meas}(\Omega)$ as $N \to \infty$, and setting

$$\mathcal{F} = \{\varphi_N \in L^{r'(x)}(\Omega) : \varphi_N \equiv 0 \text{ a.e. in } \Omega\backslash E(N)\},$$

we shall show that $\mathcal{F}$ is dense in $L^{r'(x)}(\Omega)$. Let $f \in L^{r'(x)}(\Omega)$, we set

$$f_N(x) = \begin{cases} f(x) & \text{if } x \in E(N), \\ 0 & \text{if } x \in \Omega\backslash E(N). \end{cases}$$

Then

$$\begin{aligned}
\rho_{r'(x)}(f_N - f) &= \int_\Omega |f_N(x) - f(x)|^{r'(x)}dx \\
&= \int_{E(N)} |f_N(x) - f(x)|^{r'(x)}dx + \int_{\Omega\backslash E(N)} |f_N(x) - f(x)|^{r'(x)}dx \\
&= \int_{\Omega\backslash E(N)} |f(x)|^{r'(x)}dx \\
&= \int_\Omega |f(x)|^{r'(x)}\chi_{\Omega\backslash E(N)}dx
\end{aligned}$$

Taking $\psi_N(x) = |f(x)|^{r'(x)}\chi_{\Omega\backslash E(N)}$ for almost every x in Ω, we obtain

$$\psi_N \to 0 \text{ a.e. in } \Omega \quad \text{and} \quad |\psi_N| \le |f|^{r'(x)}.$$

Using the dominated convergence theorem, we have $\rho_{r'(x)}(f_N - f) \to 0$ as $N \to \infty$; therefore $f_N \to f$ in $L^{r'(x)}(\Omega)$. Consequently $\mathcal{F}$ is dense in $L^{r'(x)}(\Omega)$. Now we shall show that

$$\lim_{n\to\infty} \int_\Omega \varphi(x)\big(g_n(x) - g(x)\big)dx = 0, \quad \forall\, \varphi \in \mathcal{F}.$$

Since $\varphi \equiv 0$ in $\Omega \setminus E(N)$, it suffices to prove that

$$\int_{E(N)} \varphi(x)(g_n(x) - g(x))dx \to 0 \quad \text{as } n \to \infty.$$

We set $\phi_n = \varphi\big(g_n - g\big)$. Since $|\varphi(x)||g_n(x) - g(x)| \le |\varphi(x)|$ a.e. in $E(N)$ and $\phi_n \to 0$ a.e. in Ω, thanks to the dominated convergence theorem, we deduce $\phi_n \to 0$ in $L^1(\Omega)$. Which implies that

$$\lim_{n\to\infty} \int_\Omega \varphi(x)(g_n(x) - g(x))dx = 0, \quad \forall\, \varphi \in \mathcal{F}$$

Now, by the density of $\mathcal{F}$ in $L^{r'(x)}(\Omega)$, we conclude that

$$\lim_{n\to\infty}\int_\Omega \varphi g_n dx = \int_\Omega \varphi g dx, \quad \forall \varphi \in L^{r'(x)}(\Omega).$$

Finally $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega)$. $\square$

Lemma 3.4. *Assume* (H1)–(H4)*, and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p(x)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,p(x)}(\Omega)$ *and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\,dx \to 0. \tag{3.10}$$

Then, $u_n \to u$ *in* $W_0^{1,p(x)}(\Omega)$.

Proof. Let $D_n = [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)$. Then by (H2), D_n is a positive function, and by (3.10) $D_n \to 0$ in $L^1(\Omega)$. Extracting a subsequence, still denoted by u_n, we can write $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ which implies $u_n \to u$ a.e. in Ω, Similarly $D_n \to 0$ a.e. in Ω. Then there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \setminus B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $k(x) < \infty$, $u_n(x) \to u(x)$, $D_n(x) \to 0$.

Defining $\xi_n = \nabla u_n(x)$, $\xi = \nabla u(x)$, we have

$$\begin{aligned} D_n(x) &= [a(x,u_n,\xi_n) - a(x,u_n,\xi)](\xi_n - \xi)\\ &= a(x,u_n,\xi_n)\xi_n + a(x,u_n,\xi)\xi - a(x,u_n,\xi_n)\xi - a(x,u_n,\xi)\xi_n\\ &\geq \alpha|\xi_n|^{p(x)} + \alpha|\xi|^{p(x)} - \beta(k(x) + |u_n|^{p(x)-1} + |\xi_n|^{p(x)-1})|\xi|\\ &\quad - \beta(k(x) + |u_n|^{p(x)-1} + |\xi|^{p(x)-1})|\xi_n|\\ &\geq \alpha|\xi_n|^{p(x)} - C_x\big[1 + |\xi_n|^{p(x)-1} + |\xi_n|\big], \end{aligned} \tag{3.11}$$

where C_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \to u(x)$ we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then by a standard argument $|\xi_n|$ is bounded uniformly with respect to n, indeed (3.11) becomes

$$D_n(x) \geq |\xi_n|^{p(x)}\big(\alpha - \frac{C_x}{|\xi_n|^{p(x)}} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p(x)-1}}\big). \tag{3.12}$$

If $|\xi_n| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$ which gives a contradiction. Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and by the continuity of a we obtain

$$[a(x,u(x),\xi^*) - a(x,u(x),\xi)](\xi^* - \xi) = 0. \tag{3.13}$$

In view of (H2), we have $\xi^* = \xi$. The uniqueness of the cluster point implies

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e.in } \Omega. \tag{3.14}$$

Since the sequence $a(x,u_n,\nabla u_n)$ is bounded in $(L^{p'(x)}(\Omega))^N$, and $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e. in Ω, Lemma 3.3 implies

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{in } (L^{p'(x)}(\Omega))^N \text{ a.e. in } \Omega. \tag{3.15}$$

We set $\bar{y}_n = a(x,u_n,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,u,\nabla u)\nabla u$. As in [3] we can write

$$\bar{y}_n \to \bar{y} \text{in} L^1(\Omega).$$

By (H3) we have

$$\alpha|\nabla u_n|^{p(x)} \leq a(x,u_n,\nabla u_n)\nabla u_n.$$

Let $z_n = |\nabla u_n|^{p(x)}$, $z = |\nabla u|^{p(x)}$, $y_n = \frac{\bar{y}_n}{\alpha}$, and $y = \frac{\bar{y}}{\alpha}$. Then by Fatou's lemma,

$$\int_\Omega 2y\,dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|\,dx; \tag{3.16}$$

i.e., $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|dx$. Then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|dx \leq 0, \tag{3.17}$$

this implies

$$\nabla u_n \to \nabla u \quad \text{in } (L^{p(x)}(\Omega))^N. \tag{3.18}$$

Hence $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, which completes the present proof. □

For $v \in W_0^{1,p(x)}(\Omega)$, we associate the Nemytskii operator F with respect to f, defined by

$$F(v, \nabla v)(x) = f(x, v, \nabla v)) \quad \text{a.e. } x \text{ in } \Omega. \tag{3.19}$$

Lemma 3.5. *The mapping $v \mapsto F(v, \nabla v)$ is continuous from the space $W_0^{1,p(x)}(\Omega)$ to the space $L^{p'(x)}(\Omega)$.*

Proof. By (H4), we have

$$|f(x, r, \xi)| \leq g(x) + |r|^{\eta(x)} + |\xi|^{\delta(x)}, \tag{3.20}$$

thus, as in [2],

$$|f(x, r, \xi)|^{p'(x)} \leq 2^{2(p'^+ - 1)}\Big(g(x)^{p'(x)} + |r|^{p'(x)\eta(x)} + |\xi|^{p'(x)\delta(x)}\Big). \tag{3.21}$$

Let E be a measurable subset of Ω. Then

$$\int_E |f(x, v, \nabla v)|^{p'(x)} dx \leq C\Big(\int_E g(x)^{p'(x)} dx + \int_E |v|^{p'(x)\eta(x)} dx + \int_E |\nabla v|^{p'(x)\delta(x)} dx\Big),$$

with $0 \leq \eta(x) < p(x) - 1$ implying $0 \leq p'(x)\eta(x) < p(x)$ and

$$0 \leq \delta(x) < \frac{p(x) - 1}{p'(x)} \Rightarrow 0 \leq p'(x)\delta(x) < p(x) - 1. \tag{3.22}$$

For any sequence $(v_n)_n$ such that $v_n \to v$ in $W_0^{1,p(x)}(\Omega)$, we shall show that $F(v_n, \nabla v_n) \to F(v, \nabla v)$ in $W_0^{1,p(x)}(\Omega)$. We have $v_n \to v$ in $W_0^{1,p(x)}(\Omega)$ implies that

$$v_n \to v \quad \text{a.e. in } \Omega,$$
$$\nabla v_n \to \nabla v \quad \text{a.e. in } \Omega.$$

Since f is a carathéodory function,

$$|f(x, v_n, \nabla v_n)|^{p'(x)} \to |f(x, v, \nabla v)|^{p'(x)} \quad \text{a.e. in } \Omega,$$
$$|f(x, v_n, \nabla v_n)|^{p'(x)} \leq C\Big(g(x)^{p'(x)} + |v_n|^{p'(x)\eta(x)} + |\nabla v_n|^{p'(x)\delta(x)}\Big),$$

and

$$\begin{aligned} &C\Big(g(x)^{p'(x)} + |v_n|^{p'(x)\eta(x)} + |\nabla v_n|^{p'(x)\delta(x)}\Big) \\ &\to C\Big(g(x)^{p'(x)} + |v|^{p'(x)\eta(x)} + |\nabla v|^{p'(x)\delta(x)}\Big), \end{aligned}$$

Hence, by Vitali's theorem we deduce that

$$f(x, v_n, \nabla v_n) \to f(x, v, \nabla v) \quad \text{in } L^{p'(x)}(\Omega); \tag{3.23}$$

i.e., $v \mapsto F(v, \nabla v)$ is continuous. $\square$

4. Existence result

Consider the problem

$$\begin{gathered} -\operatorname{div} a(x, u, \nabla u) = f(x, u, \nabla u) \quad \text{in } D'(\Omega), \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{4.1}$$

Theorem 4.1. *Under the assumptions (H1)–(H4), there exists at least one solution $u \in W_0^{1,p(x)}(\Omega)$ of the problem* (4.1).

Remark 4.2. (1) Theorem 4.1, generalizes to Sobolev spaces with variables exponent the analogous statement in [1]. (2) Theorem 4.1, generalizes the analogous one in [11], in the sense that in [11] the authors have considered the particular case $Au = -\triangle_{p(x)} u$ and $f = f(x, u)$. (3) In the case where $p(x) = p = cte$ in the theorem 4.1 we obtain the results of [10] and [3].

Proof of the Theorem 4.1. This proof is done in two steps.
Step 1 We show that the operator $B : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$B(v) := A(v) - f(x, v, \nabla v)$$

is calculus variational.
Assertion 1. Let

$$B(u, v) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla v) - f(x, u, \nabla u).$$

then $B(v) = B(v, v)$ for all $v \in W_0^{1,p(x)}(\Omega)$.
Assertion 2. The operator $v \mapsto B(u, v)$ is bounded for all $u \in W_0^{1,p(x)}(\Omega)$.

Let $\psi \in W_0^{1,p(x)}(\Omega)$, we have

$$\langle B(u, v), \psi\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla v) \frac{\partial \psi}{\partial x_i} dx - \int_\Omega f(x, u, \nabla u)\psi(x) dx. \tag{4.2}$$

From Hölder's inequality, the growth condition (H1) and as in (3.5), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla v) \frac{\partial \psi}{\partial x_i} dx \\
&= \int_\Omega a(x, u, \nabla v) \nabla \psi dx \\
&\le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) |a(x, u, \nabla v)|_{(L^{p'(x)}(\Omega))^N} |\nabla \psi|_{(L^{p(x)}(\Omega))^N} \\
&\le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \Big(\int_\Omega |a(x, u, \nabla v)|^{p'(x)} dx \Big)^{1/\gamma} \|\psi\| \\
&\le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big) \Big(\int_\Omega \Big[\beta(k(x) + |u|^{p(x)-1} + |\nabla v|^{p(x)-1})\Big]^{p'(x)} dx \Big)^{1/\gamma} \|\psi\|
\end{aligned}$$

$$\le C'\Big(\int_\Omega k(x)^{p'(x)}dx + \int_\Omega |u|^{p(x)}dx + \int_\Omega |\nabla v|^{p(x)}dx\Big)^{1/\gamma}\|\psi\|,$$

where

$$\gamma = \begin{cases} p'^- & \text{if } |a(x,u,\nabla v)|_{(L^{p'(x)}(\Omega))^N} > 1,\\ p'^+ & \text{if } |a(x,u,\nabla v)|_{(L^{p'(x)}(\Omega))^N} \le 1,\end{cases}$$

we recall that $\|\psi\|$ its equivalent to the norm $|\nabla\psi|_{p(x)}$ on $W_0^{1,p(x)}(\Omega)$ (see Remark(2.6)). We have, $k \in L^{p'(x)}(\Omega)$, $u \in W_0^{1,p(x)}(\Omega)$ and $v \in W_0^{1,p(x)}(\Omega)$. Therefore,

$$\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla v)\frac{\partial\psi}{\partial x_i} \le C\|\psi\|. \tag{4.3}$$

Similarly,

$$\begin{aligned}\int_\Omega f(x,u,\nabla u)\psi dx &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)|f(x,u,\nabla u)|_{L^{p'(x)}(\Omega)}|\psi|_{L^{p(x)}(\Omega)}\\ &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\big(\int_\Omega |f(x,u,\nabla u)|^{p'(x)}dx\Big)^{1/\alpha}\|\psi\|,\end{aligned}$$

where

$$\alpha = \begin{cases} p'^- & \text{if } |f(x,u,\nabla u)|_{L^{p'(x)}(\Omega)} > 1,\\ p'^+ & \text{if } |f(x,u,\nabla u)|_{L^{p'(x)}(\Omega)} \le 1.\end{cases}$$

Then, by (H4),

$$\begin{aligned}&\int_\Omega f(x,u,\nabla u)\psi dx\\ &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\|\psi\|\big[\int_\Omega (g(x) + |u|^{\eta(x)} + |\nabla u|^{\delta(x)})^{p'(x)}dx\big]^{1/\alpha}\\ &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\|\psi\|2^{2(p'^+-1)\frac{1}{\alpha}}\big[\int_\Omega (g(x)^{p'(x)} + |u|^{\eta(x)p'(x)} + |\nabla u|^{\delta(x)p'(x)})dx\big]^{1/\alpha}\\ &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\|\psi\|2^{\frac{2(p'^+-1)}{\alpha}}\big[\int_\Omega g(x)^{p'(x)}dx + \int_\Omega |u|^{\eta(x)p'(x)}dx\\ &\quad + \int_\Omega |\nabla u|^{\delta(x)p'(x)})dx\big]^{1/\alpha}\\ &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big)\|\psi\|2^{\frac{2(p'^+-1)}{\alpha}}\big[\int_\Omega g(x)^{p'(x)}dx + |u|^\beta_{L^{p'\eta}} + |\nabla u|^\theta_{L^{p'\delta}}\big]^{1/\alpha},\end{aligned}$$

where

$$\beta = \begin{cases} (\eta p')^+ & \text{if } |u|_{L^{p'\eta}} > 1\\ (\eta p')^- & \text{if } |u|_{L^{p'\eta}} \le 1\,,\end{cases} \quad \theta = \begin{cases} (\delta p')^+ & \text{if } |\nabla u|_{L^{p'\delta}} > 1\\ (\delta p')^- & \text{if } |\nabla u|_{L^{p'\delta}} \le 1\,.\end{cases}$$

Since $0 \le \eta(x) < p(x) - 1$, this implies $0 \le \eta(x)p'(x) < p(x)$. Then there exists a constant $C_1 > 0$ such that

$$|u|_{L^{p'\eta}} \le C_1|u|_{L^{p(x)}(\Omega)} \tag{4.4}$$

and $0 \le \delta(x) < (p(x) - 1)/p'(x)$, this implies $0 \le \delta(x)p'(x) < p(x) - 1 < p(x)$. Then there exists a constant $C_2 > 0$ such that

$$|\nabla u|_{L^{p'\delta}} \le C_2|\nabla u|_{L^{p(x)}(\Omega)}\,. \tag{4.5}$$

Since $u \in W_0^{1,p(x)}(\Omega)$, there exists a constant $C_3 > 0$ such that

$$\int_\Omega f(x,u,\nabla u)\psi dx \le C_3\|\psi\|\,. \tag{4.6}$$

Therefore, there exists a constant $C_0 > 0$ such that

$$|\langle B(u,v),\psi\rangle| \le C_0\|\psi\| \quad \text{for all } u,v \in W_0^{1,p(x)}(\Omega); \tag{4.7}$$

i.e., $\langle B(u,v),\psi\rangle$ is bounded in $W_0^{1,p(x)}(\Omega) \times W_0^{1,p(x)}(\Omega)$.

We claim that $v \mapsto B(u,v)$ is hemicontinuous for all $u \in W_0^{1,p(x)}(\Omega)$; i.e., the operator $\lambda \mapsto \langle B(u,v_1+\lambda v_2),\psi\rangle$ is continuous for all $v_1,v_2,\psi \in W_0^{1,p(x)}(\Omega)$. For this, we need Lemma 3.3. Since a_i is a carathéodory function,

$$a_i(x,u,\nabla(v_1+\lambda v_2)) \to a_i(x,u,\nabla v_1) \quad \text{a.e. in } \Omega \text{ as } \lambda \mapsto 0. \tag{4.8}$$

and, by (H1),

$$|a(x,u,\nabla(v_1+\lambda v_2))| \le \beta(k(x) + |u|^{p(x)-1} + |\nabla(v_1+\lambda v_2)|^{p(x)-1})\,. \tag{4.9}$$

Further, $(a(x,u,\nabla(v_1+\lambda v_2)))_\lambda$ is bounded in $(L^{p'(x)}(\Omega))^N$; thus, by Lemma 3.3,

$$a(x,u,\nabla(v_1+\lambda v_2)) \rightharpoonup a(x,u,\nabla v_1) \quad \text{in } (L^{p'(x)}(\Omega))^N \text{ as } \lambda \to 0, \tag{4.10}$$

Hence,

$$\begin{aligned}
&\lim_{\lambda\to 0}\langle B(u,v_1+\lambda v_2),\psi\rangle\\
&= \lim_{\lambda\to 0}\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla(v_1+\lambda v_2))\frac{\partial\psi}{\partial x_i}dx - \int_\Omega f(x,u,\nabla u)\psi dx\\
&= \sum_{i=1}^N \int_\Omega a_i(x,u,\nabla v_1)\frac{\partial\psi}{\partial x_i}dx - \int_\Omega f(x,u,\nabla u)\psi dx\\
&= \langle B(u,v_1),\psi\rangle \text{ for all } v_1,v_2,\psi \in W_0^{1,p(x)}(\Omega)
\end{aligned}$$

Similarly, we show that $u \mapsto B(u,v)$ is bounded and hemicontinuous for all $v \in W_0^{1,p(x)}(\Omega)$. Indeed. By (H4), we have $(f(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)))_\lambda$ is bounded in $L^{p'(x)}(\Omega)$, and since f is a carathéodory function,

$$f(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)) \to f(x,u_1,\nabla u_1) \quad \text{as } \lambda \to 0, \tag{4.11}$$

Hence, Lemma 3.3 gives

$$f(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)) \rightharpoonup f(x,u_1,\nabla u_1) \quad \text{in } L^{p'(x)}(\Omega) \text{ as } \lambda \to 0, \tag{4.12}$$

On the other hand, as in(4.10), we have

$$a(x,u_1+\lambda u_2,\nabla v) \rightharpoonup a(x,u_1,\nabla v) \quad \text{in } L^{p'(x)}(\Omega) \text{ as } \lambda \to 0. \tag{4.13}$$

Combining (4.12) and (4.13), we conclude that $u \mapsto B(u,v)$ is bounded and hemicontinuous.

Assertion 3. From (H2), we have

$$\langle B(u,u) - B(u,v), u-v\rangle = \sum_{i=1}^N \int_\Omega (a_i(x,u,\nabla u) - a_i(x,u,\nabla v))\big(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\big)dx > 0 \tag{4.14}$$

Assertion 4. Assume that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, and $\langle B(u_n,u_n) - B(u_n,u), u_n - u\rangle \to 0$ as $n \to \infty$, we claim that $B(u_n,v) \rightharpoonup B(u,v)$ in $W^{-1,p'(x)}(\Omega)$. We have $\langle B(u_n,u_n) - B(u_n,u), u_n - u\rangle \to 0$ as $n \to \infty$,

$$\begin{aligned}
&\langle \sum_{i=1}^N -\big[\frac{\partial}{\partial x_i} a_i(x,u_n,\nabla u_n) + a_i(x,u_n,\nabla u)\big], u_n - u\rangle \\
&= \sum_{i=1}^N \int_\Omega \big[a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u)\big]\big(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\big)dx \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

Then by Lemma 3.4, we have $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ and it follows from Lemma 3.5 that

$$f(x,u_n,\nabla u_n) \to f(x,u,\nabla u) \quad \text{in } L^{p'(x)}(\Omega). \tag{4.15}$$

since $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $v \in W_0^{1,p(x)}(\Omega)$, by Lemma 3.2, $a_i(x,u_n,\nabla v) \to a_i(x,u,\nabla v)$ in $L^{p'(x)}(\Omega)$. Consequently,

$$\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial \psi}{\partial x_i}dx \to \int_\Omega a_i(x,u,\nabla v)\frac{\partial \psi}{\partial x_i}dx. \tag{4.16}$$

On the other hand, we have $f(x,u_n,\nabla u_n) \to f(x,u,\nabla u)$ in $L^{p'(x)}(\Omega)$, thus weakly. Since $\psi \in W_0^{1,p(x)}(\Omega)$, we have $\psi \in L^{p(x)}(\Omega)$. Then

$$\int_\Omega f(x,u_n,\nabla u_n)\psi dx \to \int_\Omega f(x,u,\nabla u)\psi dx \quad \text{as } n \to \infty$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \langle B(u_n,v),\psi\rangle &= \lim_{n\to\infty} \Big(\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla v)\frac{\partial \psi}{\partial x_i}dx - \int_\Omega f(x,u_n,\nabla v_n)\psi dx\Big) \\
&= \sum_{i=1}^N \int_\Omega a_i(x,u,\nabla v)\frac{\partial \psi}{\partial x_i}dx - \int_\Omega f(x,u,\nabla u)\psi dx \\
&= \langle B(u,v),\psi\rangle \quad \text{for all } \psi \in W_0^{1,p(x)}(\Omega).
\end{aligned}$$

Assertion 5. Assume $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $B(u_n,v) \rightharpoonup \psi$ in $W^{-1,p'(x)}(\Omega)$. We claim that $\langle B(u_n,v),u_n\rangle \to \langle \psi,u\rangle$. Thanks to $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, we obtain by Lemma 3.2,

$$a_i(x,u_n,\nabla v) \to a_i(x,u,\nabla v) \quad \text{in } L^{p'(x)}(\Omega) \text{ as } n \to \infty. \tag{4.17}$$

Such that

$$\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial u_n}{\partial x_i}dx \to \int_\Omega a_i(x,u,\nabla v)\frac{\partial u}{\partial x_i}dx. \tag{4.18}$$

Hence together with

$$\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla v)\frac{\partial u}{\partial x_i}dx - \int_\Omega f(x,u_n,\nabla u_n)u dx \to \langle \psi,u\rangle, \tag{4.19}$$

we have

$$\langle B(u_n,v),u_n\rangle = \sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla v)\frac{\partial u_n}{\partial x_i}dx - \int_\Omega f(x,u_n,\nabla u_n)u_n dx$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \Big[\int_\Omega a_i(x, u_n, \nabla v)\big(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\big) dx + \int_\Omega a_i(x, u_n, \nabla v) \frac{\partial u}{\partial x_i} dx \Big] \\
&\quad - \int_\Omega f(x, u_n, \nabla u_n) u dx - \int_\Omega f(x, u_n, \nabla u_n)(u_n - u) dx.
\end{aligned}
$$

But in view of (4.17) and (4.18), we obtain

$$
\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla v)\big(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\big) dx \to 0. \tag{4.20}
$$

On the other hand, by Hölder's inequality,

$$
\begin{aligned}
&\int_\Omega |f(x, u_n, \nabla u_n)(u_n - u)| dx \\
&\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big) |f(x, u_n, \nabla u_n)|_{L^{p'(x)}(\Omega)} |u_n - u|_{L^{p(x)}(\Omega)} \\
&\le C |u_n - u|_{L^{p(x)}(\Omega)} \to 0 \quad \text{as } n \to \infty;
\end{aligned}
$$

i.e.,

$$
\int_\Omega f(x, u_n, \nabla u_n)(u_n - u) dx \to 0 \quad \text{as } n \to \infty. \tag{4.21}
$$

Thanks to (4.19), (4.20) and (4.21), we conclude that

$$
\lim_{n\to\infty} \langle B(u_n, v), u_n \rangle = \langle \psi, u \rangle. \tag{4.22}
$$

Step 2 We claim that the operator B satisfies the coercivity condition

$$
\lim_{\|v\|\to\infty} \frac{\langle B(v), v\rangle}{\|v\|} = +\infty. \tag{4.23}
$$

Since

$$
\langle B(v), v \rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, v, \nabla v) \ \frac{\partial v}{\partial x_i} \, dx - \int_\Omega f(x, v, \nabla v) v \, dx, \tag{4.24}
$$

Then, by (H3),

$$
\langle Bv, v \rangle \ge \alpha \|v\|^{p(x)} - \int_\Omega f(x, v, \nabla v) v \, dx \tag{4.25}
$$

In view of (H4),

$$
\int_\Omega f(x, v, \nabla v) v dx \le \int_\Omega g(x)|v| \, dx + \int_\Omega |v|^{\eta(x)+1} dx + \int_\Omega |\nabla v|^{\delta(x)} |v| dx \tag{4.26}
$$

Thanks to Hölder's inequality, we have

$$
\int_\Omega g(x)|v| dx \le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big) |g|_{L^{p'(x)}(\Omega)} |v|_{L^{p(x)}(\Omega)} \le C_0 \|v\| \tag{4.27}
$$

on the other hand,

$$
\int_\Omega |v|^{\eta(x)+1} dx \le \begin{cases} |v|^{\eta^+ + 1}_{L^{\eta(x)+1}(\Omega)} & \text{if } |v|_{L^{\eta(x)+1}(\Omega)} > 1, \\ |v|^{\eta^- + 1}_{L^{\eta(x)+1}(\Omega)} & \text{if } |v|_{L^{\eta(x)+1}(\Omega)} \le 1, \end{cases}
$$

Thus,

$$
\int_\Omega |v|^{\eta(x)+1} \le |v|^{\beta}_{L^{\eta(x)+1}(\Omega)}, \tag{4.28}
$$

where

$$\beta = \begin{cases} \eta^+ + 1 & \text{if } |v|_{L^{\eta(x)+1}(\Omega)} > 1, \\ \eta^- + 1 & \text{if } |v|_{L^{\eta(x)+1}(\Omega)} \le 1, \end{cases}$$

since $0 \le \eta(x) < p(x) - 1$ implies $1 \le \eta(x) + 1 < p(x)$, consequently, (4.28) becomes

$$\int_\Omega |v|^{\eta(x)+1} dx \le C_1 |v|^\beta_{L^{p(x)}(\Omega)} \le C_1 \|v\|^\beta \quad \text{with } \beta < p^-. \tag{4.29}$$

Further, by Hölder's inequality,

$$\begin{aligned} \int_\Omega |\nabla v|^{\delta(x)} |v| dx &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big) \big\| \nabla v|^{\delta(x)} \big|_{L^{p'(x)}(\Omega)} |v|_{L^{p(x)}(\Omega)} \\ &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big) \Big(\int_\Omega |\nabla v|^{\delta(x) p'(x)} dx \Big)^{1/\gamma} |v|_{L^{p(x)}(\Omega)} \\ &\le \big(\frac{1}{p^-} + \frac{1}{p'^-}\big) \Big(\int_\Omega |\nabla v|^{\theta} dx \Big)^{1/\gamma} |v|_{L^{p(x)}(\Omega)}, \end{aligned}$$

where

$$\gamma = \begin{cases} p'^- & \text{if } \big\| \nabla v|^{\delta(x)} \big|_{L^{p'(x)}(\Omega)} > 1, \\ p'^+ & \text{if } \big\| \nabla v|^{\delta(x)} \big|_{L^{p'(x)}(\Omega)} \le 1, \end{cases} \quad \theta = \begin{cases} \delta^+ p'^+ & \text{if } |\nabla v| > 1, \\ \delta^- p'^- & \text{if } |\nabla v| \le 1. \end{cases}$$

Then

$$\int_\Omega |\nabla v|^{\delta(x)} |v| dx \le C \big(|v|_{W_0^{1,\theta}(\Omega)} \big)^{\theta/\gamma} |v|_{L^{p(x)}(\Omega)}, \tag{4.30}$$

since $0 \le \delta(x) < (p(x) - 1)/p'(x)$ implies $0 \le \delta(x) p'(x) < p(x) - 1$, and

$$0 \le \delta^+ < \big(\frac{p-1}{p'}\big)^- = \frac{p^- - 1}{p'^+} \Longrightarrow 0 \le \delta^+ p'^+ < p^- - 1,$$

and

$$0 \le \delta^- p'^- < \frac{(p^- - 1)}{p'^+} p'^- \le p^- - 1.$$

Therefore, $0 \le \theta < p^- - 1 < p(x)$. On the other hand,

$$0 \le \frac{\theta}{p'^+} < \frac{p^- - 1}{p'^+} \quad \text{and} \quad 0 \le \frac{\theta}{p'^-} < \frac{p^- - 1}{p'^-}\,.$$

Thus

$$\int_\Omega |\nabla v|^{\delta(x)} |v| dx \le C_2 \|v\|^{\theta/\gamma} \|v\| \tag{4.31}$$

Combining (4.25), (4.27), (4.29), and (4.31), we deduce that

$$\frac{\langle B(v), v \rangle}{\|v\|} \ge \alpha \|v\|^{p(x)-1} - C_0 - C_1 \|v\|^{\beta - 1} - C_2 \|v\|^{\theta/\gamma}\,. \tag{4.32}$$

Then we have

$$0 \le \frac{\theta}{p'^+} < \frac{p^- - 1}{p'^+}, \quad 0 \le \frac{\theta}{p'^-} < \frac{p^- - 1}{p'^-}, \quad \frac{p^- - 1}{p'^+} \le \frac{p^- - 1}{p'^-}\,;$$

Thus,

$$0 \le \frac{\theta}{\gamma} < \frac{p^- - 1}{p'^-} < p^- - 1\,. \tag{4.33}$$

Since $\beta - 1 < p^- - 1$, we conclude that

$$\frac{\langle B(v), v \rangle}{\|v\|} \ge \alpha \|v\|^{p(x)-1} - C_0 - C_1 \|v\|^{\beta - 1} - C_2 \|v\|^{\theta/\gamma} \to +\infty \quad \text{as } \|v\| \to +\infty.$$

Finally, by a classical theorem in [13], the problem (4.1) has a solution, so the proof of theorem 4.1 is achieved.

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