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POSITIVE PERIODIC SOLUTIONS FOR THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. For several classes of third-order constant coefficient linear differential equations we obtain existence and uniqueness of periodic solutions utilizing explicit Green's functions. We discuss an iteration method for constant coefficient nonlinear differential equations and provide new conditions for the existence of periodic positive solutions for third-order time-varying nonlinear and neutral differential equations.

1. Introduction

Let $X = C[0,\omega]$ with norm $\|x\| = \max_{t\in[0,\omega]} |x(t)|$. We denote $C_\omega^+ = \{u(t) \in X,\ u(t) > 0,\ u(t+\omega) = u(t)\}$, and $C_\omega^- = \{u(t) \in X,\ u(t) < 0,\ u(t+\omega) = u(t)\}$. We study the existence of positive periodic solutions for certain classes of third-order differential equations. Third-order differential equations arise in a variety of areas in agriculture, biology, economics and physics [9, 6, 14, 15, 21] and attract a lot of attention of many researchers [5, 16, 4, 7, 8, 17, 27, 11, 23, 25, 22, 3, 13] and the reference therein. In the study of higher-order (in particular third-order) differential equations, the naive idea to translate the equation into a first order system of differential equations by defining $x_1 = x,\ x_2 = x',\ x_3 = x'', \dots$ (see [20, 18, 24, 19]), works well for showing existence of periodic solutions, however, it does not obviously lead to existence proofs for positive periodic solutions, since the condition $x = x_1 \ge 0$ of positivity for the higher order equation is different from the natural positivity condition $(x_1, x_2, \dots) \ge 0$ for the corresponding system. Another approach which is frequently used is to transform the third-order equation into a corresponding integral equation and to establish the existence of positive periodic solutions based on a fixed point theorem in cones. Following this path one needs an explicit representation of the Green's function for corresponding ordinary equation, see [1, 2]. In [1], R. Agarwal gave the explicit Green's function for the nth-order and $2m$th-order differential equations. In addition, Anderson studied the Green's function for the third-order boundary value problem in [2],

$$u'''(t) = 0, \quad t_1 \le t \le t_3$$

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$$u(t_1) = u'(t_2) = 0, \quad \gamma u(t_3) + \delta u''(t_3) = 0$$

The singular nonlinear third-order periodic boundary value problem

$$\begin{gathered} u'''(t) + \rho^3 u(t) = f(t, u(t)), \quad t \in [0, 2\pi], \\ u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2, \end{gathered} \tag{1.1}$$

has been investigated in [16, 25, 4], where $\rho \in (0, 1/\sqrt{3})$ is a constant, $f : (0, 2\pi) \times (0, +\infty) \to \mathbb{R}^+$. By employing the Green's function for the equation

$$\begin{gathered} u'' - \rho u' + \rho^2 u = 0, \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{gathered}$$

the existence and multiplicity of positive solutions of (1.1) were established. However, the direct Green's function of (1.1) was not constructed.

Motivated by these excellent works, we give the explicit forms of the Green's functions for several differential third-order equations with the ω-periodic boundary value conditions and then provide sufficient conditions for the existence of positive periodic solutions.

This article is divided into six parts. In order to get the main result, we first consider the above four types of third-order constant coefficient linear differential equations and present their Green's functions and properties for those equations in Section 2. In Section 3, by applying the Banach fixed-point theorem and the results of Section 2, we obtain existence and uniqueness of solutions and an iteration method for the following constant coefficient nonlinear differential equations

$$u''' - \rho^3 u = f(t, u), \tag{1.2}$$

where $f \in C([0, \omega] \times \mathbb{R}, \mathbb{R})$. In Section 4, we study third-order time-varying nonlinear differential equations,

$$u''' - a(t)u = f(t, u), \tag{1.3}$$

$$u''' + a(t)u = f(t, u), \tag{1.4}$$

$a > 0$, $f \in C([0, \omega] \times [0, \infty), [0, \infty))$. We provide sufficient conditions for the existence of positive solutions for linear versions of equations (1.3) and (1.4). In Section 5, we go one step further and discuss a more general third-order nonlinear differential equation

$$y''' + p(t)y'' + q(t)y' + c(t)y = g(t, y). \tag{1.5}$$

Here $p, q, c \in C(\mathbb{R}, \mathbb{R})$, $g \in (\mathbb{R} \times [0, \infty), [0, \infty))$, and $g(t, y) > 0$ for $y > 0$; p, q, c, g are ω-periodic functions in t for some period $\omega > 0$. In Section 6 we study a neutral functional differential equation

$$(x(t) - cx(t - \tau(t)))''' + a(t)x(t) = f(t, x(t - \tau(t))), \tag{1.6}$$

and present an existence result for positive periodic solutions for this equation with an ω-periodic function $\tau \in C(\mathbb{R}, \mathbb{R})$, constants ω, c with $|c| < 1$, $a \in C_\omega^+$, $f \in C(\mathbb{R} \times [0, \infty), [0, \infty))$ and $f(t, x)$ is ω-periodic in t.

2. Green's Functions

Theorem 2.1. *For $\rho > 0$ and $h \in X$, the equation*

$$\begin{gathered} u''' - \rho^3 u = h(t), \\ u(0) = u(\omega), \quad u'(0) = u'(\omega), \quad u''(0) = u''(\omega) \end{gathered} \tag{2.1}$$

has a unique ω-periodic solution which is of the form

$$u(t) = \int_0^\omega G_1(t,s)(-h(s))\mathrm{d}s, \tag{2.2}$$

where

$$G_1(t,s) = \begin{cases} \frac{2\exp(\frac{1}{2}\rho(s-t))[\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6})-\exp(-\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s-\omega)+\frac{\pi}{6})]}{3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))} \\ +\frac{\exp(\rho(t-s))}{3\rho^2(\exp(\rho\omega)-1)}, \quad \text{if } 0 \le s \le t \le \omega \\ \frac{2\exp(\frac{1}{2}\rho(s-t-\omega))[\sin(\frac{\sqrt{3}}{2}\rho(t-s+\omega)+\frac{\pi}{6})-\exp(-\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6})]}{3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))} \\ +\frac{\exp(\rho(t+\omega-s))}{3\rho^2(\exp(\rho\omega)-1)}, \quad \text{if } 0 \le t \le s \le \omega \end{cases} \tag{2.3}$$

Proof. It is easy to check that the associated homogeneous equation of (2.1) has the solution $v(t) = c_1\exp(\rho t) + \exp(-\frac{\rho t}{2})(c_2\cos\frac{\sqrt{3}\rho}{2}t + c_3\sin\frac{\sqrt{3}\rho}{2}t)$. The only periodic solution of the associated homogeneous equation of (2.1) is the trivial solution; i.e., $c_1, c_2, c_3 = 0$. This follows by assuming that $v(t)$ is periodic; we immediately get that $c_1 = 0$ and by assuming that $c_2^2 + c_3^2 > 0$ and choosing φ such that $\sin\varphi = \frac{c_2}{\sqrt{c_2^2+c_3^2}}$, $\cos\varphi = \frac{c_3}{\sqrt{c_2^2+c_3^2}}$, we obtain

$$\begin{aligned} \frac{v(t)}{\sqrt{c_2^2+c_3^2}} &= \exp\big(-\frac{\rho t}{2}\big)\big(\sin\varphi\cos\frac{\sqrt{3}\rho}{2}t + \cos\varphi\sin\frac{\sqrt{3}\rho}{2}t\big) \\ &= \exp\big(-\frac{\rho t}{2}\big)\sin\big(\varphi + \frac{\sqrt{3}\rho}{2}t\big) \end{aligned}$$

which for $t \to \infty$ contradicts periodicity of v, proving that $c_2 = c_3 = 0$.

Applying the method of variation of parameters, we obtain

$$\begin{gathered} c_1'(t) = \frac{\exp(-\rho t)}{3\rho^2}h(t), \quad c_2'(t) = \frac{\frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}\rho t}{2} - \frac{1}{3}\cos\frac{\sqrt{3}\rho t}{2}}{\rho^2}\exp(\frac{\rho t}{2})h(t), \\ c_3'(t) = \frac{-\frac{1}{3}\sin\frac{\sqrt{3}\rho t}{2} - \frac{\sqrt{3}}{3}\cos\frac{\sqrt{3}\rho t}{2}}{\rho^2}\exp(\frac{\rho t}{2})h(t), \end{gathered}$$

and then

$$\begin{gathered} c_1(t) = c_1(0) + \int_0^t \frac{\exp(-\rho s)}{3\rho^2}h(s)\,\mathrm{d}s, \\ c_2(t) = c_2(0) + \int_0^t \frac{\frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}\rho s}{2} - \frac{1}{3}\cos\frac{\sqrt{3}\rho s}{2}}{\rho^2}\exp(\frac{\rho s}{2})h(s)\,\mathrm{d}s, \\ c_3(t) = c_3(0) + \int_0^t \frac{-\frac{1}{3}\sin\frac{\sqrt{3}\rho s}{2} - \frac{\sqrt{3}}{3}\cos\frac{\sqrt{3}\rho s}{2}}{\rho^2}\exp(\frac{\rho s}{2})h(s)\,\mathrm{d}s. \end{gathered}$$

$$
\begin{aligned}
u(t) &= c_1(t)\exp(\rho t) + \exp(-\frac{\rho t}{2})(c_2(t)\cos\frac{\sqrt{3}\rho}{2}t + c_3(t)\sin\frac{\sqrt{3}\rho}{2}t) \\
&= c_1(0)\exp(\rho t) + c_2\exp(-\frac{\rho t}{2})\cos(\frac{\sqrt{3}}{2}\rho t) + c_3(0)\exp(-\frac{\rho t}{2})\sin(\frac{\sqrt{3}}{2}\rho t) \\
&\quad + \int_0^t \frac{\exp(\rho(t-s))}{3\rho^2}h(s)\,\mathrm{d}s + \int_0^t \frac{\sin(\frac{\sqrt{3}}{2}\rho(s-t)-\frac{\pi}{6})}{6\rho^2}\exp(\frac{\rho}{2}(s-t))h(s)\,\mathrm{d}s
\end{aligned}
$$

Noting that $u(0)=u(\omega)$, $u'(0)=u'(\omega)$, $u''(0)=u''(\omega)$, we obtain

$$
c_1(0) = \int_0^\omega \frac{\exp(\rho(\omega-s))}{3\rho^2(1-\exp(\rho\omega))}h(s)\,\mathrm{d}s,
$$

$$
c_2(0) = \int_0^\omega \frac{2\exp(\frac{\rho(s-\omega)}{2})[\exp(-\frac{\rho\omega}{2})\sin(\frac{\pi}{6}-\frac{\sqrt{3}\rho s}{2}) - \sin(\frac{\pi}{6}-\frac{\sqrt{3}\rho(s-\omega)}{2})]}{3\rho^2(\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos\frac{\sqrt{3}\rho\omega}{2}+1)}h(s)\,\mathrm{d}s,
$$

$$
c_3(0) = \int_0^\omega \frac{2\exp(\frac{\rho(s-\omega)}{2})[\exp(-\frac{\rho\omega}{2})\cos(\frac{\pi}{6}-\frac{\sqrt{3}\rho s}{2}) - \cos(\frac{\pi}{6}-\frac{\sqrt{3}\rho(s-\omega)}{2})]}{3\rho^2(\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos\frac{\sqrt{3}\rho\omega}{2}+1)}h(s)\,\mathrm{d}s,
$$

Therefore,

$$
\begin{aligned}
u(t) &= c_1(t)\exp(\rho t) + \exp(-\frac{\rho t}{2})(c_2(t)\cos\frac{\sqrt{3}\rho}{2}t + c_3(t)\sin\frac{\sqrt{3}\rho}{2}t) \\
&= \int_0^t \Big\{2\exp(\frac{1}{2}\rho(s-t))\big[\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6}) - \exp(-\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s \\
&\quad -\omega)+\frac{\pi}{6})\big]/\big[3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))\big] \\
&\quad + \frac{\exp(\rho(t-s))}{3\rho^2(1-\exp(\rho\omega))}\Big\}h(s)\,\mathrm{d}s \\
&\quad + \int_t^\omega \Big\{2\exp(\frac{1}{2}\rho(s-t-\omega))[\sin(\frac{\sqrt{3}}{2}\rho(t-s+\omega)+\frac{\pi}{6}) - \exp(-\frac{1}{2}\rho\omega) \\
&\quad \times\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6})\big]/\big[3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))\big] \\
&\quad + \frac{\exp(\rho(t+\omega-s))}{3\rho^2(1-\exp(\rho\omega))}\Big\}h(s)\,\mathrm{d}s \\
&= \int_0^\omega G_1(t,s)h(s)\,\mathrm{d}s
\end{aligned}
$$

where $G_1(t,s)$ is defined as in (2.3).

By direct calculation, we obtain the solution u satisfies the periodic boundary value condition of the problem (2.1). □

Similarly, we have the following result.

Theorem 2.2. *For $\rho > 0$ and $h \in X$ the equation*

$$
\begin{gathered}
u''' + \rho^3 u = h(t), \\
u(0)=u(\omega), \quad u'(0)=u'(\omega), \quad u''(0)=u''(\omega)
\end{gathered} \tag{2.4}
$$

has a unique ω-periodic solution

$$
u(t) = \int_0^\omega G_2(t,s)h(s)\mathrm{d}s, \tag{2.5}
$$

where

$$G_2(t,s)=\begin{cases}\frac{2\exp(\frac{1}{2}\rho(t-s))[\sin(\frac{\sqrt{3}}{2}\rho(t-s)-\frac{\pi}{6})-\exp(\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s-\omega)-\frac{\pi}{6})]}{3\rho^2(1+\exp(\rho\omega)-2\exp(\frac{1}{2}\rho\omega)\cos(\frac{\sqrt{3}}{2}\rho\omega))}\\ +\frac{\exp(\rho(s-t))}{3\rho^2(1-\exp(-\rho\omega))}, \quad \text{if } 0\le s\le t\le\omega\\ \frac{2\exp(\frac{1}{2}\rho(t+\omega-s))[\sin(\frac{\sqrt{3}}{2}\rho(t+\omega-s)-\frac{\pi}{6})-\exp(\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s)-\frac{\pi}{6})]}{3\rho^2(1+\exp(\rho\omega)-2\exp(\frac{1}{2}\rho\omega)\cos(\frac{\sqrt{3}}{2}\rho\omega))}\\ +\frac{\exp(\rho(s-t-\omega))}{3\rho^2(1-\exp(-\rho\omega))}, \quad \text{if } 0\le t\le s\le\omega\end{cases} \tag{2.6}$$

Theorem 2.3. *For $\rho>0$ and $h\in X$ the equation*

$$\begin{gathered} u'''-3\rho u''+3\rho^2u'-\rho^3u=h(t),\\ u(0)=u(\omega),\quad u'(0)=u'(\omega),\quad u''(0)=u''(\omega)\end{gathered} \tag{2.7}$$

has a unique ω-periodic solution

$$u(t)=\int_0^\omega G_3(t,s)(-h(s))\mathrm{d}s, \tag{2.8}$$

where

$$G_3(t,s)=\begin{cases}\frac{[(s-t)\exp(\rho\omega)+\omega-s+t]^2+\omega^2\exp(\rho\omega)}{2(\exp(\rho\omega)-1)^3}\exp(\rho(t+\omega-s)), & 0\le t\le s\le\omega\\ \frac{[(s-t+\omega)\exp(\rho\omega)-s+t]^2+\omega^2\exp(\rho\omega)}{2(\exp(\rho\omega)-1)^3}\exp(\rho(t-s)), & 0\le s\le t\le\omega\end{cases} \tag{2.9}$$

Proof. In this case, the associated homogeneous equation of (2.7) has solutions

$$u(t)=c_1\exp(\rho t)+c_2t\exp(\rho t)+c_3t^2\exp(\rho t).$$

Analogously, by applying the method of variation of parameters we obtain

$$\begin{gathered} c_1'(t)=\frac{h(t)t^2}{2\exp(\rho t)},\quad c_2'(t)=\frac{-h(t)t}{\exp(\rho t)},\quad c_3'(t)=\frac{h(t)}{2\exp(\rho t)},\\ c_1(t)=c_1(0)+\int_0^t\frac{h(s)s^2}{2\exp(\rho s)}\,\mathrm{d}s,\quad c_2(t)=c_2(0)+\int_0^t\frac{-h(s)s}{\exp(\rho s)}\,\mathrm{d}s,\\ c_3(t)=c_3(0)+\int_0^t\frac{h(s)}{2\exp(\rho s)}\,\mathrm{d}s.\end{gathered}$$

Noting that $u(0)=u(\omega)$, $u'(0)=u'(\omega)$, $u''(0)=u''(\omega)$, we obtain

$$\begin{gathered} c_1(0)=\int_0^\omega\frac{h(s)\exp(\rho(\omega-s))[(s\exp(\rho\omega)+\omega-s)^2+\omega^2\exp(\rho\omega)]}{2(1-\exp(\rho\omega))^3}\,\mathrm{d}s,\\ c_2(0)=\int_0^\omega\frac{h(s)\exp(\rho(\omega-s))(\omega-s+s\exp(\rho\omega))}{(1-\exp(\rho\omega))^2}\,\mathrm{d}s,\\ c_3(0)=\int_0^\omega\frac{h(s)\exp(\rho(\omega-s))}{2(1-\exp(\rho\omega))}\,\mathrm{d}s.\end{gathered}$$

Therefore,

$$\begin{aligned} u(t)&=c_1\exp(\rho t)+c_2t\exp(\rho t)+c_3t^2\exp(\rho t)\\ &=\int_0^t\frac{[(s-t+\omega)\exp(\rho\omega)-s+t]^2+\omega^2\exp(\rho\omega)}{2(\exp(\rho\omega)-1)^3}\exp(\rho(t-s))(-h(s))\,\mathrm{d}s\\ &\quad+\int_t^\omega\frac{[(s-t)\exp(\rho\omega)+\omega-s+t]^2+\omega^2\exp(\rho\omega)}{2(\exp(\rho\omega)-1)^3}\\ &\quad\times\exp(\rho(t+\omega-s))(-h(s))\,\mathrm{d}s\end{aligned}$$

$$= \int_0^\omega G_3(t,s)(-h(s))\,\mathrm{d}s$$

Where $G_3(t,s)$ is as defined in (2.9). $\square$

A dual version of Theorem 2.3 which can be proved similarly.

Theorem 2.4. *For $\rho > 0$ and $h \in X$ the equation*

$$\begin{gathered} u''' + 3\rho u'' + 3\rho^2 u' + \rho^3 u = h(t), \\ u(0) = u(\omega), \quad u'(0) = u'(\omega), \quad u''(0) = u''(\omega) \end{gathered} \tag{2.10}$$

has a unique ω-periodic solution

$$u(t) = \int_0^\omega G_4(t,s)h(s)\mathrm{d}s, \tag{2.11}$$

where

$$G_4(t,s) = \begin{cases} \frac{[(s-t)\exp(-\rho\omega)+\omega-s+t]^2+\omega^2\exp(-\rho\omega)}{2(1-\exp(-\rho\omega))^3}\exp(-\rho(t+\omega-s)), \\ \quad \text{if } 0 \le t \le s \le \omega \\ \frac{[(s-t+\omega)\exp(-\rho\omega)-s+t]^2+\omega^2\exp(-\rho\omega)}{2(1-\exp(-\rho\omega))^3}\exp(-\rho(t-s)), \\ \quad \text{if } 0 \le s \le t \le \omega \end{cases} \tag{2.12}$$

Now we present the properties of the Green's functions for (2.1), (2.4), (2.7), (2.10). For convenience we use the abbreviations

$$\begin{gathered} A_1 = \frac{1}{3\rho^2(\exp(\rho\omega)-1)}, \quad B_1 = \frac{3+2\exp(-\frac{\rho\omega}{2})}{3\rho^2(1-\exp(-\frac{\rho\omega}{2}))^2}, \\ A_2 = \frac{\omega^2(1+\exp(\rho\omega))}{2(\exp(\rho\omega)-1)^3}, \quad B_2 = \frac{\omega^2\exp(2\rho\omega)(1+\exp(\rho\omega))}{2(\exp(\rho\omega)-1)^3}. \end{gathered}$$

Theorem 2.5. *$\int_0^\omega G_1(t,s)\mathrm{d}s = 1/\rho^3$ and if $\sqrt{3}\rho\omega < 4\pi/3$ holds, then $0 < A_1 < G_1(t,s) \le B_1$ for all $t \in [0,\omega]$ and $s \in [0,\omega]$.*

Proof. Let

$$H_1(t,s) = \frac{\exp(\rho(t-s))}{3\rho^2[\exp(\rho\omega)-1]}, \quad {}^*_1(t,s) = \frac{\exp(\rho(t+\omega-s))}{3\rho^2[\exp(\rho\omega)-1]},$$

$$\begin{aligned} H_2(t,s) = \Big(&2\exp(\frac{1}{2}\rho(s-t))\big[\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6}) - \exp(-\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s \\ &-\omega)+\frac{\pi}{6})\big]\Big)\Big/\Big(3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))\Big), \end{aligned}$$

$$\begin{aligned} &H_2^*(t,s) \\ &= \Big(2\exp(\frac{1}{2}\rho(s-t-\omega))\big[\sin(\frac{\sqrt{3}}{2}\rho(t-s+\omega)+\frac{\pi}{6}) - \exp(-\frac{1}{2}\rho\omega) \\ &\quad \times\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6})\big]\Big)\Big/\Big(3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))\Big). \end{aligned}$$

A direct computation shows that $\int_0^\omega G_1(t,s)\mathrm{d}s = 1/\rho^3$. It is easy to see that $H_1(t,s) > 0$ for $s \in [0,t]$ and $H_1^*(t,s) > 0$ for $s \in [t,\omega]$ and $\exp(-\rho\omega)+1-$

$2\exp(-\frac{\rho\omega}{2})\cos\frac{\sqrt{3}\rho\omega}{2} > [1-\exp(-\frac{\rho\omega}{2})]^2 > 0$. For convenience, we denote $\theta = \frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6}$,

$$\begin{aligned}
g_1(t,s) &= \sin\big(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6}\big) - \exp\big(-\frac{\rho\omega}{2}\big)\sin\big(\frac{\sqrt{3}}{2}\rho(t-s-\omega)+\frac{\pi}{6}\big)\\
&= \sin(\theta) - \exp(-\frac{\rho\omega}{2})\sin(\theta-\frac{\sqrt{3}}{2}\rho\omega),
\end{aligned}$$

$$\begin{aligned}
g_1^*(t,s) &= \sin\big(\frac{\sqrt{3}}{2}\rho(t-s+\omega)+\frac{\pi}{6}\big) - \exp(-\frac{\rho\omega}{2})\sin\big(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6}\big)\\
&= \sin(\theta+\frac{\sqrt{3}}{2}\rho\omega) - \exp(-\frac{\rho\omega}{2})\sin\theta.
\end{aligned}$$

If $g_1(t,s) > 0$ and $g_1^*(t,s) > 0$, then obviously $H_2(t,s) > 0$, $H_2^*(t,s) > 0$ and $G_1(t,s) > 0$.

For $0 \le s \le t \le \omega$, since $\sqrt{3}\rho\omega < 4\pi/3$, we have

$$\begin{gathered}
\frac{\pi}{6} \le \theta \le \frac{\sqrt{3}}{2}\rho\omega + \frac{\pi}{6} < \frac{5\pi}{6},\\
-\frac{\pi}{2} < \frac{\pi}{6} - \frac{\sqrt{3}}{2}\rho\omega \le \theta - \frac{\sqrt{3}}{2}\rho\omega \le \frac{\pi}{6}.
\end{gathered}$$

(i) For $-\frac{\pi}{2} < \theta - \frac{\sqrt{3}}{2}\rho\omega \le 0$, then $\sin\theta > 0$, $\sin(\theta-\frac{\sqrt{3}}{2}\rho\omega) < 0$, we obtain $g_1(t,s) > 0$

(ii) For $0 < \theta - \frac{\sqrt{3}}{2}\rho\omega \le \frac{\pi}{6}$, we have $\sin\theta > 0$, $\sin(\theta - \frac{\sqrt{3}}{2}\rho\omega) > 0$, and

$$0 < \frac{\sqrt{3}}{4}\rho\omega \le \theta - \frac{\sqrt{3}}{4}\rho\omega \le \frac{\pi}{6} + \frac{\sqrt{3}}{4}\rho\omega < \frac{\pi}{2}.$$

$$\begin{aligned}
g_1(t,s) &= \sin(\theta) - \exp(-\frac{\rho\omega}{2})\sin(\theta - \frac{\sqrt{3}}{2}\rho\omega)\\
&\ge \sin\theta - \sin(\theta - \frac{\sqrt{3}}{2}\rho\omega)\\
&= \sin(\theta - \frac{\sqrt{3}}{4}\rho\omega + \frac{\sqrt{3}}{4}\rho\omega) - \sin(\theta - \frac{\sqrt{3}}{4}\rho\omega - \frac{\sqrt{3}}{4}\rho\omega)\\
&= 2\cos(\theta - \frac{\sqrt{3}}{4}\rho\omega)\sin(\frac{\sqrt{3}}{4}\rho\omega) > 0
\end{aligned}$$

For $0 \le t \le s \le \omega$,

$$\begin{gathered}
-\frac{\pi}{2} < -\frac{\sqrt{3}}{2}\rho\omega + \frac{\pi}{6} \le \theta \le \frac{\pi}{6},\\
\frac{\pi}{6} \le \theta + \frac{\sqrt{3}}{2}\rho\omega \le \frac{\pi}{6} + \frac{\sqrt{3}}{2}\rho\omega < \frac{5}{6}\pi.
\end{gathered}$$

(i) For $-\frac{\pi}{2} < \theta \le 0$, we have $\sin\theta < 0$, $\sin(\theta + \frac{\sqrt{3}}{2}\rho\omega) > 0$, then $g_1^*(t,s) > 0$.

(ii)For $0 < \theta \le \frac{\pi}{6}$, we have $\sin\theta > 0$, $\sin(\theta + \frac{\sqrt{3}}{2}\rho\omega) > 0$, and

$$0 < \theta + \frac{\sqrt{3}}{4}\rho\omega < \frac{\pi}{2}$$

$$g_1^*(t,s) = \sin(\theta + \frac{\sqrt{3}}{2}\rho\omega) - \exp(-\frac{\rho\omega}{2})\sin\theta$$

$$
\begin{aligned}
&\geq \sin(\theta + \frac{\sqrt{3}}{2}\rho\omega) - \sin\theta \\
&= \sin(\theta + \frac{\sqrt{3}}{4}\rho\omega + \frac{\sqrt{3}}{4}\rho\omega) - \sin(\theta + \frac{\sqrt{3}}{4}\rho\omega - \frac{\sqrt{3}}{4}\rho\omega) \\
&= 2\cos(\theta + \frac{\sqrt{3}}{4}\rho\omega)\sin(\frac{\sqrt{3}}{4}\rho\omega) > 0
\end{aligned}
$$

If $\sqrt{3}\rho\omega < 4\pi/$, we obtain $g_1(t,s) > 0$ and $g_1^*(t,s) > 0$, proving that $G(t,s) > 0$ for all $t \in [0,\omega]$ and $s \in [0,\omega]$.

Next we compute a lower and an upper bound for $G_1(t,s)$ for $s \in [0,\omega]$. We have

$$
\begin{aligned}
A_1 = \frac{1}{3\rho^2(\exp(\rho\omega)-1)} &\leq \frac{\exp(\rho(t+\omega-s)}{3\rho^2(\exp(\rho\omega)-1)} < G_1(t,s) \\
&\leq \frac{\exp(\rho(t+\omega-s))}{3\rho^2[\exp(\rho\omega)-1]} + \frac{\exp(\frac{\rho(s-t-\omega)}{2})[2+2\exp(-\frac{\rho\omega}{2})]}{3\rho^2[\exp(-\rho\omega)+1-2\exp(-\frac{\rho\omega}{2})\cos\frac{\sqrt{3}\rho\omega}{2}]} \\
&\leq \frac{\exp(\rho\omega)}{3\rho^2[\exp(\rho\omega)-1]} + \frac{2+2\exp(-\frac{\rho\omega}{2})}{3\rho^2[\exp(-\rho\omega)+1-2\exp(-\frac{\rho\omega}{2})\cos\frac{\sqrt{3}\rho\omega}{2}]} \\
&\leq \frac{1}{3\rho^2[1-\exp(-\rho\omega)]} + \frac{2+2\exp(-\frac{\rho\omega}{2})}{3\rho^2[1-\exp(\frac{-\rho\omega}{2})]^2} \\
&\leq \frac{3+2\exp(-\frac{\rho\omega}{2})}{3\rho^2[1-\exp(\frac{-\rho\omega}{2})]^2} = B_1
\end{aligned}
$$

and the proof is complete. ∎

Similarly, the following dual theorem can be proved.

Theorem 2.6. *$\int_0^\omega G_2(t,s)\mathrm{d}s = 1/\rho^3$ and if $\sqrt{3}\rho\omega < 4\pi/3$ holds, then $0 < A_1 < G_2(t,s) \leq B_1$ for all $t \in [0,\omega]$ and $s \in [0,\omega]$.*

The next theorem provides bounds for G_3.

Theorem 2.7. *$\int_0^\omega G_3(t,s)\mathrm{d}s = 1/\rho^3$ and $0 < A_2 \leq G_3(t,s) \leq B_2$ for all $t \in [0,\omega]$ and $s \in [0,\omega]$.*

Proof. A direct computation shows that $\int_0^\omega G_3(t,s)\mathrm{d}s = 1/\rho^3$. Next we compute bounds for $G_3(t,s)$. For convenience rewrite

$$
G_3(t,s) = \begin{cases} \frac{\exp(\rho(\omega-s+t))}{2(\exp(\rho\omega)-1)^3} H_3(t,s), & 0 \leq t \leq s \leq \omega \\ \frac{\exp(\rho(t-s))}{2(\exp(\rho\omega)-1)^3} H_3^*(t,s), & 0 \leq s \leq t \leq \omega \end{cases}
$$

with

$$
\begin{gathered}
H_3(t,s) = [(s-t)(\exp(\rho\omega)-1)+\omega]^2 + \omega^2\exp(\rho\omega), \quad 0 \leq t \leq s \leq \omega \\
H_3^*(t,s) = [(s-t+\omega)\exp(\rho\omega)-s+t]^2 + \omega^2\exp(\rho\omega), \quad 0 \leq s \leq t \leq \omega
\end{gathered}
$$

Since

$$
\begin{gathered}
\frac{\partial H_3(t,s)}{\partial s} = 2[(s-t)(\exp(\rho\omega)-1)+\omega](\exp(\rho\omega)-1) > 0, \quad 0 \leq t \leq s \leq \omega, \\
\frac{\partial H_3^*(t,s)}{\partial s} = 2[(s-t+\omega)\exp(\rho\omega)-s+t](\exp(\rho\omega)-1) > 0, \quad 0 \leq s \leq t \leq \omega,
\end{gathered}
$$

the functions $s \mapsto H_3(t,s)$ and $s \mapsto H_3^*(t,s)$ are increasing. Recalling that

$$H_3(t,t) = \omega^2 + \omega^2 \exp(\rho\omega) = \omega^2(1+\exp(\rho\omega)),$$
$$H_3(t,\omega) = [(\omega - t)\exp(\rho\omega) + t]^2 + \omega^2 \exp(\rho\omega) \le \omega^2 \exp(2\rho\omega) + \omega^2 \exp(\rho\omega),$$
$$\omega^2 + \omega^2 \exp(\rho\omega) \le H_3^*(t,0) = [(\omega - t)\exp(\rho\omega) + t]^2 + \omega^2 \exp(\rho\omega),$$
$$H^*(t,t) = \omega^2 \exp(2\rho\omega) + \omega^2 \exp(\rho\omega).$$

and

$$0 < H_3(t,t) \le H_3(t,s) < H_3(t,\omega) \le \omega^2 \exp(2\rho\omega) + \omega^2 \exp(\rho\omega),$$
$$0 < \omega^2 + \omega^2 \exp(\rho\omega) \le H_3^*(t,0) \le H_3^*(t,s) \le H^*(t,t),$$

Using these inequalities we obtain $0 < A_2 \le G_3(t,s) \le B_2$ and the proof is complete. □

Similarly, we obtain the following analog theorem.

Theorem 2.8. *$\int_0^\omega G_4(t,s)\mathrm{d}s = 1/\rho^3$ and $0 < A_2 \le G_4(t,s) \le B_2$ for all $t \in [0,\omega]$ and $s \in [0,\omega]$.*

3. Periodic Solution of (1.2)

For reference we briefly recall Banach's fixed point theorem and related error estimates.

Lemma 3.1 ([28]). *Let M be a closed nonempty set in the Banach space X and $A : M \to M$ a k-contractive operator; i.e., there exists k, $0 \le k < 1$, with*

$$\|Au - Av\| \le k\|u - v\| \quad \text{for all } u, v \in M. \tag{3.1}$$

Consider the operator equation

$$u = Au, \quad u \in M, \tag{3.2}$$

and for any $u_0 \in M$ the iteration

$$u_{n+1} = Au_n, \quad n = 0, 1, \dots \tag{3.3}$$

Then the following statements hold:

(i) Existence and uniqueness: *there exists a unique u^* which solves (3.2); i.e., $Au^* = u^*$.*

(ii) Convergence of the iteration method: *For all $u_0 \in M$ one has $\lim_{n\to\infty} u_n = u^*$.*

(iii) Error estimates: *For all $n = 0, 1, \dots$, one has the so-called a priori error estimate*

$$\|u_n - u\| \le k^n(1-k)^{-1}\|u_1 - u_0\|, \tag{3.4}$$

and the so-called a posteriori error estimate

$$\|u_{n+1} - u\| \le k(1-k)^{-1}\|u_{n+1} - u_n\|. \tag{3.5}$$

(iv) Rate of convergence: *For all $n = 0, 1, \dots$, one has $\|u_{n+1} - u\| \le k\|u_n - u\|$.*

Now we consider (1.2). Let X be defined as in Section 1. Define an operator $D : X \to X$ by

$$Du(t) = \int_0^\omega G_1(t,s) f(s, u(s))\mathrm{d}s.$$

By Theorem 2.1, we know that the periodic solution problem of (1.2) is equal to the fixed point problem $u = Du$. For any $u_0 \in X$ define the sequence (u_n) by

$$u_{n+1}(t) = \int_0^\omega G_1(t,s)f(s,u_n(s))\mathrm{d}s, \quad n = 0,1,\dots. \tag{3.6}$$

We introduce the abbreviation:

$$\vartheta = \frac{2}{3\rho\sqrt{1+\exp(-\rho\omega) - 2\exp(-\frac{1}{2}\rho\omega)\cos(\frac{\sqrt{3}}{2}\rho\omega)}} + \frac{\exp(\rho\omega)}{3\rho(\exp(\rho\omega)-1)}. \tag{3.7}$$

Theorem 3.2. *Assume that the partial derivative $f_u \in C([0,\omega]\times\mathbb{R},\mathbb{R})$ and there is a number l such that $|f_u(t,u)| \le l$ for all $t \in [0,\omega]$, $u \in \mathbb{R}$. Then if $l\vartheta\omega < \rho$, the following statements hold:*

- (i) *the original problem* (1.2) *has a unique solution $u \in X$;*
- (ii) *the sequence (u_n) constructed by* (3.6) *converges to u in X;*
- (iii) *for all $n = 0,1,2,\dots$, we obtain for $k := \frac{l\vartheta\omega}{\rho}$ the following error estimates*

$$\|u_n - u\| \le k^n(1-k)^{-1}\|u_1 - u_0\|, \quad \|u_{n+1} - u\| \le k^n(1-k)^{-1}\|u_{n+1} - u_n\|.$$

Proof. By calculation, we obtain: For $0 \le t \le s \le \omega$,

$$\begin{aligned}
&|G_1(t,s)| \\
&\le \Big|\frac{\exp(\rho(t+\omega-s))}{3\rho^2[\exp(\rho\omega)-1]}\Big| \\
&\quad + \Big|\frac{\exp(\frac{\rho(s-t-\omega)}{2})[2\sin(\frac{\pi}{6} - \frac{\sqrt{3}\rho}{2}(s-t-\omega)) - 2\exp(-\frac{\rho\omega}{2})\sin(\frac{\pi}{6} - \frac{\sqrt{3}\rho}{2}(s-t))]}{3\rho^2[\exp(-\rho\omega)+1-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}\rho\omega}{2})]}\Big| \\
&\le \Big|\frac{\exp(\rho\omega)}{3\rho^2[\exp(\rho\omega)-1]}\Big| + \Big|\frac{2\exp(\frac{\rho(s-t-\omega)}{2})\sin(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\rho(s-t)+\varphi_1)}{3\rho^2\sqrt{\exp(-\rho\omega)+1-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}\rho\omega}{2})}}\Big| \\
&\le \frac{\exp(\rho\omega)}{3\rho^2[\exp(\rho\omega)-1]} + \frac{2}{3\rho^2\sqrt{\exp(-\rho\omega)+1-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}\rho\omega}{2})}} = \frac{\vartheta}{\rho}.
\end{aligned}$$

Similarly, we can obtain: For $0 \le s \le t \le \omega$,

$$|G_1(t,s)| \le \frac{\vartheta}{\rho}.$$

So for all $t \in [0,\omega]$ and $s \in [0,\omega]$, we have $|G_1(t,s)| \le \frac{\vartheta}{\rho}$.

On the other hand, for any $t \in [0,\omega]$, and $u,v \in \mathbb{R}$, by the mean value theorem, there exists a $w \in \mathbb{R}$ such that

$$|f(s,u) - f(s,v)| \le |f_u(s,w)||u-v| \le l|u-v|.$$

Therefore,

$$\begin{aligned}
|Du - Dv| &= \Big|\int_0^\omega G_1(t,s)[f(s,u(s)) - f(s,v(s))]\mathrm{d}s\Big| \\
&\le \int_0^\omega |G_1(t,s)||f(s,u(s)) - f(s,v(s))|\mathrm{d}s \\
&\le l|u-v|\int_0^\omega |G_1(t,s)|\mathrm{d}s \le \frac{l\vartheta\omega}{\rho}|u-v|;
\end{aligned}$$

i.e., $\|Du - Dv\| \leq k\|u - v\|$ for all $u, x \in X$. Let $M := X$, the assertions follow directly from Lemma 3.1. $\square$

Similarly, we can obtain the corresponding results for the equations

$$u''' + \rho^3 u = f(t, u),$$
$$u''' - 3\rho u'' + 3\rho^2 u' - \rho^3 u = f(t, u),$$
$$u''' + 3\rho u'' + 3\rho^2 u' + \rho^3 u = f(t, u),$$

where $f \in C([0, \omega] \times \mathbb{R}, (0, +\infty))$.

4. Positive Solutions of (1.3), (1.4)

In this section, we consider the existence of positive periodic solutions for the third-order nonlinear differential equations with ω-periodic boundary value condition

$$u''' - a(t)u = f(t, u), \tag{4.1}$$

and

$$u''' + a(t)u = f(t, u), \tag{4.2}$$

where $a \in C([0, \omega], (0, \infty))$, and $f \in C([0, \omega] \times [0, \infty), [0, \infty))$, and $f(t, u) > 0$ for $u > 0$. We introduce the following abbreviations

$$a^* = \max\{a(t) : t \in [0, \omega]\}, \quad a_* = \min\{a(t) : t \in [0, \omega]\}, \quad rho = \sqrt[3]{a^*},$$
$$\overline{f}_0 = \lim_{x \to 0^+} \sup_{t \in [0,\omega]} \frac{f(t, x)}{x}, \quad \underline{f}_\infty = \lim_{x \to \infty} \inf_{t \in [0,\omega]} \frac{f(t, x)}{x}.$$

Let X be defined as in the beginning of Section 1. Moreover, define a cone K_0 in X by $K_0 = \{x \in X : x(t) > \theta\|x\|\}$, where $0 < \theta = \frac{a_* A_1}{a^* B_1} < 1$ and for $r > 0$ define $K_{0r} = \{x \in K_0 : \|x\| < r\}$ and $\partial K_{0r} = \{x \in K_0 : \|x\| = r\}$.

First, we study the following equation corresponding to (4.1)

$$u''' - a(t)u = h(t), \quad h \in C_\omega^- \tag{4.3}$$

Define operators $T_1, B_1 : X \to X$ by

$$(T_1 h)(t) = \int_0^\omega G_1(t, s)(-h(s))\mathrm{d}s, \quad (B_1 u)(t) = (a(t) - a^*)u(t). \tag{4.4}$$

Clearly, if $\sqrt{3}\rho\omega < 4\pi/3$ holds, then T_1, B_1 are completely continuous, $(T_1 h)(t) > 0$ for $-h(t) > 0$ and $\|B_1\| \leq a^* - a_*$. By Theorem 2.1, the solution of (4.3) can be written in the form

$$u(t) = (T_1 h)(t) + (T_1 B_1 u)(t). \tag{4.5}$$

And for $\|T_1 B_1\| \leq \|T_1\|\|B_1\| \leq \frac{1}{a^*}(a^* - a_*) < 1$, we have

$$u(t) = (I - T_1 B_1)^{-1}(T_1 h)(t). \tag{4.6}$$

Define an operator $P_1 : X \to X$ by

$$(P_1 h)(t) = (I - T_1 B_1)^{-1}(T_1 h)(t). \tag{4.7}$$

Obviously, for any $h \in C_\omega^-$, $u(t) = (P_1 h)(t)$ is the unique positive solution of (4.3).

Lemma 4.1. *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, then P_1 is completely continuous and*

$$(T_1 h)(t) \leq (P_1 h)(t) \leq \frac{a^*}{a_*}\|(T_1 h)(t)\|, \quad \textit{for all } h \in C_\omega^-. \tag{4.8}$$

Proof. Since $\|T_1B_1\| \le \|T_1\|\|B_1\| \le 1 - \frac{a_*}{a^*} < 1$, by a Neumann expansion of P_1, we have

$$\begin{aligned} P_1 &= (I - T_1B_1)^{-1}T_1 \\ &= (I + T_1B_1 + (T_1B_1)^2 + \dots + (T_1B_1)^n + \dots)T_1 \\ &= T_1 + T_1B_1T_1 + (T_1B_1)^2T_1 + \dots + (T_1B_1)^nT_1 + \dots \end{aligned} \tag{4.9}$$

By (4.9), and recalling that $\|T_1B_1\| \le 1 - \frac{a_*}{a^*}$ and $(T_1h)(t) > 0$, we obtain

$$(T_1h)(t) \le (P_1h)(t) \le \frac{a^*}{a_*}\|(T_1h)(t)\|, \quad h \in C_\omega^+. \tag{4.10}$$

From [10], we obtain that T_1 is completely continuous and P_1 is also completely continuous. □

Define an operator $Q_1 : X \to X$ by

$$Q_1u(t) = P_1(-f(t,u)). \tag{4.11}$$

From the continuity of P_1, it is easy to verify that Q_1 is completely continuous in X. Comparing (4.1) with (4.3), we see that the existence of solutions for equation (4.1) is equivalent to the existence of fixed-points for the equation $u = Q_1u$.

Lemma 4.2. $Q_1(K_0) \subset K_0$.

Proof. It is easy to verify that $Q_1u(t+\omega) = Q_1u(t)$. For $u \in K_0$, we have from Lemma 4.1 that

$$\begin{aligned} Q_1u(t) &= P_1(-f(t,u)) \ge T_1(-f(t,u)) \\ &= \int_0^\omega G_1(t,s)f(s,u(s))\mathrm{d}s > A_1\int_0^\omega f(s,u(s))\mathrm{d}s, \end{aligned}$$

on the other hand

$$\begin{aligned} Q_1u(t) &= P_1(-f(t,u)) \le \frac{a^*}{a_*}\|T_1(-f(t,u))\| \\ &= \frac{a^*}{a_*}\max_{t\in[0,\omega]}\int_0^\omega G_1(t,s)f(s,u(s))\mathrm{d}s \le \frac{a^*B_1}{a_*}\int_0^\omega f(s,u(s))\mathrm{d}s. \end{aligned}$$

Therefore,

$$Q_1u(t) > \frac{a_*A_1}{a^*B_1}\|Q_1u\| = \theta\|Q_1u\|;$$

i.e., $Q_1(K_0) \subset K_0$. □

Next, we study the following equation corresponding to (4.2).

$$u''' + a(t)u = h(t), \quad h \in C_\omega^+ \tag{4.12}$$

Define operators $T_2, B_2 : X \to X$ by

$$(T_2h)(t) = \int_0^\omega G_2(t,s)h(s)\mathrm{d}s, \quad (B_2u)(t) = (a^* - a(t))u(t). \tag{4.13}$$

If $\sqrt{3}\rho\omega < 4\pi/3$ holds, T_2, B_2 are completely continuous, $(T_2h)(t) > 0$ for $h(t) > 0$ and $\|B_2\| \le a^* - a_*$. By Theorem 2.2, the solution of (4.12) can be written in the form

$$u(t) = (T_2h)(t) + (T_2B_2u)(t). \tag{4.14}$$

And for $\|T_2B_2\| \le \|T_2\|\|B_2\| \le \frac{1}{a^*}(a^* - a_*) < 1$, we have

$$u(t) = (I - T_2B_2)^{-1}(T_2h)(t). \tag{4.15}$$

Define an operator $P_2 : X \to X$ by

$$(P_2h)(t) = (I - T_2B_2)^{-1}(T_2h)(t). \tag{4.16}$$

Obviously, for any $h \in C_\omega^+$, $u(t) = (P_2h)(t)$ is the unique positive solution of (4.12). Similar to Lemma 4.1, we can prove the following result.

Lemma 4.3. *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, then P_2 is completely continuous and*

$$(T_2h)(t) \le (P_2h)(t) \le \frac{a^*}{a_*}(T_2h)(t), \quad \textit{for all } h \in C_\omega^+. \tag{4.17}$$

Define an operator $Q_2 : X \to X$ by

$$Q_2u(t) = P_2(f(t,u)). \tag{4.18}$$

Clearly, Q_2 is completely continuous in X. Comparing (4.2) with (4.12), it is clear that the existence of solutions for equation (4.2) is equivalent to the existence of fixed-points for the equation $u = Q_2u$.

Lemma 4.4. $Q_2(K_0) \subset K_0$.

Proof. From the definition of Q_2, it is easy to verify that $Q_2u(t+\omega) = Q_2u(t)$. For $u \in K_0$, we have from Lemma 4.3 that

$$Q_2u(t) = P_2(f(t,u)) \ge T_2(f(t,u)) = \int_0^\omega G_2(t,s)f(s,u(s))\mathrm{d}s > A_1\int_0^\omega f(s,u(s))\mathrm{d}s,$$

on the other hand

$$\begin{aligned} Q_2u(t) = P_2(f(t,u)) &\le \frac{a^*}{a_*}\|T_2f(t,u)\| \\ &= \frac{a^*}{a_*}\max_{t\in[0,\omega]}\int_0^\omega G_2(t,s)f(s,u(s))\mathrm{d}s \le \frac{a^*B_1}{a_*}\int_0^\omega f(s,u(s))\mathrm{d}s. \end{aligned}$$

Therefore,

$$Q_2u(t) > \frac{a_*A_1}{a^*B_1}\|Q_2u\| = \theta\|Q_2u\|;$$

i.e., $Q_2(K_0) \subset K_0$. $\square$

Lemma 4.5 ([12])**.** *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous operator such that $Tx \ne x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$, so*

(i) *if $\|Tx\| \ge \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$;*
(ii) *if $\|Tx\| \le \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.*

Theorem 4.6. *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, and $\overline{f}_0 = 0, \underline{f}_\infty = \infty$, then (4.1) has at least one positive solution.*

Proof. If $\overline{f}_0 = 0$, we can choose $0 < r_1 < 1$, such that $f(t,u) \le \varepsilon u$ for $0 \le u \le r_1, t \in [0,\omega]$, where the constant $\varepsilon > 0$ satisfies

$$\frac{a^*B_1}{a_*}\varepsilon\omega < 1.$$

By recalling the proof of Lemma 4.1, we obtain that

$$\|Q_1 u\| \leq \frac{a^* B_1}{a_*} \int_0^\omega f(s,u(s))\mathrm{d}s \leq \frac{a^* B_1}{a_*}\varepsilon \int_0^\omega u(s)\mathrm{d}s \leq \frac{a^* B_1}{a_*}\varepsilon\omega\|u\| < \|u\|$$

for $u \in \partial K_{0r_1}$, $t \in [0,\omega]$. On the other hand, if $\underline{f}_\infty = \infty$, there exists a constant $\tilde{H} > r_1$ such that $f(t,u) \geq \eta u$ for $u \geq \tilde{H}, t \in [0,\omega]$, where the constant $\eta > 0$ satisfies $A_1\eta\omega\theta > 1$, and again from the proof of Lemma 4.1, one can easily see that

$$Q_1 u > A_1 \int_0^\omega f(s,u(s))\mathrm{d}s > A_1\eta \int_0^\omega u(s)\mathrm{d}s > A_1\eta\omega\theta\|u\| > \|u\|$$

for $u \in \partial K_{0\tilde{H}}$, $t \in [0,\omega]$. By Lemma 4.5, we know that $i(Q_1, K_{0r_1}, K_0) = 1, i(Q_1, K_{0\tilde{H}}, K_0) = 0$, i.e. $i(Q_1, K_{0\tilde{H}} \backslash \bar{K}_{0r_1}, K_0) = -1$, and Q_1 has a fixed point in $K_{0\tilde{H}} \backslash \bar{K}_{0r_1}$. Consequently, (4.1) has a positive ω-periodic solution for $r_1 < u < \tilde{H}$. $\square$

By Lemmas 4.3 and 4.4, we obtain the following result.

Theorem 4.7. *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, and $\overline{f}_0 = 0$, $\underline{f}_\infty = \infty$, then (4.2) has at least one positive solution.*

5. Positive Solutions for (1.5)

Theorem 5.1. *If $\frac{1}{3}p^2(t) + p'(t) = q(t)$, then (1.5) can be transformed into*

$$u'''(t) + b(t)u(t) = f(t,u), \tag{5.1}$$

where

$$\begin{gathered} b(t) = -\frac{1}{3}p''(t) + \frac{2}{27}p^3(t) - \frac{1}{3}p(t)q(t) + c(t), \\ f(t,u) = \frac{g(t, u\exp(-\int \frac{p(t)}{3}\mathrm{d}t))}{\exp(-\int \frac{p(t)}{3}\mathrm{d}t)}, \end{gathered} \tag{5.2}$$

$b \in C([0,\omega],\mathbb{R})$*;* $f \in C([0,\omega] \times [0,\infty), [0,\infty))$.

Proof. Let $y = ux$, then

$$y' = u'x + ux', \quad y'' = u''x + 2u'x' + ux'', \quad y''' = u'''x + 3u''x' + 3u'x'' + ux'''. \tag{5.3}$$

Substituting (5.3) into (1.5) yields

$$\begin{aligned} &u'''x + [3x' + p(t)x]u'' + [3x'' + 2p(t)x' + q(t)x]u' + [x''' + p(t)x'' + q(t)x' + c(t)x]u \\ &= g(t,ux), \end{aligned}$$

or equivalently for $x \neq 0$,

$$\begin{aligned} &u''' + \frac{3x' + p(t)x}{x}u'' + \frac{3x'' + 2p(t)x' + q(t)x}{x}u' + \frac{x''' + p(t)x'' + q(t)x' + c(t)x}{x}u \\ &= \frac{g(t,ux)}{x}. \end{aligned}$$

It is easy to verify that $3x' + p(t)x = 0$ and $3x'' + 2p(t)x' + q(t)x = 0$ if $x = \exp(-\int \frac{p(t)}{3}\mathrm{d}t)$ and $\frac{1}{3}p^2(t) + p'(t) = q(t)$. Hence (1.5) can be transformed to

$$u'''(t) + b(t)u(t) = f(t,u),$$

where $b(t), f(t,u)$ are given in (5.2). It is easy to see that $b \in C([0,\omega],\mathbb{R})$; $f \in C([0,\omega] \times [0,\infty), [0,\infty))$, $f(t,u) > 0$ for $u > 0$. $\square$

By Theorem 5.1, under the assumption that $\frac{1}{3}p^2(t)+p'(t)=q(t)$, we know that if u is a positive solution for (5.1), then $y=u\exp(-\int\frac{p(t)}{3}\mathrm{d}t)$ is also a positive solution for (1.5). Next we discuss Eq (5.1) in two cases *(i)* $b\in C([0,\omega],(0,\infty))$ and *(ii)* $b\in C([0,\omega],(-\infty,0))$.

Case (i): $b\in C([0,\omega],(-\infty,0))$. In this case, (5.1) is equivalent to

$$u'''-a(t)u=f(t,u), \tag{5.4}$$

with $a(t)=-b(t)$, clearly $a\in C([0,\omega],(0,\infty))$, and $a^*=\max\{-b(t):t\in[0,\omega]\}$, $a_*=\min\{-b(t):t\in[0,\omega]\}$.

Case (ii): $b\in C([0,\omega],(0,\infty))$. In this case, (5.1) is equivalent to

$$u'''+a(t)u=f(t,u) \tag{5.5}$$

here $a(t)=b(t)$. Finally, by recalling the proofs for the existence of positive solutions of (5.4) and (5.5) in Section 3 and by applying Theorem 5.1 we obtain

Theorem 5.2. *If* $\frac{1}{3}p^2(t)+p'(t)=q(t)$ *and* $\sqrt{3}\rho\omega<4\pi/3$ *hold,* $\overline{f}_0=0,\underline{f}_\infty=\infty$, *then* (1.5) *has at least one positive solution.*

We illustrate our results with an example.

Example 5.3. Consider the third-order differential equation

$$\begin{aligned} &y'''+\sin ty''+\Big(\frac{1}{3}\sin^2 t+\cos t\Big)y'\\ &+\Big(-\frac{1}{1000}\exp(\sin t)-\frac{1}{3}\sin t+\frac{1}{27}\sin^3 t+\frac{1}{3}\sin t\cos t\Big)y\\ &=\frac{1}{1000}\exp(\sin t)y^2. \end{aligned} \tag{5.6}$$

Comparing with (1.5), we are lead to the definitions

$$\begin{gathered} p(t)=\sin t,\quad q(t)=\frac{1}{3}\sin^2 t+\cos t,\\ c(t)=-\frac{1}{1000}\exp(\sin t)-\frac{1}{3}\sin t+\frac{1}{27}\sin^3 t-\frac{1}{3}\sin t\cos t,\\ g(t,y)=\frac{1}{1000}\exp(\sin t)y^2. \end{gathered}$$

It is easy to see that $\frac{1}{3}p^2(t)+p'(t)=q(t)$. Then by Theorem 5.1, we can transform (5.6) into

$$u'''+b(t)u=f(t,u), \tag{5.7}$$

where $b(t)=-\frac{1}{1000}\exp(\sin t)$ and $f(t,u)=\frac{1}{1000}u^2\exp\big(\sin t+\frac{\cos t}{3}\big)$.

Since $\sqrt{3}\rho\omega=1.5188<4\pi/3$, and noticing that $\overline{f}_0=0,\underline{f}_\infty=\infty$, we know from Theorem 5.2 that (5.7) has a positive solution u, and then (5.6) has a positive solution $y=u\exp(\frac{\cos s}{3})$.

6. Positive Periodic Solution for (1.6)

Equation (1.6) can be rewritten as

$$(x(t)-cx(t-\tau(t)))'''+a(t)(x(t)-cx(t-\tau(t)))=f(t,x(t-\tau(t)))-ca(t)x(t-\tau(t)). \tag{6.1}$$

With $y(t)=x(t)-cx(t-\tau(t))$ Equation (6.1) can be transformed into

$$y'''+a(t)y(t)=f(t,x(t-\tau(t)))-ca(t)x(t-\tau(t)) \tag{6.2}$$

Define $P : X \to X$ by

$$(Ph)(t) = (I - TB)^{-1}(Th)(t), \tag{6.3}$$

where T, B are defined as T_2, B_2 in Section 3. Define operators $Q, S : X \to X$ by

$$(Qx)(t) = P(f(t, x(t-\tau(t))) - ca(t)x(t-\tau(t))), \quad (Sx)(t) = cx(t-\tau(t)). \tag{6.4}$$

From (6.2) and (6.4) and the results of Section 3, we know that the existence of periodic solutions for (1.6) is equivalent to the existence of solutions for the operator equation

$$Qx + Sx = x \quad \text{in } X. \tag{6.5}$$

Lemma 6.1 ([26]). *Let X be a Banach space, assume K is a bounded closed convex subset of X and $Q, S : K \to X$ satisfy the following assumptions:*

(i) $Qx + Sy \in K, \forall x, y \in K$,
(ii) *S is a contractive operator,*
(iii) *Q is a completely continuous operator in K.*

Then $Q + S$ has a fixed point in K.

Theorem 6.2. *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, $c \in (0, 1)$, and $ca_* \le f(t, x) - ca(t)x \le a^*$ for all $t \in [0, \omega]$ and for all $x \in [\frac{ca_*}{(1-c)a^*}, \frac{a^*}{(1-c)a_*}]$, then (1.6) has at least one positive ω-periodic solution $x(t)$ with $0 < \frac{ca_*}{(1-c)a^*} \le x(t) \le \frac{a^*}{(1-c)a_*}$.*

Proof. Define $K_1 = \{x \in X : x \in [\frac{ca_*}{(1-c)a^*}, \frac{a^*}{(1-c)a_*}]\}$. Obviously, K_1 is a bounded closed convex set in X. Since P is completely continuous, so is Q. Besides, it is easy to see that S is contractive if $|c| < 1$. Now we prove that $Qx + Sy \in K_1$ for all $x, y \in K_1$. By Lemma 4.3, we obtain

$$\begin{aligned}
&Qx(t) + Sy(t)\\
&= P(f(t, x(t-\tau(t))) - ca(t)x(t-\tau(t))) + cy(t-\tau(t))\\
&\le \frac{a^*}{a_*}\|T(f(t, x(t-\tau(t))) - ca(t)x(t-\tau(t)))\| + cy(t-\tau(t))\\
&\le \frac{a^*}{a_*}\max_{t\in[0,\omega]}\int_t^{t+\omega} G_2(t,s)(f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s)))ds + cy(t-\tau(t))\\
&\le \frac{a^*}{a_*}\max_{t\in[0,\omega]}\int_t^{t+\omega} G_2(t,s)a^* ds + c\frac{a^*}{(1-c)a_*}\\
&= \frac{a^*}{a_*}a^*\frac{1}{a^*} + \frac{ca^*}{(1-c)a_*} = \frac{a^*}{(1-c)a_*}
\end{aligned} \tag{6.6}$$

On the other hand,

$$\begin{aligned}
&Qx(t) + Sy(t)\\
&= P(f(t, x(t-\tau(t))) - ca(t)x(t-\tau(t))) + cy(t-\tau(t))\\
&\ge T(f(t, x(t-\tau(t))) - ca(t)x(t-\tau(t)) + cy(t-\tau(t))\\
&\ge \int_t^{t+\omega} G_2(t,s)(f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s)))ds + cy(t-\tau(t))\\
&\ge \frac{1}{a^*}ca_* + \frac{c^2a_*}{(1-c)a^*} = \frac{ca_*}{(1-c)a^*}
\end{aligned} \tag{6.7}$$

Combining (6.6) and (6.7), we obtain $Qx + Sy \in K_1$, for all $x, y \in K_1$. By Lemma 6.1 we obtain that $Q+S$ has a fixed point $x \in K_1$; i.e., (1.6) has a positive ω-periodic solution $x(t)$ with $0 < \frac{ca_*}{(1-c)a^*} \le x(t) \le \frac{a^*}{(1-c)a_*}$. $\square$

Example 6.3. Consider the equation

$$
\begin{aligned}
&\big(x(t) - \frac{1}{2}x(t-\cos^2 t)\big)''' + \frac{1}{1000}(1 - \frac{1}{2}\sin^2 t)x(t) \\
&= \frac{1}{1000}(1 - \frac{3}{4}\sin^2 t) + \frac{1}{2000}(1 - \frac{1}{2}\sin^2 t)x(t-\cos^2 t),
\end{aligned}
$$

here $c = \frac{1}{2}$, $a(t) = \frac{1}{1000}(1 - \frac{1}{2}\sin^2 t)$ and $\tau(t) = \cos^2 t$. Obviously $a \in C(\mathbb{R}, (0,\infty))$ is a π-periodic function with $a^* = \frac{1}{1000}$, $a_* = \frac{1}{2000}$, and then $\rho = \frac{1}{10}$. Noticing that $\frac{\sqrt{3}\pi}{10} < 4\pi/3$ holds. Moreover, it is easy to see that $\frac{1}{4000} \le f(t,x) - ca(t)x = \frac{1}{1000}(1 - \frac{3}{4}\sin^2 t) \le \frac{1}{1000}$. Then by Theorem 6.2, we know the equation has at least one positive solution x with $\frac{1}{2} \le x(t) \le 4$.

Theorem 6.4. *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, $c = 0$, and $0 < f(t, x(t-\tau(t))) \le a^*$ for all $t \in [0,\omega]$ and for all $x \in [0, a^*/a_*]$, then* (1.6) *has at least one positive ω-periodic solution x with $0 < x(t) \le a^*/a_*$.*

Proof. By (6.4), $S = 0$. We define $K_2 = \{x \in X : x \in [0, \frac{a^*}{a_*}]\}$. Similarly as in the proof of Theorem 6.2 we obtain that (1.6) has at least one nonnegative ω-periodic solution $x(t)$ with $0 \le x(t) \le \frac{a^*}{a_*}$. Since $F(x) > 0$, it is easy to see from (6.5) and (6.7), that $x(t) > 0$; i.e., (1.6) has at least one positive ω-periodic solution $x(t)$ with $0 < x(t) \le \frac{a^*}{a_*}$. $\square$

Example 6.5. Consider the equation

$$
x'''(t) + \frac{1}{1000}\big(1 - \frac{1}{2}\sin^2 t\big)x(t) = \frac{1}{1000}\big(1 - \frac{1}{3}\sin^2 t\big) - \frac{x(t-\cos^2 t)}{4000}.
$$

as in Example 6.3, we obtain $\frac{\sqrt{3}\pi}{10} < 4\pi/3$ holds. Moreover, $0 < \frac{1}{6000} \le f(t, x(t - \tau(t))) = \frac{1}{1000}(1 - \frac{1}{3}\sin^2 t) - \frac{x(t-\cos^2 t)}{4000} \le \frac{1}{1000}$ for all $t \in [0,\pi]$ and $x \in [0,2]$. All assumptions of Theorem 6.4 are satisfied and hence the equation has at least one positive solution $x(t)$ with $0 < x(t) \le 2$.

Theorem 6.6. *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, $c \in (-\frac{a_*}{a^*}, 0)$, and $-ca^* < f(t,x) - ca(t)x \le a_*$ for all $t \in [0,\omega]$ and $x \in [0,1]$, then* (1.6) *has at least one positive ω-periodic solution x with $0 < x(t) \le 1$.*

Proof. As in the proof of Theorem 6.2, define $K_3 = \{x \in X : x \in [0,1]\}$ and then K_3 is a bounded closed convex set in X. Q is a completely continuous, and S is contractive since $|c| < 1$. Now we prove $Qx + Sy \in K_3$ for all $x, y \in K_3$. By Lemma 4.3, we obtain

$$
\begin{aligned}
&(Qx)(t) + (Sy)(t) \\
&= P(f(t, x(t-\tau(t))) - ca(t)x(t-\tau(t))) + cy(t-\tau(t)) \\
&\le \frac{a^*}{a_*}\|T(f(t, x(t-\tau(t))) - ca(t)x(t-\tau(t)))\| + cy(t-\tau(t)) \\
&\le \frac{a^*}{a_*}\max_{t\in[0,\omega]}\int_t^{t+\omega} G_2(t,s)(f(s, x(s-\tau(s))) - ca(s)x(s-\tau(s)))ds + cy(t-\tau(t))
\end{aligned}
$$

$$\le \frac{a^*}{a_*}\int_t^{t+\omega} G_2(t,s)a_* ds = 1. \tag{6.8}$$

On the other hand, by Theorem 2.6 and Lemma 4.1

$$\begin{aligned}
&(Qx)(t) + (Sy)(t)\\
&= P(f(t,x(t-\tau(t))) - ca(t)x(t-\tau(t))) + cy(t-\tau(t))\\
&\ge \int_t^{t+\omega} G_2(t,s)(f(s,x(s-\tau(s))) - ca(s)x(s-\tau(s)))ds + cy(t-\tau(t))\\
&> -ca^*\frac{1}{a^*} + c = 0.
\end{aligned} \tag{6.9}$$

Combining (6.8) and (6.9), we obtain $Qx + Sy \in K_3$ for any $x, y \in K_3$. By Lemma 6.1, we obtain that (1.6) has at least one nonnegative ω-periodic solution $x(t)$ with $0 \le x(t) \le 1$. Since $F(x) > -ca^*$, by (6.9), we obtain $x(t) > 0$. So (1.6) has at least one positive ω-periodic solution x with $0 < x(t) \le 1$. $\square$

Example 6.7. Consider the equation

$$\begin{aligned}
&\big(x(t) + \frac{1}{4}x(t-\cos^2 t)\big)''' + \frac{1}{1000}\big(1 - \frac{1}{3}\sin^2 t\big)x(t)\\
&= \big(1 - \frac{1}{3}\sin^2 t\big)\big[\frac{1}{1500} - \frac{1}{4000}x(t-\cos^2 t)\big].
\end{aligned}$$

As in Example 6.2, we can verify that all the assumptions of Theorem 6.6 hold, then the equation has at least one positive solution x with $0 < x(t) \le 1$.

Remark 6.8. In a similar way, we can discuss the third order neutral differential equation

$$(x(t) - cx(t-\tau(t)))''' - a(t)x(t) = f(t, x(t-\tau(t))),$$

with $a \in C(\mathbb{R}, (0,\infty))$, $\tau \in C(\mathbb{R},\mathbb{R})$, $f \in C(\mathbb{R}\times[0,\infty), [0,\infty))$, $a(t), \tau(t)$ are ω-periodic functions, $f(t,x)$ is ω-periodic in t, and ω, c are constants with $|c| < 1$.

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