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VANISHING p-CAPACITY OF SINGULAR SETS FOR p-HARMONIC FUNCTIONS

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Abstract. In this article, we study a counterpart of the removable singularity property of p-harmonic functions. It is shown that p-capacity of the singular set of any p-harmonic function vanishes, and such function is always weakly $N(p-1)/(N-p)$-integrable. Several related results are also shown.

1. Introduction

This article estimates the size of the singular sets and the local behavior of solutions to some (quasilinear) elliptic equations of second order. The equations to be treated here are general enough to include those studied by Serrin in his milestone paper [11]. The size of singular sets is measured by the capacity, and the local behavior of the solution is described by the weak L^q norm, for appropriate q.

First, given $1 \le p < N$, we put

$$K^p = \{f \in L^{p^*}(\mathbb{R}^N, \mathbb{R}) : \nabla f \in L^p(\mathbb{R}^N, \mathbb{R}^N)\},$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. Also define

$$\mathrm{Cap}_p(A) = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla f|^p dx : f \ge 0,\ f \in K^p,\ A \subset \{f(x) \ge 1\}^\circ \Big\},$$

where $A \subset \mathbb{R}^N$ is a given subset, and B° indicates the interior of the set $B \subset \mathbb{R}^N$. The operator $\mathrm{Cap}_p(A)$ is called *p-capacity* of A in short, and provides an outer measure to $\mathbb{R}^N$; see [3] for more information.

Next, given an open set Ω in $\mathbb{R}^N$ with $N \ge 3$ and $1 < q < \infty$, the weak L^q space on Ω, denoted by $L^q_w(\Omega)$, is defined by

$$L^q_w(\Omega) = \{u \in L^1_{\rm loc}(\Omega) : \|u\|_{L^q_w(\Omega)} < +\infty\}$$

and

$$\|u\|_{L^q_w(\Omega)} = \sup\{|K|^{-1+1/q} \int_K |u| dx : K \subset \Omega \text{ compact}\},$$

where $|K|$ indicates the N-dimensional Lebesgue measure of K. Thus, we obtain $|x|^{-\alpha} \in L^{N/\alpha}_w(B)$ and $|x|^{-\alpha} \notin L^{N/\alpha}(B)$ for $0 < \alpha < N$ and $B = B_1(0)$. This

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L^q_w-space is sometimes called the Marcinkiewicz space or Lorentz $L^{q,\infty}$ space; see [18, 19].

The singular set is indicated by a closed set $\Sigma \subset \Omega$ in this paper. In the linear case, we study the second order elliptic operator of divergence form defined on $\Omega\setminus\Sigma$; i.e.,

$$Lu = \sum_{i,j=1}^{N} D_i(a_{ij}(x)D_j u) + c(x)u$$

satisfying the strict ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \tag{1.1}$$

for any $x \in \Omega \setminus \Sigma$ and $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, where $\delta > 0$ is a constant and $a_{ij}(x) = a_{ji}(x), c(x)$ are bounded measurable functions.

The function $u = u(x)$ discussed in the following theorem is defined on $\Omega \setminus \Sigma$, and is locally Hölder continuous there by the result of DeGiorgi, Nash, and Moser [6]. The crucial assumption is as follows:

- (A) There is $s_0 > 0$ such that $\Omega_{s_0} \subset\subset \Omega$, Ω_{s_0} has a Lipschitz boundary, and Ω_s is open for any $s \geq s_0$, where $\Omega_s = \{x \in \Omega \setminus \Sigma : |u(x)| > s\} \cup \Sigma$.

This means that Σ is an actual singular set of $u = u(x)$, and henceforth $u = u(x)$ is identified with a function defined on Ω, taking $|u| = +\infty$ on Σ unless otherwise stated.

Theorem 1.1. *Let $c(x) \geq 0$ a.e. $x \in \Omega \setminus \Sigma$, and $u = u(x) \in H^1_{\rm loc}(\Omega \setminus \Sigma)$ be a solution to*

$$Lu = 0 \quad in\, \Omega \setminus \Sigma$$

satisfying (A)*. Then, it holds that* $\mathrm{Cap}_2(\Sigma) = 0$ *and* $u \in L_w^{N/(N-2)}(\Omega)$.

There is an analogous result for the parabolic equation [10], i.e., the blow-up set $D(t) = \{x \in \Omega \mid u(x,t) = +\infty\} \subset\subset \Omega$ of the solution $u = u(x,t)$ to the differential inequality $u_t - \Delta u \geq 0$ is negligible with respect to the N-dimensional Lebesgue measure for a.e. t. See also [14] for further developments on this subject.

The next theorem is the simplest form of our result on the quasilinear case. Here, we obtain $u \in C^{1,\alpha}_{\rm loc}(\Omega \setminus \Sigma)$ by a theorem of Tolksdorf, DiBenedetto, and Lewis [17, 2, 9], and therefore, $\partial\Omega_{s_0}$ is smooth if s_0 is a regular value of u.

Theorem 1.2. *Let $1 < p < N$ and $u = u(x) \in W^{1,p}_{\rm loc}(\Omega \setminus \Sigma)$ be a solution to*

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad in\ \Omega \setminus \Sigma \tag{1.2}$$

satisfying (A)*. Then* $\mathrm{Cap}_p(\Sigma) = 0$ *and* $u \in L_w^{N(p-1)/(N-p)}(\Omega)$.

Remark 1.3. Henceforth, $B^m_r(z)$ denotes the m-dimensional ball centered at z with radius r. We have two typical examples related to Theorem 1.2. First, if $\Omega = B^N_1(0)$ and $1 < p < N$, then $u(x) = |x|^{(p-N)/(p-1)}$ is a solution to (1.2) for $\Sigma = \{0\}$, and actually, it holds that $\mathrm{Cap}_p(\Sigma) = 0$ and $u \in L_w^{N(p-1)/(N-p)}(\Omega)$ if $\frac{2N}{N+1} < p < N$.

Next, if $p < m < N$ is an integer, $\Omega = B^m_1(0) \times \mathbb{R}^{N-m}$, and

$$\tilde{x} = (x_1, \dots, x_m, \underbrace{0, \dots, 0}_{N-m})$$

for $x = (x_1, \dots, x_m, x_{m+1}, \dots, x_N)$, then $u(x) = |\tilde{x}|^{(p-m)/(p-1)}$ is a solution to (1.2) for $\Sigma = \{0\} \times \mathbb{R}^{N-m}$. In this example, the assumption (A) does not hold in the strict sense, because Ω_s is not bounded for any $s \geq 0$. But, if $H^{N-m}(A)$ denotes the $(N-m)$-dimensional Hausdorff measure of A, then it holds that $H^{N-m}(\Sigma \cap B_R^N(0)) < +\infty$ for any $R > 0$. This implies $\mathrm{Cap}_m(\Sigma \cap B_R^N(0)) = 0$ from the general theory and hence $\mathrm{Cap}_m(\Sigma) = 0$. More precisely, we have

$$\mathrm{Cap}_p(A) \leq CH^{N-p}(A)$$

and $H^{N-p}(A) < +\infty$ implies $\mathrm{Cap}_p(A) = 0$ for any $1 < p < N$. It holds also that

$$u \in L_w^{m(p-1)/(m-p)}(\Omega) \subset L_w^{N(p-1)/(N-p)}(\Omega)$$

because $m(p-1)/(m-p) \geq N(p-1)/(N-p)$ by $p < m \leq N$.

Remark 1.4. The above theorems may be compared to the removable singularity property studied by many authors. First, from the classical theorem by Carleson [1, p.88] if $u = u(x) \in L^\infty_{\rm loc}(\Omega)$ is a solution to

$$\Delta u = 0 \quad \text{in } \Omega \setminus \Sigma,$$

then the set Σ is removable (that is, there exists a harmonic function $\tilde{u}$ defined in Ω such that $\tilde{u} = u$ on $\Omega \setminus \Sigma$) if and only if $\mathrm{Cap}_2(\Sigma) = 0$.

This fact is extended to p-harmonic functions. Indeed, in [8, Theorem 7.36], Heinonen, Kilpeläinen, and Martio considered a more general equation

$$\operatorname{div} A(x, \nabla u) = 0 \tag{1.3}$$

with the vector function A satisfying the growth condition $A(x,\xi) \simeq |\xi|^{p-1}$. They proved that if $u \in L^\infty_{\rm loc}(\Omega)$ is a solution of (1.3) in $\Omega \setminus \Sigma$, then Σ is removable if and only if $\mathrm{Cap}_p(\Sigma) = 0$.

These results are concerned with bounded solutions, while Serrin [11] proved the following theorem concerning θ-integrable solutions (and [12] for the linear case): Let u be a continuous solution to the quasilinear elliptic equation of divergence form

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \tag{1.4}$$

in $\Omega \setminus \Sigma$, where A and B satisfy certain structural conditions admitting the p-Laplace equation as a typical example. First, if $u \in W^{1,p}_{\rm loc}(\Omega \setminus \Sigma)$ is a weak solution to (1.4) then u is locally Hölder continuous in $\Omega \setminus \Sigma$. One of the main theorem of [11], now says that if $\mathrm{Cap}_q(\Sigma) = 0$ for $1 < p \leq q \leq N$ and $u \in L^\theta(\Omega \setminus \Sigma)$ for $\theta > q(p-1)/(q-p)$, then Σ is removable; that is, there is continuous $\tilde{u}$ defined on all of Ω such that $\tilde{u} = u$ on $\Omega \setminus \Sigma$.

In the other result of [11], if $u = u(x)$ is a solution to (1.4) in $\Omega \setminus \{0\}$ with $B \equiv 0$ and $1 < p < N$, and satisfies $u \geq L$ for some $L > 0$, then either $\Sigma = \{0\}$ is removable or $u(x) \simeq |x|^{(p-N)/(p-1)} \to +\infty$ as $|x| \to 0$. We see that $u \in L_w^{N(p-1)/(N-p)}(\Omega)$ holds in the latter case.

The singular set Σ of our theorems are not removable. However, this set must be small measured by the capacity, just because it is an actual singular set of the solution. The solution, on the other hand, is neither locally bounded in Ω nor θ-integrable in $\Omega \setminus \Sigma$ for some θ from the results quoted above, but still obeys a profile of weak integrability in Ω.

This weak integrability is slightly worse than the condition for which Serrin's removability theorem holds, and is just the same as the one of the fundamental solution to the p-harmonic equation.

Remark 1.5. The solution in our theorems is assumed to be only in $W^{1,p}_{\rm loc}(\Omega\setminus\Sigma)$. In contrast with this, if there is $u=u(x)\in W^{1,p}(\Omega_0\setminus\Sigma)$ satisfying (A), then it follows that $\mathrm{Cap}_p(\Sigma)=0$, where $\Omega_0=\Omega_{s_0}$. In other words, under the cost of global p-integrability on $\Omega_0\setminus\Sigma$ with its first derivatives, this u does not need to be a solution to any equation to infer $\mathrm{Cap}_p(\Sigma)=0$. Here, $\Gamma_0=\partial\Omega_0$ may not be Lipschitz continuous.

In fact, since $|u|=+\infty$ on Σ, we obtain $\min\{|u|,s\}=s$ on Σ for $s>s_0$. Now, we define $f_s=f_s(x)\in K^p$ by

$$f_s(x)=\begin{cases}\frac{1}{s-s_0}\big(\min\{|u(x)|,s\}-s_0\big) & x\in\Omega_0\\ 0 & x\in\Omega_0^c.\end{cases}$$

Then $\Sigma\subset\{x\in\mathbb{R}^N:\ f_s(x)=1\}^\circ$ and

$$\nabla f_s=\begin{cases}\frac{1}{s-s_0}\nabla|u| & \text{on } \Omega_0\setminus\Omega_s\\ 0 & \text{on } \Omega_s\cup\Omega_0^c,\end{cases}$$

which implies

$$\begin{aligned}\mathrm{Cap}_p(\Sigma)&\le\int_{\mathbb{R}^N}|\nabla f_s|^p dx=\frac{1}{(s-s_0)^p}\int_{\Omega_0\setminus\Omega_s}|\nabla u|^p dx\\ &\le\frac{1}{(s-s_0)^p}\int_{\Omega_0\setminus\Sigma}|\nabla u|^p dx=o(1)\end{aligned}$$

as $s\to+\infty$ by $u\in W^{1,p}(\Omega_0\setminus\Sigma)$.

Here, we note two properties related to the above consideration. First, any function in $W^{1,p}(\Omega_0\setminus\Sigma)$ is identified with the one in $W^{1,p}(\Omega_0)$ if $\mathrm{Cap}_p(\Sigma)=0$, and therefore, each $u\in W^{1,p}(\Omega_0\setminus\Sigma)$ satisfying (A) (with $\Gamma_0=\partial\Omega_0$ not necessarily Lipschitz continuous) belongs to $W^{1,p}(\Omega_0)$. Next, $\mathrm{Cap}_p(\Sigma)=0$ follows from

$$\int_{\Omega_0\setminus\Omega_s}|\nabla u|^p dx=o(s^p)\quad \text{as } s\to+\infty \tag{1.5}$$

if $u\in W^{1,p}_{\rm loc}(\Omega\setminus\Sigma)$ satisfies (A). This fact is often used in the rest of the present paper.

If the solution $u=u(x)$ is sufficiently smooth on $\Omega\setminus\Sigma$, our theorems have a simple proof using classical co-area formula, Sard's lemma, and isoperimetric inequality. This argument is described in §2 for the reader's convenience. In the general case without regularity, we follow the argument of Talenti [15] to compensate the lack of smoothness of the solution. See §3.

2. Regular case

In the regular case, there is a transparent proof of Theorem 1.1. This section is devoted to the description of the main idea of the proof, restricted to this case. Thus, we treat the solution $u=u(x)$ to $\Delta u=0$ in $\Omega\setminus\Sigma$ satisfying (A).

Since u is smooth in $\Omega\setminus\Sigma$ in this case, we may assume that $s_0>0$ is a regular value of $|u|=|u|(x)$ by Sard's lemma. Let $\Omega_0=\Omega_{s_0}$. Then $\Gamma_0=\partial\Omega_0$ is smooth and the disjoint union of the boundaries of $\Omega_0^{\pm} = \{x \in \Omega_0 \setminus \Sigma \mid \pm u(x) > s_0\} \cup \Sigma$. We obtain $u \in H^1_{\rm loc}(\overline{\Omega}_0 \setminus \Sigma)$ and

$$\Delta u = 0, \quad |u| > s_0 \quad \text{in } \Omega_0 \setminus \Sigma, \quad |u| = s_0 \quad \text{on } \Gamma_0. \tag{2.1}$$

Furthermore, for any $s > s_0$,

$$\varphi_s = (\operatorname{sgn} u) \cdot \max\{s - |u|, 0\} \tag{2.2}$$

satisfies $\varphi_s \in H^1(\overline{\Omega}_0 \setminus \Sigma)$, $\operatorname{supp} \varphi_s \subset \overline{\Omega_0} \setminus \Sigma$,

$$\varphi_s|_{\Gamma_0} = (\operatorname{sgn} u) \cdot (s - s_0), \quad \varphi_s = 0 \quad \text{on } \Omega_s \setminus \Sigma,$$

and

$$\nabla \varphi_s = \begin{cases} -(\nabla u) & \text{on } \Omega_0 \setminus \overline{\Omega}_s \\ 0 & \text{on } \Omega_s \setminus \Sigma. \end{cases}$$

Testing this on (2.1), we obtain

$$\int_{\Omega_0 \setminus \Omega_s} |\nabla u|^2 dx = (s - s_0) K = o(s^2) \tag{2.3}$$

as $s \to +\infty$, where

$$K = -\int_{\Gamma_0} (\operatorname{sgn} u) \frac{\partial u}{\partial \nu} dH^{N-1}$$

and ν is the outer unit normal to Γ_0. Since Γ_0 is smooth, the above $K > 0$ is defined in the classical sense. This implies $\operatorname{Cap}_2(\Sigma) = 0$ by (2.3). See (1.5) of Remark 1.5.

Next, differentiating both sides of (2.3), we have

$$-\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u|^2 dx = \frac{d}{ds} \int_{\Omega_0 \setminus \Omega_s} |\nabla u|^2 dx = K$$

for $s \in (s_0, s')$, where $s' > s_0$ is arbitrary. Since $u = u(x)$ is smooth on $\Omega \setminus \Sigma$, Sard's lemma guarantees that the set of critical values of u has the one-dimensional Lebesgue measure 0. Then, from the co-area formula, we obtain

$$K = -\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u|^2 dx = \int_{\{|u| = s\}} |\nabla u| dH^{N-1} \quad \text{a.e. } s \in (s_0, s'). \tag{2.4}$$

We apply the co-area formula also to $\mu(s) = |\Omega_s| = \int_{\Omega_s} dx$. Again, Sard's lemma assures

$$-\mu'(s) = \int_{\{|u| = s\}} |\nabla u|^{-1} dH^{N-1} \quad \text{a.e. } s > s_0. \tag{2.5}$$

By (2.4), (2.5), and the Schwarz inequality

$$\Big(\int_{\{|u| = s\}} dH^{N-1} \Big)^2 \le \int_{\{|u| = s\}} |\nabla u| dH^{N-1} \cdot \int_{\{|u| = s\}} |\nabla u|^{-1} dH^{N-1},$$

now we obtain

$$H^{N-1}(\{|u| = s\})^2 \le K \cdot (-\mu'(s)) \quad \text{a.e. } s \in (s_0, s'). \tag{2.6}$$

The classical isoperimetric inequality in $\mathbb{R}^N$, on the other hand, implies

$$N C_N^{1/N} H^N(\Omega_s)^{(N-1)/N} \le H^{N-1}(\{|u| = s\}),$$

where C_N is the volume of N-dimensional unit ball. Combining this with (2.6), it follows that

$$N^2 C_N^{2/N} \mu(s)^{2(N-1)/N} \le K \cdot (-\mu'(s));$$

that is,

$$C(N,K) \le \mu(s)^{-2(N-1)/N} \cdot (-\mu'(s)) \quad \text{a.e. } s \in (s_0, s') \tag{2.7}$$

for $C(N,K) = N^2 C_N^{2/N} K^{-1}$.

If we define

$$\phi(\mu) = \frac{N}{N-2}\mu^{-(N-2)/N}, \tag{2.8}$$

then

$$\frac{d}{ds}\phi(\mu(s)) = \mu(s)^{-2(N-1)/N} \cdot (-\mu'(s)),$$

and therefore, (2.7) is written as

$$C(N,K) \le \frac{d}{ds}\phi(\mu(s)) \quad \text{a.e. } s \in (s_0, s').$$

Integrating both sides from s_0 to s' and rewriting s' to s, we obtain

$$\begin{gathered} C(N,K)(s-s_0) + \phi(\mu(s_0)) \le \phi(\mu(s)), \\ \phi(\mu(s))^{-1} \le \{C(N,K)(s-s_0) + \phi(\mu(s_0))\}^{-1}. \end{gathered} \tag{2.9}$$

Here, we used

$$\int_{s_0}^{s'} \frac{d}{ds}\phi(\mu(s))ds \le \phi(\mu(s')) - \phi(\mu(s_0)),$$

assured by the fact that $s \mapsto \phi(\mu(s))$ is non-decreasing. We note that the distribution function $\mu = \mu(s)$ is not necessarily absolutely continuous in s even if $u = u(x)$ is smooth in x. More precisely, it is only right-continuous and even discontinuous points can arise.

Multiplying both sides by s in (2.9), now we have

$$\begin{gathered} \big(\frac{N-2}{N}\big)s\mu(s)^{(N-2)/N} \le \frac{s}{C(N,K)(s-s_0) + \phi(\mu(s_0))} \\ s^{N/(N-2)}\mu(s) \le \big(\frac{N}{N-2}\big)^{N/(N-2)} s^{N/(N-2)}\{C(N,K)(s-s_0) + \phi(\mu(s_0))\}^{-N/(N-2)} \end{gathered}$$

for $s > s_0$, and therefore,

$$s^{N/(N-2)}\mu(s) = O(1) \quad \text{as } s \to +\infty.$$

This implies $u \in L_w^{N/(N-2)}(\Omega)$. See [5].

3. Irregular case

In the irregular case, we use the co-area formula and the isoperimetric inequality associated with the perimeter. Such tools were adopted by Talenti [15] in the proof of his comparison theorem to overcome the lack of smoothness of the solution. To begin with, we collect several facts concerning the perimeter used in later arguments.

First, the co-area formula to functions of bounded variation Fleming and Rishel [4] is applicable to $u \in W^{1,1}_{\rm loc}(\Omega \setminus \Sigma)$, and it holds that

$$-\frac{d}{ds}\int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u| dx = P(\Omega_s) \quad \text{a.e. } s_0 < s < s'. \tag{3.1}$$

The right-hand side abbreviates $P(\Omega_s, \mathbb{R}^N)$, where for the measurable set $E \subset \mathbb{R}^N$ and an open set $U \subset \mathbb{R}^N$, $P(E, U)$ denotes DeGiorgi's perimeter of E in U; i.e.,

$$P(E,U) = \sup \Big\{ \int_E \operatorname{div} \vec{g} dx : \vec{g} \in C_0^\infty(U, \mathbb{R}^N), \max_{x\in U} : \vec{g}(x)| \le 1 \Big\}.$$

A measurable set $E \subset \mathbb{R}^N$ satisfying $P(E) < +\infty$ is called a Caccioppoli set, or a set of finite perimeter in $\mathbb{R}^N$. It is a set whose indicator function has a bounded total variation on $\mathbb{R}^N$. See [7]. DeGiorgi's isoperimetric inequality is concerned with these Caccioppoli sets in $\mathbb{R}^N$. More precisely, if E is such a set, then

$$NC_N^{1/N} |E|^{(N-1)/N} \le P(E). \tag{3.2}$$

Finally, we use the general trace lemma. See [16, Chapter I, Theorem 1.2] or [13, Lemma 1.2.2] for the proof.

Lemma 3.1. *If $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $1 < q < \infty$, $q' = \frac{q}{q-1}$,*

$$E_q(\Omega) = \{\vec{v} \in L^q(\Omega)^N : \operatorname{div}\ \vec{v} \in L^q(\Omega)\},$$

and

$$\|\vec{v}\|_{E_q} = \Big(\|\vec{v}\|_q^q + \|\operatorname{div}\ \vec{v}\|_q^q \Big)^{1/q},$$

then there is a bounded linear operator, called the generalized normal component trace,

$$\Gamma_\nu : \vec{v} \in E_q(\Omega) \mapsto \Gamma_\nu \vec{v} \in W^{-1/q,q}(\partial\Omega) = \big(W^{1/q,q'}(\partial\Omega)\big)^*$$

such that $g \mapsto \langle g, \Gamma_\nu \vec{v} \rangle$ is compatible to the functional

$$g \in W^{1/q,q'}(\partial\Omega) \mapsto \int_{\partial\Omega} g(x)\nu(x) \cdot \vec{v}(x) dH^{N-1}$$

defined for $\vec{v} \in C^\infty(\overline{\Omega})^N$ and the exterior unit normal ν. It holds that

$$(\vec{v}, \nabla\varphi) + (\operatorname{div}\vec{v}, \varphi) = \langle \Gamma_\nu \vec{v}, \gamma_0 \varphi \rangle$$

for $\vec{v} \in E_q(\Omega)$ and $\varphi \in W^{1,q'}(\Omega)$, where

$$\gamma_0 : W^{1,q'}(\Omega) \to W^{1/q,q'}(\partial\Omega)$$

is the usual trace operator.

Proof of Theorem 1.1. As in the previous section, we note that $u \in H^1_{\rm loc}(\overline{\Omega}_0 \setminus \Sigma)$ satisfies

$$Lu = 0,\ |u| > s_0 \quad \text{in } \Omega_0 \setminus \Sigma, \quad |u| = s_0 \quad \text{on } \Gamma_0$$

for $\Omega_0 = \Omega_{s_0}$ and $\Gamma_0 = \partial\Omega_0$. Since $s_0 > 0$, this Γ_0 is the disjoint union of the Lipschitz boundaries of $\Omega_0^\pm = \{x \in \Omega_0 \setminus \Sigma : \pm u(x) > s_0\} \cup \Sigma$, and testing this by $\varphi_s = \varphi_s(x)$ defined in (2.2) is permitted. We obtain

$$\sum_{i,j=1}^N \int_{\Omega_0\setminus\Omega_s} a_{ij} D_j u D_i u dx = (s - s_0)K - \int_{\Omega_0\setminus\Omega_s} c|u|(s - |u|) dx, \tag{3.3}$$

where

$$K = -\langle \frac{\partial u}{\partial \nu_L}, \operatorname{sgn} u \rangle_{H^{-1/2}(\Gamma_0), H^{1/2}(\Gamma_0)}$$

and

$$\frac{\partial u}{\partial \nu_L} = \sum_{i,j=1}^{N} \nu_i a_{ij} D_j u \in H^{-1/2}(\Gamma_0) \equiv W^{-1/2,2}(\Gamma_0)$$

is the general trace of

$$\vec{v} = \Big(\sum_{j=1}^{N} a_{ij} D_j u \Big)_{i=1,\dots,N} \in E_{2,\mathrm{loc}}(\overline{\Omega_0} \setminus \Sigma).$$

We emphasize that $\operatorname{sgn} u = \pm 1$ exclusively on each component of Γ_0, because $u = u(x)$ is continuous in $\Omega \setminus \Sigma$. Using (1.1) and $c \geq 0$, we obtain

$$\delta \int_{\Omega_0 \setminus \Omega_s} |\nabla u|^2 dx \leq (s - s_0) K = o(s^2)$$

as $s \to +\infty$, and hence $\mathrm{Cap}_2(\Sigma) = 0$.

To show $u \in L_w^{N/(N-2)}(\Omega)$, we use the fact that

$$g_s(x) = c(x)|u(x)|(s - |u(x)|)$$

is non-negative and non-decreasing in s for each $x \in \Omega_0 \setminus \Omega_s$. The set $\Omega_0 \setminus \Omega_s$ is also non-decreasing in s, and therefore, the function

$$s \mapsto I(s) = \int_{\Omega_0 \setminus \Omega_s} c|u|(s - |u|) dx$$

is non-decreasing. Thus, differentiating both sides of (3.3), we obtain

$$\begin{aligned} \frac{d}{ds} \int_{\Omega_0 \setminus \Omega_s} \sum_{i,j=1}^{N} a_{ij} D_j u D_i u dx &= -\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} \sum_{i,j=1}^{N} a_{ij} D_j u D_i u dx \\ &\leq K \quad \text{a.e. } s \in (s_0, s'), \end{aligned} \tag{3.4}$$

where $s' > s_0$ is arbitrary.

The next lemma is a key ingredient of the proof, where $\mu(s) = |\Omega_s|$.

Lemma 3.2. *It holds that*

$$-\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u| dx \leq (-\mu'(s))^{1/2} \Big(-\frac{d}{ds} \int_{\Omega_s \setminus \overline{\Omega}_{s'}} \delta^{-1} \sum_{i,j=1}^{N} a_{ij} D_j u D_i u dx \Big)^{1/2} \tag{3.5}$$

a.e. $s \in (s_0, s')$.

Proof. First, the mapping

$$s \in (s_0, s') \mapsto \int_{\Omega_s \setminus \overline{\Omega}_{s'}} |\nabla u| dx$$

is non-increasing. Given $0 < h \ll 1$ in $s < s+h < s'$, we take its differential quotient. In fact, by the Schwarz inequality and (1.1), we obtain

$$\begin{aligned}
&\frac{1}{h}\Big[\int_{\Omega_s\setminus\overline{\Omega}_{s'}}|\nabla u|dx-\int_{\Omega_{s+h}\setminus\overline{\Omega}_{s'}}|\nabla u|dx\Big]\\
&=\frac{1}{h}\int_{\Omega_s\setminus\Omega_{s+h}}|\nabla u|dx\\
&\le\Big(\frac{1}{h}\int_{\Omega_s\setminus\Omega_{s+h}}dx\Big)^{1/2}\Big(\frac{1}{h}\int_{\Omega_s\setminus\Omega_{s+h}}|\nabla u|^2dx\Big)^{1/2}\\
&\le\Big(\frac{\mu(s)-\mu(s+h)}{h}\Big)^{1/2}\Big(\frac{1}{h}\int_{\Omega_s\setminus\Omega_{s+h}}\delta^{-1}\sum_{i,j=1}^N a_{ij}D_juD_iudx\Big)^{1/2}\\
&=\big(-\mu'(s)\big)^{1/2}\Big(-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}\delta^{-1}\sum_{i,j=1}^N a_{ij}D_juD_iudx\Big)^{1/2}+o(1)
\end{aligned}$$

as $h \downarrow 0$, and hence (3.5) follows. □

Now, we continue the proof of Theorem 1.1. It holds that

$$NC_N^{1/N}\mu(s)^{(N-1)/N}\le P(\Omega_s)=-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}|\nabla u|dx\quad\text{a.e. } s\in(s_0,s') \tag{3.6}$$

by (3.2)-(3.1). Combining this with (3.5), we obtain

$$N^2C_N^{2/N}\le\mu(s)^{-2(N-1)/N}(-\mu'(s))\Big(-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}\delta^{-1}\sum_{i,j=1}^N a_{ij}D_juD_iudx\Big)$$

for a.e. $s \in (s_0, s')$. Then (3.4) guarantees

$$C(N,K)\le\frac{d}{ds}\phi(\mu(s))\quad\text{a.e. } s\in(s_0,s')$$

for $\phi = \phi(\mu)$ defined by (2.8), where $C(N,K) = \delta N^2 C_N^{2/N} K^{-1}$.

At this stage, we can follow the argument in the previous section, and obtain $u \in L_w^{N/(N-2)}(\Omega)$.

Proof of Theorem 1.2. Testing

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)=0,\quad |u|>s_0\quad\text{in }\Omega_0\setminus\Sigma,\quad |u|=s_0\quad\text{on }\Gamma_0$$

by $\varphi_s = \varphi_s(x)$ of (2.2) is permitted similarly, and then we obtain

$$\int_{\Omega_0\setminus\Omega_s}|\nabla u|^pdx=(s-s_0)K=o(s^p)\quad\text{as } s\to+\infty, \tag{3.7}$$

where

$$K=-\langle|\nabla u|^{p-2}\frac{\partial u}{\partial\nu},\operatorname{sgn}u\rangle_{W^{-1/p',p'}(\Gamma_0),W^{1-1/p,p}(\Gamma_0)}$$

for $\frac{1}{p'} + \frac{1}{p} = 1$. Thus, it holds that $\mathrm{Cap}_p(\Sigma) = 0$. Differentiating (3.7) with respect to s, on the other hand, we obtain also

$$-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}|\nabla u|^pdx=\frac{d}{ds}\int_{\Omega_0\setminus\Omega_s}|\nabla u|^pdx=K\quad\text{a.e. } s\in(s_0,s'), \tag{3.8}$$

for $s' > s_0$ arbitrarily fixed. Then, the following lemma takes place of Lemma 3.2, which is proven by Hölder's inequality instead of the Schwarz inequality.

Lemma 3.3. *It holds that*

$$-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}|\nabla u|dx \le (-\mu'(s))^{1/p'}\Big(-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}|\nabla u|^p dx\Big)^{1/p}$$

for a.e. $s\in(s_0,s')$.

Inequality (3.6), on the other hand, is derived from DeGiorgi's isoperimetric inequality and Fleming-Rishel's co-area formula. This inequality, therefore, is applicable even to this case, and we obtain

$$NC_N^{1/N}\mu(s)^{(N-1)/N} \le (-\mu'(s))^{1/p'}\Big(-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}|\nabla u|^p dx\Big)^{1/p},$$

and hence

$$N^{p'}C_N^{p'/N} \le \mu(s)^{-p'(N-1)/N}(-\mu'(s))\Big(-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}}|\nabla u|^p dx\Big)^{p'/p}$$

for a.e. $s\in(s_0,s')$. Combining this with (3.8), we have

$$N^{p'}C_N^{p'/N} \le \mu(s)^{-\frac{p'(N-1)}{N}}K^{p'/p}\cdot(-\mu'(s));$$

i.e.,

$$C(N,K)\le \frac{d}{ds}\phi(\mu(s)) \quad \text{a.e. } s\in(s_0,s'), \tag{3.9}$$

for

$$\phi(\mu)=\frac{N(p-1)}{N-p}\mu^{-\frac{N-p}{N(p-1)}} \quad\text{and}\quad C(N,K)=N^{p'}C_N^{p'/N}K^{-p'/p}. \tag{3.10}$$

Integrating (3.9) from s_0 to s', rewriting s' to s, and noting the monotonicity of $s\mapsto\phi(\mu(s))$, we obtain

$$C(N,K)(s-s_0)+\phi(\mu(0))\le\phi(\mu(s)),$$
$$\phi(\mu(s))^{-1}\le\big(C(N,K)(s-s_0)+\phi(\mu(0))\big)^{-1}.$$

Multiplying both sides by s, now we have

$$\frac{N-p}{N(p-1)}\cdot s\mu(s)^{(N-p)/N(p-1)}\le\frac{s}{C(N,K)(s-s_0)+\phi(\mu(0))}$$
$$\Big(\frac{N-p}{N(p-1)}\Big)^{\frac{N(p-1)}{N-p}}\cdot s^{\frac{N(p-1)}{N-p}}\mu(s)\le s^{\frac{N(p-1)}{N-p}}\{C(N,K)(s-s_0)+\phi(\mu(0))\}^{-\frac{N(p-1)}{N-p}}.$$

This implies

$$s^{\frac{N(p-1)}{N-p}}\mu(s)=O(1)\quad\text{as } s\to\infty, \tag{3.11}$$

and hence $u\in L_w^{N(p-1)/(N-p)}(\Omega)$.

4. Generalizations

This section is devoted to several generalizations. First, we show the following result.

Theorem 4.1. *Regardless of the sign of* $c=c(x)$*, it holds that* $\mathrm{Cap}_2(\Sigma)=0$ *and* $\log(1+|u|)\in L_w^{N/(N-2)}(\Omega)$ *in Theorem 1.1.*

Proof. The second term of the right-hand side of (3.3),

$$-\int_{\Omega_0\setminus\Omega_s} c|u|(s-|u|)dx = -I(s)$$

is estimated from above by

$$-I(s) \le \|c_-\|_\infty \int_{\Omega_0\setminus\Omega_s} |u|(s-|u|)dx \le s^2\|c_-\|_\infty \int_{\Omega_0\setminus\Omega_s} \frac{|u|}{s}dx$$

where $c(x) = c_+(x) - c_-(x)$, $c_\pm \ge 0$.

Recall that by assumption (A), Ω_0 is a compact subset of Ω. Since

$$\int_{\Omega_0\setminus\Omega_s} \frac{|u|}{s}dx = \int_{\Omega_0} I_{\Omega_s^c}(x)\frac{|u|}{s}dx$$

and

$$\big|I_{\Omega_s^c}(x)\frac{|u|}{s}\big| \le 1 \in L^1(\Omega_0), \quad I_{\Omega_s^c}(x)\frac{|u(x)|}{s} \to 0 \quad \text{a.e. } x\in\Omega_0$$

as $s\to\infty$, where I_A is the indicator function of a set A, we obtain

$$\int_{\Omega_0\setminus\Omega_s} \frac{|u|}{s}dx = o(1)$$

from the dominated convergence theorem. Going back to (3.3), we have

$$\int_{\Omega_0\setminus\Omega_s} |\nabla u|^2 dx = o(s^2).$$

Hence $\mathrm{Cap}_2(\Sigma) = 0$. Also we have

$$\int_{\Omega_0\setminus\Omega_s} c|u|(s-|u|)dx = o(s^2).$$

Since $c = c(x) \in L^\infty(\Omega\setminus\Sigma)$, the above $I = I(s)$ is a function of bounded variation in s. Given $0 < h \ll 1$ in $s < s+h < s'$, we obtain

$$\begin{aligned}
&-\Big(\frac{I(s+h)-I(s)}{h}\Big)\\
&= -\frac{1}{h}\Big(\int_{\Omega_0\setminus\Omega_{s+h}} c|u|(s+h-|u|)dx - \int_{\Omega_0\setminus\Omega_s} c|u|(s-|u|)dx\Big)\\
&= -\frac{1}{h}\int_{\Omega_s\setminus\Omega_{s+h}} c|u|(s+h-|u|)dx - \int_{\Omega_0\setminus\Omega_s} c|u|dx\\
&\le \|c_-\|_\infty\Big(\int_{\Omega_s\setminus\Omega_{s+h}} |u|dx + \int_{\Omega_0\setminus\Omega_s} |u|dx\Big)\\
&= \|c_-\|_\infty \int_{\Omega_0\setminus\Omega_{s+h}} |u|dx\\
&\le \|c_-\|_\infty|\Omega_0|(s+h).
\end{aligned}$$

Therefore,

$$\frac{d}{ds}\int_{\Omega_0\setminus\Omega_s} \sum_{i,j=1}^N a_{ij}D_j u D_i u\,dx \le K + \|c_-\|_\infty|\Omega_0|s \quad \text{a.e. } s\in(s_0,s').$$

Using this instead of (3.4), we obtain

$$\phi(\mu(s))^{-1} \le C \left(\log(s+1)\right)^{-1},$$
$$\left(\log(s+1)\right)^{N/(N-2)} \mu(s) = O(1) \quad \text{as } s \to +\infty$$

for $\phi = \phi(\mu)$ defined by (2.8). The last equality implies that $\log(1+|u|) \in L_w^{N/(N-2)}(\Omega)$. □

Similar results are also valid to the problems formulated by Serrin [11]. Treating a simple case, we take the mappings $A : \Omega \setminus \Sigma \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $B : \Omega \setminus \Sigma \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that $x \mapsto A(x,z,\xi)$ and $x \mapsto B(x,z,\xi)$ are measurable for each $(z,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(z,\xi) \mapsto A(x,z,\xi)$ and $(z,\xi) \mapsto B(x,z,\xi)$ are continuous in (z,ξ) for a.e. $x \in \Omega \setminus \Sigma$. We assume the ellipticity

$$A(x,z,\xi) \cdot \xi \ge \delta |\xi|^p$$

and the growth rates

$$\begin{gathered} |A(x,z,\xi)| \le \Lambda |\xi|^{p-1}, \\ |B(x,z,\xi)| \le a(x)|\xi|^{p-1} + b(x)|z|^{p-1} \end{gathered} \tag{4.1}$$

for $(z,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega \setminus \Sigma$, where $\delta > 0$, $1 < p < N$, and $\Lambda > 0$ are constants, and $a, b \in L^\infty_{\rm loc}(\Omega \setminus \Sigma)$. As Serrin [11] proved among other things, in this case the solution $u = u(x) \in W^{1,p}_{\rm loc}(\Omega \setminus \Sigma)$ to

$$\operatorname{div} A(x,u,\nabla u) = B(x,u,\nabla u) \quad \text{in } \Omega \setminus \Sigma \tag{4.2}$$

is locally Hölder continuous. Then, we obtain the following result.

Theorem 4.2. *If $u \in W^{1,p}_{\rm loc}(\Omega \setminus \Sigma)$ is a solution to (4.2) satisfying (A), then $\operatorname{Cap}_p(\Sigma) = 0$ and $u \in L_w^{N(p-1)/(N-p)}(\Omega)$, provided that*

$$(\operatorname{sgn} z) \cdot B(x,z,\xi) \le 0 \tag{4.3}$$

for any $(\xi,z) \in \mathbb{R}^N \times \mathbb{R}$ and a.e. $x \in \Omega \setminus \Sigma$.

In the other case without (4.3), we obtain $\operatorname{Cap}_p(\Sigma) = 0$ if the second relation of (4.1) is slightly strengthened; i.e., any $\varepsilon > 0$ admits $C_\varepsilon > 0$ such that

$$|B(x,z,\xi)| \le \varepsilon |\xi|^{p-1} + C_\varepsilon |z|^{p-1} \tag{4.4}$$

for $(z,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega \setminus \Sigma$. If $\varepsilon = 0$ is attained in (4.4), then $\log(1+|u|) \in L_w^{\frac{N(p-1)}{N-p}}(\Omega)$ follows furthermore.

Now we check several key points.

(1) Testing

$$\begin{gathered} \operatorname{div} A(x,u,\nabla u) = B(x,u,\nabla u), \quad |u| > s_0 \quad \text{in } \Omega_0 \setminus \Sigma \\ |u| = s_0 \quad \text{on } \Gamma_0 \end{gathered}$$

with $\varphi_s = \varphi_s(x)$ of (2.2), we obtain

$$\int_{\Omega_0 \setminus \Omega_s} A(x,u,\nabla u) \cdot \nabla u dx = (s - s_0)K + \int_{\Omega_0 \setminus \Omega_s} B(x,u,\nabla u)(\operatorname{sgn} u)(s - |u|) dx, \tag{4.5}$$

where $K = -\langle \Gamma_\nu A(x,u,\nabla u), \operatorname{sgn} u \rangle_{W^{-1/p',p'}(\Gamma_0), W^{1-1/p,p}(\Gamma_0)}$. If (4.3) holds, then the second term of the right-hand side of (4.5) is non-positive, and this implies

$$\delta \int_{\Omega_0 \setminus \Omega_s} |\nabla u|^p dx \le (s - s_0)K = o(s^p) \quad \text{as } s \to +\infty.$$

In the other case of (4.4), we obtain

$$\begin{aligned}
&\delta \int_{\Omega_0\setminus\Omega_s} |\nabla u|^p dx \\
&\le (s-s_0)K + \varepsilon\cdot s\int_{\Omega_0\setminus\Omega_s} |\nabla u|^{p-1}dx + C_\varepsilon \int_{\Omega_0\setminus\Omega_s} |u|^{p-1}(s-|u|)dx \\
&\le o(s^p) + \varepsilon s|\Omega_0|^{1/p}\Big(\int_{\Omega_0\setminus\Omega_s} |\nabla u|^p dx\Big)^{(p-1)/p} \\
&\le o(s^p) + \frac{\delta}{2}\int_{\Omega_0\setminus\Omega_s} |\nabla u|^p dx + C\varepsilon^p s^p.
\end{aligned}$$

Here, as before, we estimate

$$\int_{\Omega_0\setminus\Omega_s} |u|^{p-1}(s-|u|)dx \le s^p \int_{\Omega_0\setminus\Omega_s} (\frac{|u|}{s})^{p-1}dx = o(s^p)$$

by the dominated convergence theorem. Then $\mathrm{Cap}_p(\Sigma)=0$ follows.

(2) When (4.3) holds, we differentiate (4.5) in s, using the monotonicity of

$$s\mapsto I(s) = \int_{\Omega_0\setminus\Omega_s} B(x,u,\nabla u)(\operatorname{sgn} u)(s-|u|)dx.$$

Then

$$\begin{aligned}
&\frac{d}{ds}\int_{\Omega_0\setminus\Omega_s} A(x,u,\nabla u)\cdot\nabla u dx \\
&= -\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}} A(x,u,\nabla u)\cdot\nabla u dx \\
&= K + \frac{d}{ds}\int_{\Omega_0\setminus\Omega_s} B(x,u,\nabla u)(\operatorname{sgn} u)(s-|u|)dx \\
&\le K \quad \text{a.e. } s\in(s_0,s').
\end{aligned} \tag{4.6}$$

Even in the other case without (4.3), $s\mapsto I(s)$ is a function of bounded variation, and it holds that

$$\frac{I(s+h)-I(s)}{h} \le \int_{\Omega_0\setminus\Omega_{s+h}} |B(x,u,\nabla u)|dx$$

for $s<s+h<s'$. Therefore, if $\varepsilon=0$ is attained in (4.4), we obtain

$$\begin{aligned}
\frac{d}{ds}\int_{\Omega_0\setminus\Omega_s} A(x,u,\nabla u)\cdot\nabla u dx &\le K + C_0\int_{\Omega_0\setminus\Omega_s} |u|^{p-1}dx \\
&\le K + C_0|\Omega_0|s^{p-1} \quad \text{a.e. } s\in(s_0,s').
\end{aligned} \tag{4.7}$$

(3) We establish Talenti's inequality; i.e.,

$$-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}} |\nabla u|dx \le (-\mu'(s))^{1/p'}\Big(-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}} \delta^{-1}A(x,u,\nabla u)\cdot\nabla u dx\Big)^{1/p}$$

for a.e. $s\in(s_0,s')$ and then combine this with DeGiorgi's isoperimetric inequality and Fleming-Rishel formula (3.6); i.e.,

$$N^{p'}C_N^{p'/N} \le \mu(s)^{-\frac{p'(N-1)}{N}}(-\mu'(s))\Big(-\frac{d}{ds}\int_{\Omega_s\setminus\overline{\Omega}_{s'}} \delta^{-1}A(x,u,\nabla u)\cdot\nabla u dx\Big)^{p'/p} \tag{4.8}$$

for a.e. $s\in(s_0,s')$, where $\frac{1}{p'}+\frac{1}{p}=1$ and $\mu(s)=|\Omega_s|$.

(4) If (4.6) is available, then (4.8) implies

$$\delta^{p'/p} N^{p'} C_N^{p'/N} \le \mu(s)^{-p'(N-1)/N} K^{p'/p} \cdot (-\mu'(s)).$$

In this case, we obtain

$$\frac{d}{ds}\varphi(\mu(s)) \ge c \quad \text{a.e. } s \in (s_0, s')$$

for $\phi = \phi(\mu)$ defined by (3.10) with a constant $c > 0$, and we end up with (3.11).

In the other case of (4.7), we obtain

$$\frac{d}{ds}\varphi(\mu(s)) \ge \frac{c}{1+s} \quad \text{a.e. } s \in (s_0, s').$$

Then

$$(1+\log s)^{\frac{N(p-1)}{N-p}} \mu(s) = O(1) \quad \text{as } s \to +\infty.$$

This implies $\log(1+|u|) \in L_w^{\frac{N(p-1)}{N-p}}(\Omega)$.

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